KYBERNETIKA — VOLUME 41 (2005), NUMBER 4, PAGES 425 – 434

TRANSFORMATIONS OF COPULAS

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Transformations of copulas by means of increasing bijections on the unit interval and attractors of copulas are discussed. The invariance of copulas under such transformations as well as the relationship to maximum attractors and Archimax copulas is investigated.

Keywords: copula, transformation of copulas, invariant copulas, maximum attractor, Archimax copula

AMS Subject Classification: 62E15, 62E99

1. INTRODUCTION

Sklar's Theorem [17] states that each random vector (X, Y) is characterized by some copula C in the sense that for its joint distribution H_{XY} and for the corresponding marginal distributions F_X and F_Y we have $H_{XY}(x, y) = C(F_X(x), F_Y(y))$.

In this contribution we investigate transformations of copulas by functions in one variable. Such transformations play a role in statistics: as an example, if $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are iid random vectors (characterized by some copula C) then the random vector $(\max(X_1, X_2, \ldots, X_n), \max(Y_1, Y_2, \ldots, Y_n))$ is characterized by the $\varphi_{1/n}$-transform of C in the sense of (1) below with $\varphi_{1/n}(x) = x^{1/n}$ (see [18]).

Copulas form a subclass of the class $\mathcal{V}$ of functions $V\colon [0, 1]^2 \to [0, 1]$ which are continuous, non-decreasing in each component and satisfy $\operatorname{Ran} V = [0, 1]$ (the elements of $\mathcal{V}$ are also called *binary aggregation operators* [4, 11]).

If Φ denotes the set of all increasing bijections from $[0, 1]$ to $[0, 1]$, then for each $\varphi \in \Phi$ and for each $V \in \mathcal{V}$ consider the function $V_\varphi\colon [0, 1]^2 \to [0, 1]$ given by

$$V_\varphi(x, y) = \varphi^{-1}(V(\varphi(x), \varphi(y))). \tag{1}$$

Evidently, we always have $V_\varphi \in \mathcal{V}$, and $U \le V$ implies $U_\varphi \le V_\varphi$. Moreover, for all $\varphi, \xi \in \Phi$ we always get $(V_\varphi)_\xi = V_{\varphi\circ\xi}$.

The transition from V to V_φ preserves many algebraic properties, among them commutativity and associativity as well as the existence of a neutral element, of an annihilator, of zero divisors, and of idempotent elements. Also, if V has a neutral element and/or an annihilator in the set $\{0, 1\}$, so has V_φ.

If, for $V \in \mathcal{V}$ and $\varphi \in \Phi$, we have $V_\varphi = V$ then V is called *φ-invariant* (compare [10]). As an immediate consequence, each φ-invariant $V \in \mathcal{V}$ is $\varphi_{(n)}$-invariant for each $n \in \mathbb{Z}$, where $\varphi_{(0)} = \mathrm{id}_{[0,1]}$ and, for each $n \in \mathbb{N}$, $\varphi_{(n)} = \varphi \circ \varphi_{(n-1)}$ and $\varphi_{(-n)} = (\varphi_{(n)})^{-1}$. Also, if V is both φ-invariant and ξ-invariant then it is $(\varphi \circ \xi)$-invariant. Moreover, V is φ-invariant if and only if V_ξ is $(\xi^{-1} \circ \varphi \circ \xi)$-invariant for each $\xi \in \Phi$.

The only elements of $\mathcal{V}$ which are φ-invariant for all $\varphi \in \Phi$ are the minimum, the maximum, and the two projections π_1 and π_2 given by $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$, respectively [15].

If, for $V \in \mathcal{V}$ and $\varphi \in \Phi$, the limit $\lim_{n\to\infty} V_{\varphi_{(n)}}$ exists and is an element of $\mathcal{V}$, then

$$V_\varphi^* = \lim_{n\to\infty} V_{\varphi_{(n)}}$$

is called a *φ-attractor* of V.

It is immediately seen that $U \in \mathcal{V}$ is a φ-attractor of some $V \in \mathcal{V}$ if and only if U is φ-invariant. Also, if $V \leq V_\varphi^*$ then for all $U \in \mathcal{V}$ with $V \leq U \leq V_\varphi^*$ we have $U_\varphi^* = V_\varphi^*$.

Observe that for each jointly strictly monotone $V \in \mathcal{V}$ (i. e., $V(x,y) < V(x^*,y^*)$ whenever $x < x^*$ and $y < y^*$) the diagonal section $\delta_V \colon [0,1] \to [0,1]$ given by $\delta_V(x) = V(x,x)$ is an element of Φ. Moreover, if V is also associative then V is δ_V-invariant. In this statement, the associativity assumption may not be dropped: the function $V \colon [0,1]^2 \to [0,1]$ given by $V(x,y) = \frac{1}{2}(\min(x,y) + \max(x+y-1,0))$ is jointly strictly monotone (note that V is a copula). Its diagonal section $\delta_V \in \Phi$ is given by $\delta_V(x) = \max(\frac{x}{2}, \frac{3x-1}{2})$, but $V(0.4,0.8) = 0.3 \neq 0.2 = V_{\delta_V}(0.4,0.8)$.

2. TRANSFORMATIONS OF COPULAS

Recall that a *(two-dimensional) copula* is a function $C \colon [0,1]^2 \to [0,1]$ such that $C(0,x) = C(x,0) = 0$ and $C(1,x) = C(x,1) = x$ for all $x \in [0,1]$, and C is 2-increasing, i. e., for all $x, x^*, y, y^* \in [0,1]$ with $x \leq x^*$ and $y \leq y^*$ for the volume Vol_C of the rectangle $[x,x^*] \times [y,y^*]$ we have

$$\mathrm{Vol}_C([x,x^*] \times [y,y^*]) = C(x,y) - C(x,y^*) + C(x^*,y^*) - C(x^*,y) \geq 0. \qquad (2)$$

Important examples of copulas are the *Fréchet–Hoeffding bounds* M and W given by $M(x,y) = \min(x,y)$ and $W(x,y) = \max(x+y-1,0)$, respectively, and the *product* Π given by $\Pi(x,y) = x \cdot y$. Obviously, each copula C satisfies $W \leq C \leq M$.

Clearly, $a \in [0,1]$ is an idempotent element of C if and only if $\varphi(a)$ is an idempotent element of C_φ, and M is the only copula which is φ-invariant for each $\varphi \in \Phi$.

In general, the fact that C is a copula is neither necessary nor sufficient for C_φ being a copula.

Example 2.1. For $\varphi \in \Phi$ defined by $\varphi(x) = x^2$ we have that W_φ, which is given by $W_\varphi(x,y) = \sqrt{\max(x^2+y^2-1,0)}$, is not Lipschitz (see [9, Example 1.26]) and, therefore, not a copula. However, for $\xi \in \Phi$ defined by $\xi(x) = \sqrt{1-(1-x)^2}$ the transformation $(W_\varphi)_\xi$ is a copula (see [13, Table 4.1, (4.2.2)]).

Now we first are interested under which conditions C_φ is a copula and under which conditions a copula C is φ-invariant.

Example 2.2. For $p \in]0,\infty[$ consider the function $\varphi_p \in \Phi$ defined by $\varphi_p(x) = x^p$.

(i) The product Π is φ_p-invariant for each $p \in]0,\infty[$.

(ii) The Fréchet–Hoeffding lower bound W is φ_p-invariant only if $p = 1$, and W_{φ_p} is a copula only if $p \in]0,1]$.

The following result follows from [12, Theorem 7]:

Proposition 2.3. Assume that $V \in \mathcal{V}$ is associative and has neutral element 1, and let $\varphi \in \Phi$. Then V_φ is a copula if and only if for all $x, y, z \in [0,1]$

$$|\varphi^{-1}(V(x,z)) - \varphi^{-1}(V(y,z))| \le |\varphi^{-1}(x) - \varphi^{-1}(y)|. \tag{3}$$

Observe that for an associative copula C and $\varphi \in \Phi$ the function C_φ is a copula if and only if C satisfies (3) for all $x, y, z \in [0,1]$. This can be seen either from Proposition 2.3 or directly from the fact that an associative function $V \in \mathcal{V}$ with neutral element 1 (i. e., a continuous t-norm [9]) is a copula if and only if it is 1-Lipschitz.

Theorem 2.4. For each $\varphi \in \Phi$ the following are equivalent:

(i) The function φ is concave.

(ii) For each copula C the function C_φ is a copula.

Proof. In order to show that (ii) implies (i), recall that W is an Archimedean copula and that $t_W \colon [0,1] \to [0,\infty]$ given by $t_W(x) = 1 - x$ is an additive generator of W. Then W_φ is also an Archimedean copula with additive generator $t_{W,\varphi} = 1 - \varphi$. However, because of [16, Theorem 6.3.3], additive generators of Archimedean copulas are necessarily convex functions, and thus $\varphi = 1 - t_{W,\varphi}$ is a concave bijection.

Conversely, it suffices to show that C_φ is 2-increasing. Fix arbitrary elements $x, x^*, y, y^* \in [0,1]$ with $x \le x^*$ and $y \le y^*$. Then also $\varphi(x), \varphi(x^*), \varphi(y), \varphi(y^*) \in [0,1]$ with $\varphi(x) \le \varphi(x^*)$ and $\varphi(y) \le \varphi(y^*)$, and the fact that C is 2-increasing implies

$$C(\varphi(x^*), \varphi(y^*)) - C(\varphi(x^*), \varphi(y)) \ge C(\varphi(x), \varphi(y^*)) - C(\varphi(x), \varphi(y)).$$

From the monotonicity of C it follows that $C(\varphi(x^*), \varphi(y)) \ge C(\varphi(x), \varphi(y))$. As a consequence of the convexity of the function φ^{-1} we obtain

$$\begin{aligned} &\varphi^{-1}(C(\varphi(x), \varphi(y^*))) - \varphi^{-1}(C(\varphi(x), \varphi(y))) \\ &\le \varphi^{-1}(C(\varphi(x^*), \varphi(y)) + C(\varphi(x), \varphi(y^*)) - C(\varphi(x), \varphi(y))) - \varphi^{-1}(C(\varphi(x), \varphi(y))) \\ &\le \varphi^{-1}(C(\varphi(x^*), \varphi(y^*))) - \varphi^{-1}(C(\varphi(x^*), \varphi(y))), \end{aligned}$$

i. e., $C_\varphi(x, y^*) - C_\varphi(x, y) \le C_\varphi(x^*, y^*) - C_\varphi(x^*, y)$, thus proving that C_φ is 2-increasing. □

Example 2.5. Fix $\varepsilon, \delta \geq 0$. Then the function $C_{\varepsilon,\delta}\colon [0,1]^2 \to [0,1]$ defined by

$$C_{\varepsilon,\delta}(x,y) = xy\left(1 + x^{\varepsilon}y^{\delta}(1-x)(1-y)\right)$$

is a (non-associative) copula, and for each $n \in \mathbb{N}$ also $(C_{\varepsilon,\delta})_{\varphi_{1/n}}$ given by

$$(C_{\varepsilon,\delta})_{\varphi_{1/n}}(x,y) = xy\left(1 + x^{\varepsilon/n}y^{\delta/n}\left(1-x^{1/n}\right)\left(1-y^{1/n}\right)\right)^n$$

is a copula. Observe that for all $(x,y) \in [0,1]^2$ we have

$$\lim_{n\to\infty} (C_{\varepsilon,\delta})_{\varphi_{1/n}}(x,y) = xy.$$

Moreover, for each $n > 1$ the product Π is the $\varphi_{1/n}$-attractor of $C_{\varepsilon,\delta}$.

Note that, in general, transformations do not preserve the structure of copulas. For example, a transformation of a shuffle of M [13] is not necessarily a shuffle of M, and also topological properties of the support of a copula may be changed by a transformation.

3. INVARIANT COPULAS

Denote by Φ_c the set of all concave functions in Φ. Clearly, if $\psi \in \Phi_c$ then also $\psi_{(n)} \in \Phi_c$ for each $n \in \mathbb{N}$. Therefore, we only look for copulas which are ψ-attractors for some $\psi \in \Phi_c$. Let us start with some negative result whose proof is straightforward.

Lemma 3.1. Let $C \neq M$ be a copula which satisfies at least one of the following properties:

(i) There exists an idempotent element of C in $]0,1[$, and 1 is not an accumulation point of the set of idempotent elements in $]0,1[$.

(ii) There exists $u \in]0,1[$ such that each $x \in [u,1]$ is an idempotent element of C.

(iii) There exists a zero divisor of C.

Then C is ψ-invariant only if $\psi = \mathrm{id}_{[0,1]}$.

On the other hand, if for some $\psi \in \Phi_c$ we have $W_\psi^* = M$ then we obtain $C_\psi^* = M$ for each copula C.

Lemma 3.2. If $\psi \in \Phi_c$ satisfies $\psi'(1^-) = 0$ then no Archimedean copula is a ψ-attractor.

Proof. Fix $\psi \in \Phi \setminus \{\mathrm{id}_{[0,1]}\}$ and suppose that some Archimedean copula C with continuous additive generator $t\colon [0,1] \to [0,\infty]$ is a ψ-attractor or, equivalently, satisfies $C_\psi = C$. Then

$$C(x,y) = C_\psi(x,y) = (t\circ\psi)^{(-1)}(t\circ\psi(x) + t\circ\psi(y))$$

implies that $t \circ \psi$ is an additive generator of C, i.e., we have $t \circ \psi = k \cdot t$ for some $k \in]0,1[\cup]1,\infty[$. Then $t(0) = \infty$, i.e., t is a bijection and we get $\psi(x) = t^{-1}(k \cdot t(x))$ for each $x \in [0,1]$. The convexity of t implies

$$t'(x) \leq t'(\psi(x)) = t'(t^{-1}(k \cdot t(x))) < 0$$

for all points of differentiability of t in $]0,1[$, yielding

$$\psi'(x) = \frac{k \cdot t'(x)}{t'(t^{-1}(k \cdot t(x)))} \geq k,$$

which means that necessarily $\psi'(1^-) \geq k > 0$. □

Example 3.3. Consider the function $\psi \in \Phi_c$ defined by $\psi(x) = 1 - (1-x)^2$ and observe that $\psi_{(n)}(x) = 1 - (1-x)^{2^n}$ and $\psi_{(-n)}(x) = 1 - (1-x)^{2^{-n}}$ for each $n \in \mathbb{N}$. Then for each $(x,y) \in [0,1]^2$ we obtain

$$W_{\psi_{(n)}}(x,y) = 1 - \min((1-x)^{2^n} + (1-y)^{2^n}, 1)^{2^{-n}},$$

implying $W_\psi^* = M$. Therefore $C_\psi^* = M$ for each copula C, i.e., M is the only ψ-attractor in the class of copulas. Note also that $\psi'(1^-) = 0$.

When investigating copulas which are invariant with respect to some fixed $\varphi \in \Phi$; we can restrict ourselves to transformations φ with trivial fixed points 0 and 1 only (i.e., satisfying $\varphi(x) \neq x$ for all $x \in]0,1[$), as a consequence of the following result. Recall that for $\varphi \in \Phi$ the set $[0,1] \setminus \{x \in [0,1] \mid \varphi(x) = x\}$ is open and can be written as a union of pairwise disjoint open intervals $]a_k, b_k[$, $k \in K$.

Proposition 3.4. Assume that $\varphi \in \Phi \setminus \{\mathrm{id}_{[0,1]}\}$ and let

$$[0,1] \setminus \{x \in [0,1] \mid \varphi(x) = x\} = \bigcup_{k \in K}]a_k, b_k[,$$

where the intervals $]a_k, b_k[$ are pairwise disjoint. Then a copula C is φ-invariant if and only if it can be written as ordinal sum $C = (\langle a_j, b_j, C_j \rangle)_{j \in J}$ with $K \subseteq J$, where for each $k \in K$ the copula C_k is φ_k-invariant, with $\varphi_k \colon [0,1] \to [0,1]$ being given by

$$\varphi_k(x) = \frac{\varphi(a_k + (b_k - a_k)x) - a_k}{b_k - a_k}.$$

(Note that for each $j \in J \setminus K$, C_j can be an arbitrary copula.)

4. ARCHIMAX COPULAS

The class of increasing bijections φ_p introduced in Example 2.2 contains all transformations $\varphi_{1/n}$ mentioned in the introduction. Moreover, because of the Lipschitz continuity, a copula C is φ_p-invariant for each $p \in]0,\infty[$ if and only if C is $\varphi_{1/n}$-invariant for each $n \in \mathbb{N}$.

Following [7] (compare also [5]), a copula C^* is said to be the *maximum attractor of the copula* C (or, equivalently, C *belongs to the maximum domain of attraction of* C^*) if for all $(x, y) \in [0,1]^2$ we have

$$\lim_{n\to\infty} C^n(x^{1/n}, y^{1/n}) = C^*(x, y).$$

Observe that for each $\varepsilon, \delta \geq 0$ and for the copula $C_{\varepsilon,\delta}$ considered in Example 2.5 we have $(C_{\varepsilon,\delta})^* = \Pi$, i. e., the product is a maximum attractor of $C_{\varepsilon,\delta}$.

Evidently, each copula C which is φ_p-invariant for each $p \in]0, \infty[$ is a maximum attractor of itself, i. e., $C^* = C$.

The set of all maximum attractor copulas will be denoted by $\mathcal{M}$. Putting

$$\mathcal{A} = \{A\colon [0,1] \to [0,1] \mid A \text{ is convex and } \max(x, 1-x) \leq A(x) \text{ for all } x \in [0,1]\},$$

from [14, 18] (compare also [6]) we know that each maximum attractor copula C^* can be expressed in the form

$$C^*(x, y) = e^{\log(xy)\cdot A(\frac{\log x}{\log(xy)})} \tag{4}$$

for some $A \in \mathcal{A}$. Evidently, Π is the weakest maximum attractor and M is the strongest one. The class $\mathcal{M}$ is closed under suprema and weighted geometric means. Although W belongs to the maximum domain of attraction of Π, there are copulas not belonging to any maximum domain of attraction.

Example 4.1.

(i) This example is derived from [3, pp. 166–167]: consider the strict copula C whose additive generator $t\colon [0,1] \to [0, \infty]$ is given by

$$t(x) = \log^2 x + 2^{n-5} \sin \tfrac{\log^2 x}{2^n} \qquad \text{if } n \in \mathbb{Z} \text{ and } 2^{n+1}\pi \leq \log^2 x < 2^{n+2}\pi.$$

Then we have $C^{2^n}(x^{2^{-n}}, y^{2^{-n}}) = C(x, y)$ for all $n \in \mathbb{N}$ and for all $(x, y) \in [0,1]^2$, but $C^3(x_0^{1/3}, y_0^{1/3}) \neq C(x_0, y_0)$ for some $(x_0, y_0) \in [0,1]^2$. Consequently,

$$\lim_{n\to\infty} C^{2^n}(x_0^{2^{-n}}, y_0^{2^{-n}}) = C(x_0, y_0) \neq C^3(x_0^{1/3}, y_0^{1/3}) = \lim_{n\to\infty} C^{3\cdot 2^n}(x_0^{2^{-n}/3}, y_0^{2^{-n}/3}),$$

showing that $\lim\limits_{n\to\infty} C^n(x^{1/n}, y^{1/n})$ does not exist for all $(x, y) \in [0,1]^2$.

(ii) Consider the following copula (which is an ordinal sum of infinitely many copies of W)

$$C = (\langle 2^{-2^{2k}}, 2^{-2^{2k-2}}, W\rangle)_{k\in\mathbb{Z}}.$$

Although W belongs to the maximum domain of attraction of Π, we have

$$\lim_{n\to\infty} C^{2^{2n}}(0.5^{2^{-2n}}, 0.5^{2^{-2n}}) = 0.5 \neq \tfrac{\sqrt{2}}{4} = \lim_{n\to\infty} C^{2^{2n+1}}(0.5^{2^{-2n-1}}, 0.5^{2^{-2n-1}}),$$

showing that the sequence $(C^n(0.5^{1/n}, 0.5^{1/n})))_{n\in\mathbb{N}}$ does not converge.

Now we clarify the relationship between φ_p-invariant copulas and the class $\mathcal{M}$ of maximum attractors:

Proposition 4.2. For a copula C, the following are equivalent:

(i) $C \in \mathcal{M}$.

(ii) $C_{\varphi_p} = C$ for all $p \in]0, \infty[$.

(iii) $C_{\varphi_p} = C_{\varphi_q} = C$ for some $p, q \in]0, \infty[$ such that $\frac{\log p}{\log q}$ is irrational.

Proof. The equivalence of (ii) and (iii) follows from the fact that, for $p, q \in]0, \infty[$, the set $\{mp + nq \mid m, n \in \mathbb{Z}\}$ is a dense subset of $\mathbb{R}$ if and only if $\frac{\log p}{\log q}$ is irrational (compare [1]).

As already mentioned, the class of φ_p-invariant copulas is a subclass of $\mathcal{M}$. For the converse, let $C^* \in \mathcal{M}$ be given as in (4). Then for all $p \in]0, \infty[$ and for all $(x, y) \in [0, 1]^2$

$$(C_{\varphi_p})^*(x, y) = \left(e^{\log(x^p y^p) \cdot A(\frac{\log x^p}{\log(x^p y^p)})}\right)^{1/p} = C^*(x, y),$$

showing that also (i) and (ii) are equivalent. □

Observe that $C_{\varphi_p} = C$ for some single $p \in]0, \infty[$ is not sufficient to guarantee $C \in \mathcal{M}$ (see Example 4.1 (i)).

If $t\colon [0, 1] \to [0, \infty]$ is a convex, decreasing bijection (and, therefore, an additive generator of some strict copula $C_{(t)}$) and if $A \in \mathcal{A}$ then the copula $C_{t,A}$ defined by

$$C_{t,A}(x, y) = t^{-1}\Big((t(x) + t(y)) \cdot A(\tfrac{t(x)}{t(x)+t(y)})\Big). \tag{5}$$

was called an *Archimax copula* in [5] (compare also [10]). It is obvious that the class $\mathcal{A}_t = \{C_{t,A} \mid A \in \mathcal{A}\}$ contains both M and the strict copula $C_{(t)}$, and we always have $C_{(t)} \le C_{t,A} \le M$. Moreover, $\mathcal{M} = \mathcal{A}_{-\log}$ (note that $C_{(-\log)} = \Pi$).

For a fixed convex, decreasing bijection $t\colon [0, 1] \to [0, \infty]$ and for $p \in]0, \infty[$ define $\tau_p\colon [0, 1] \to [0, 1]$ by $\tau_p(x) = t^{-1}(pt(x))$. Evidently, $\tau_p \in \Phi$ for each $p \in]0, \infty[$.

Note that a strict copula $C_{(t)}$ is φ-invariant with respect to some $\varphi \in \Phi$ if and only if $t \circ \varphi = p \cdot t$, i. e., if $\varphi = \tau_p$ for some $p \in]0, \infty[$.

In complete analogy to Proposition 4.2 we have:

Corollary 4.3. Let $t\colon [0, 1] \to [0, \infty]$ be a convex, decreasing bijection such that t^p is not convex whenever $p \in]0, 1[$. Then for each copula C the following are equivalent:

(i) $C_{\tau_p} = C$ for each $p \in]0, \infty[$.

(ii) $C_{\tau_p} = C_{\tau_q} = C$ for some $p, q \in]0, \infty[$ such that $\frac{\log p}{\log q}$ is irrational.

The proof of the following result is a matter of simple computation:

Proposition 4.4. Each Archimax copula $C_{t,A}$ is τ_p-invariant for each $p \in]0, \infty[$.

However, not each copula which is τ_p-invariant for each $p \in]0, \infty[$ is an element of $\mathcal{A}_t$: take the function $t\colon [0, 1] \to [0, \infty]$ given by $t(x) = \log^2 x$ (which generates some *Gumbel copula*, see [13, Table 4.1, (4.2.4)] and [8]); then $\tau_p(x) = x^{\sqrt{p}}$ and Π is τ_p-invariant for each $p \in]0, \infty[$, but $\Pi \notin \mathcal{A}_t$.

Putting $\mathcal{B} = \{\mathcal{A}_t \mid t\colon [0, 1] \to [0, \infty]$ is a convex, decreasing bijection$\}$, we can determine maximal elements of $\mathcal{B}$:

Theorem 4.5. Let $t\colon [0,1] \to [0,\infty]$ be a convex, decreasing bijection and define $\lambda^* = \inf\{\lambda \in]0,1[\mid t^\lambda \text{ is convex}\}$. Then $\mathcal{A}_{t^{\lambda^*}}$ is a maximal element of $\mathcal{B}$ with the property that all of its elements are τ_p-invariant for each $p \in]0,\infty[$.

Proof. Observe first that for each $\lambda \in]0,\infty[$ and for each $p \in]0,\infty[$ we have

$$(t^\lambda)^{-1}(pt^\lambda(x)) = t^{-1}\big(p^{1/\lambda}t(x)\big) = \tau_{p^{1/\lambda}}(x).$$

Based on [1] it can be shown that the only strict t-norms [9] which are τ_p-invariant for each $p \in]0,\infty[$ are generated by t^λ with $\lambda \in]0,\infty[$. Among them, only the functions t^λ with $\lambda \in [\lambda^*,\infty[$ are convex and, therefore, generate a copula. As a consequence of Proposition 4.4, only the classes $\mathcal{A}_{t^\lambda}$ with $\lambda \in [\lambda^*,\infty[$ consist of Archimax copulas which are τ_p-invariant for each $p \in]0,\infty[$.

Moreover, for each $\lambda = k \cdot \lambda^*$ with $k \in [1,\infty[$ and for each $B \in \mathcal{A}$ we obtain $C_{t^\lambda,B} = C_{t^{\lambda^*},A}$, where $A \in \mathcal{A}$ is given by

$$A(x) = (x^k + (1-x)^k)^{1/k} \cdot B^{1/k}\big(\tfrac{x^k}{x^k+(1-x)^k}\big),$$

showing that $\mathcal{A}_{t^{\lambda^*}} \supseteq \mathcal{A}_{t^\lambda}$. Moreover, no strict copula $C_{(f)}$ with additive generator $f \notin \{t^\lambda \mid \lambda \in [\lambda^*,\infty[\}$ is τ_p-invariant for all $p \in]0,\infty[$, implying $C_{(f)} \notin \mathcal{A}_{t^{\lambda^*}}$. □

Example 4.6. Consider the convex, decreasing bijection $t\colon [0,1] \to [0,\infty]$ defined by $t(x) = \frac{1}{x} - 1$ (observe that it satisfies $t'(1^-) = -1$) which generates the copula $C_{(t)}$ given by $C_{(t)} = \frac{xv}{x+y-xy}$. The corresponding family $(\tau_p)_{p\in]0,\infty[}$ is then determined by $\tau_p(x) = \frac{x}{p+(1-p)x}$. Therefore, $C_{(t)}$ is the weakest copula which is τ_p-invariant for all $p \in]0,\infty[$, and each $C_{t,A} \in \mathcal{A}_t$ is given by

$$C_{t,A}(x,y) = \frac{xy}{xy + A\big(\frac{(1-x)y}{x+y-2xy}\big) \cdot (x+y-2xy)}.$$

Note also that the functions τ_p are just multiplicative generators of the family of *Ali-Mikhail-Haq* copulas [2, 13].

Evidently, $C_{(t^{\lambda^*})}$ is the weakest associative copula which is τ_p-invariant for all $p \in]0,\infty[$. Whether it is also the weakest copula which is τ_p-invariant for all $p \in]0,\infty[$ is still an open problem. As a partial answer to this we have the following result:

Theorem 4.7. Let $t\colon [0,1] \to [0,\infty]$ be a convex, decreasing bijection such that $t'(1^-) \neq 0$. Then $C_{(t)}$ is the weakest copula which is τ_p-invariant for all $p \in]0,\infty[$.

Proof. Recall that the function $t_W\colon [0,1] \to [0,\infty]$ given by $t_W(x) = 1-x$ is an additive generator of W. Fixing $p \in \{\frac{1}{3},\frac{1}{2}\}$, it is obvious that W_{τ_p} is a nilpotent t-norm whose additive generator $t_{W,p}\colon [0,1] \to [0,\infty]$ is given by $t_{W,p}(x) = \frac{1-\tau_p(x)}{p}$.

Moreover, we have $(\tau_p)_{(n)}(x) = \tau_{p^n}$ for each $n \in \mathbb{N}$, and thus $W^*_{\tau_p}$ (if it exists) coincides with $\lim_{n\to\infty} W_{\tau_{p^n}}$. However, for all $x \in]0,1[$ we have

$$\lim_{\lambda\to 0^+} \frac{1-\tau_\lambda(x)}{\lambda} = \lim_{\lambda\to 0^+} \frac{1-t^{-1}(\lambda t(x))}{\lambda} = -\frac{t(x)}{t'(1^-)} = c\cdot t(x)$$

for some $c \in]0,\infty[$. Therefore, as a consequence of [9, Theorem 8.14] we obtain $W^*_{\tau_{1/3}} = W^*_{\tau_{1/2}} = C_{(t)}$. Then, because of Corollary 4.3, we have $W^*_{\tau_p} = C_{(t)}$ for all $p \in]0,\infty[$, and $C_{(t)}$ is the weakest copula which is τ_p-invariant for all $p \in]0,\infty[$. □

5. OPEN PROBLEMS

Problem 1. For $\psi \in \Phi_c$, do we have $C^*_\psi = M$ for each copula C if and only if $\psi'(1^-) = 0$?

Problem 2. For a convex, decreasing bijection $t\colon [0,1] \to [0,\infty]$, do we have

$$\mathcal{A}_{t^{\lambda *}} = \{C \mid C \text{ is a copula which is } \tau_p\text{-invariant for all } p \in]0,\infty[\}?$$

A potentially helpful result related to Problem 2 is the following:

Proposition 5.1. Let $C \in \mathcal{M}$ be a maximum attractor copula, let $t\colon [0,1] \to [0,\infty]$ be a convex, decreasing bijection, and put $\xi(t) = e^{-t}$. Then the aggregation operator C_ξ is a copula which belongs to $\mathcal{A}_t$. Moreover, if C is described by the dependence function $A \in \mathcal{A}$ then we have $C_\xi = C_{t,A}$.

Proof. If C is characterized by $A \in \mathcal{A}$, i.e., $C(x,y) = e^{\log(xy)\cdot A\left(\frac{\log x}{\log(xy)}\right)}$, then from

$$\begin{aligned} C_\xi(x,y) &= t^{-1}\Big(-\log e^{\log(e^{-t(x)}e^{-t(y)})\cdot A\left(\frac{\log e^{-t(x)}}{\log(e^{-t(x)}e^{-t(y)})}\right)}\Big) \\ &= t^{-1}\big((t(x)+t(y))\cdot A\big(\tfrac{t(x)}{t(x)+t(y)}\big)\big) \\ &= C_{t,A}(x,y) \end{aligned}$$

the assertion follows. □

ACKNOWLEDGEMENT

This research was supported by the COST action 274 and by grants APVT-20-003204, VEGA 1/1145/04, GA ČR 402/04/1026, MNTRS (Project "Mathematical Models of Nonlinearity, Uncertainty and Decision"), the Academy of Sciences and Arts of Vojvodina (Vojvodina Provincial Secretariat for Science and Technology Development, Project "Mathematical Models for Decision Making under Uncertain Conditions and Their Applications").

(Received February 21, 2005.)

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