DOMINATION IN THE FAMILIES
OF FRANK AND HAMACHER t-NORMS

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Domination is a relation between general operations defined on a poset. The old open problem is whether domination is transitive on the set of all t-norms. In this paper we contribute partially by inspection of domination in the family of Frank and Hamacher t-norms. We show that between two different t-norms from the same family, the domination occurs iff at least one of the t-norms involved is a maximal or minimal member of the family. The immediate consequence of this observation is the transitivity of domination on both inspected families of t-norms.

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1. INTRODUCTION

The concept of domination has been introduced within the framework of probabilistic metric spaces for triangle functions and for building cartesian products of probabilistic metric spaces [12]. Afterwards the domination of t-norms was studied in connection with construction of fuzzy equivalence relations [2, 3, 13] and construction of fuzzy orderings [1]. Recently, the concept of domination was extended to the much general class of aggregation operators [9]. The domination of aggregation operators emerges when investigating which aggregation procedures applied to the system of T-transitive fuzzy relations yield a T-transitive fuzzy relation again [9] or when seeking aggregation operators which preserves the extensionality of fuzzy sets with respect to given T-equivalence relations [10]. The most general definition of domination considered so far demands the operations to be defined on arbitrary poset [4].

Definition 1. Let \((P, \geq)\) be a poset and let \(A: P^m \rightarrow P\), \(B: P^n \rightarrow P\) be two operations defined on \(P\) with arity \(m\) and \(n\), respectively. Then we say that \(A\) dominates \(B\) (\(A \gg B\) in symbols) if each matrix \((x_{ij})\) of type \(m \times n\) over \(P\) satisfies

\[
A(B(x_{1,1}, x_{1,2}, \ldots, x_{1,n}), \ldots, B(x_{m,1}, x_{m,2}, \ldots, x_{m,n})) \geq B(A(x_{1,1}, x_{2,1}, \ldots, x_{m,1}), \ldots, A(x_{1,n}, x_{2,n}, \ldots, x_{m,n})).
\]
Let us recall that a t-norm [12, 8] is a monotone, associative and commutative binary operation $T: [0, 1]^2 \to [0, 1]$ with neutral element 1. Important examples of t-norms are: the minimum $T_M$, the product $T_P$, the Łukasiewicz t-norm $T_L$ and the drastic t-norm $T_D$ given by

$$T_M(x, y) = \min(x, y),$$
$$T_P(x, y) = xy,$$
$$T_L(x, y) = \max(0, x + y - 1),$$
$$T_D(x, y) = \begin{cases} xy & \text{max}(x, y) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We say that a t-norm $T_1$ is stronger than a t-norm $T_2$ ($T_1 \geq T_2$ in symbols) if any $x, y \in [0, 1]$ satisfy $T_1(x, y) \geq T_2(x, y)$. We use the notation $T_1 > T_2$ whenever simultaneously $T_1 \geq T_2$ and $T_1 \neq T_2$ hold. One can easily show that each t-norm is weaker than $T_M$ and stronger than $T_D$. Particularly, $T_P$ and $T_L$ satisfy $T_M > T_P > T_L > T_D$. It is obvious that $\geq$ is a partial order on the set of all t-norms, i.e., the reflexive, antisymmetric and transitive relation.

By Definition 1 we have that two t-norms $T_1$ and $T_2$ satisfy $T_1 \gg T_2$ iff for each $x, y, u, v \in [0, 1]$,

$$T_1(T_2(x, y), T_2(u, v)) \geq T_2(T_1(x, u), T_1(y, v)). \quad (1)$$

It is easy to show that each t-norm $T$ satisfies $T_M \gg T$, $T \gg T_D$ and $T \gg T$. Moreover, by [8, 11], the representative t-norms $T_P$ and $T_L$ satisfy $T_P \gg T_L$. If $T_1 \gg T_2$ then by inequality (1), the neutrality of 1 and the commutativity of t-norms we have that any $y, u \in [0, 1]$ satisfy

$$T_1(y, u) = T_1(T_2(1, y), T_2(u, 1))$$
$$\geq T_2(T_1(1, u), T_1(y, 1)) = T_2(u, y) = T_2(y, u)$$

so that $T_1 \geq T_2$, see [8]. This means that satisfaction of $T_1 \geq T_2$ is a necessary condition for $T_1 \gg T_2$ or, in other words, that domination is a subrelation of $\geq$. The converse implication does not hold as it is demonstrated by results of this paper. Domination of t-norms is moreover an antisymmetric relation which is a consequence of the fact that it is a subrelation of the antisymmetric relation $\geq$. The old open problem [12, Problem 12.11.3] is whether domination is transitive on the set of all t-norms. If it were true domination would be a partial order.

When inspecting domination, the tool of $\varphi$-transform can be helpful. Let $\varphi$ be an order isomorphism of the interval $[0, 1]$ and let $T$ be an arbitrary t-norm. Define $T_\varphi: [0, 1]^2 \to [0, 1]$ by

$$T_\varphi(x, y) = \varphi^{-1}(T(\varphi(x), \varphi(y)))$$

to be the $\varphi$-transform of $T$. It is easy to show that $T_\varphi$ is again a t-norm [8]. Moreover, for arbitrary t-norms $T_1$ and $T_2$ and for arbitrary order isomorphism $\varphi$ the satisfaction of $T_1 \gg T_2$ is equivalent to $(T_1)_\varphi \gg (T_2)_\varphi$ so that $\varphi$-transforms preserve domination [9]. Let us recall that a t-norm is strict (nilpotent) iff there
exists \( \varphi \) such that \( T = (T_\varphi)_\varphi \) (\( T = (T_L)_\varphi \)) [8]. Moreover, it is clear that each \( \varphi \)-transform of a strict (nilpotent) t-norm is again strict (nilpotent). Thus in order to characterize pairs of dominating strict (nilpotent) t-norms it suffices to characterize strict (nilpotent) t-norms dominating \( T_\varphi \) (\( T_L \)).

The following result relates domination and powers of additive generators [8]. Let \( T \) be a continuous Archimedean t-norm with additive generator \( f \) and let \( \lambda \in [0, \infty[ \) be a positive number. Define \( T^{(\lambda)} \) to be a t-norm with additive generator \( f^\lambda(x) \), i.e., the \( \lambda \)-power of \( f \). It is known that for each \( \lambda > \mu \) is \( T^{(\lambda)} \gg T^{(\mu)} \). This construction of dominating t-norms gives rise to many parametrical families of t-norms such as the Aczél–Alsina or the Dombi family.

Although the structure of domination on the set of all t-norms is still unknown, it is possible to inspect it on particular families of t-norms. One of the oldest results of this type is due to Sherwood [11] who solved the structure of domination on the family of Schweizer–Sklar t-norms. Another result of this type is the above mentioned solution of domination in the Aczél–Alsina or the Dombi family. In the next two sections we inspect another two important families - the Frank and Hamacher t-norms.

**2. FRANK t-NORMS**

Frank t-norms \( T^F_\lambda \) are given as

\[
T^F_\lambda(x,y) = \begin{cases} 
T_M(x,y) & \lambda = 0 \\
T_P(x,y) & \lambda = 1 \\
T_L(x,y) & \lambda = \infty \\
\log_\lambda \left( \frac{(\lambda^x-1)(\lambda^y-1)}{\lambda-1} + 1 \right) & \text{otherwise}
\end{cases}
\]

(2)

where \( \lambda \in [0, \infty[ \) is the characterizing parameter of the Frank t-norm. Note that the family of Frank t-norms is strictly decreasing in \( \lambda \) which means that \( T^F_{\lambda_1} > T^F_{\lambda_2} \) iff \( \lambda_1 < \lambda_2 \). In [5] M. J. Frank solved the problem of characterization of all continuous t-norms \( T \) such that the function \( F : [0,1]^2 \to [0,1] \) given by

\[
F(x,y) = x + y - T(x,y)
\]

is associative. Each \( T^F_\lambda \) solves this problem.

In what follows we find out which \( \lambda_1, \lambda_2 \in [0, \infty[ \) satisfy \( T^F_{\lambda_1} \gg T^F_{\lambda_2} \). Recall that for \( \lambda_1 = 0 \) the question is trivial as \( T^F_0 = T_M \) dominates any t-norm. Particular, for \( \lambda_1 = 1 \) and \( \lambda_2 = \infty \) the question is solved as well since \( T^F_1 = T_P \gg T_L = T^F_\infty \), see, for example, the already mentioned work of Sherwood [11]. Finally \( T^F_{\lambda_1} \gg T^F_{\lambda_2} \) cannot be satisfied for \( \lambda_1 > \lambda_2 \) due to the decreasingness of the Frank family. That's why we consider \( \lambda_1 < \lambda_2 \) in the following.

**Lemma 2.** Let \( A_n = [a_1^l, a_1^r] \times [a_2^l, a_2^r] \times \cdots \times [a_n^l, a_n^r] \), \( a_i^l < a_i^r \), \( i = 1, 2, \ldots, n \), be an \( n \)-dimensional interval. Let \( f : A_n \to \mathbb{R} \) be a real function, linear in each argument.
Moreover, let the value of $f$ be nonnegative in each vertex of $A_n$, i.e., at each point with coordinates $(b_1, b_2, \ldots, b_n)$, $b_i \in \{a_i^l, a_i^r\}$. Then $f$ is nonnegative on whole $A_n$.

**Proof.** By induction with respect to the dimension $n$. The statement is obvious for $n = 1$.

Let us assume that the claim of the lemma is true for all intervals of dimension $n - 1$ and that $A_n$ and $f$ fulfill all assumptions of the lemma. Consider arbitrary $x = (x_1, x_2, \ldots, x_n) \in A_n$. Define points

$$x_* = (x_1, x_2, \ldots, x_{n-1}, a_n^l),$$
$$x^* = (x_1, x_2, \ldots, x_{n-1}, a_n^r)$$

to be the left and right projections of the point $x$ along the last coordinate. Further define functions $f_*$ and $f^*$ by expressions

$$f_*(x_1, x_2, \ldots, x_{n-1}) = f(x_1, x_2, \ldots, x_{n-1}, a_n^l),$$
$$f^*(x_1, x_2, \ldots, x_{n-1}) = f(x_1, x_2, \ldots, x_{n-1}, a_n^r).$$

Both functions $f_*$ and $f^*$ are defined on $(n - 1)$-dimensional interval

$$A_{n-1} = [a_1^l, a_1^r] \times [a_2^l, a_2^r] \times \cdots \times [a_{n-1}^l, a_{n-1}^r]$$

and both functions are linear in each argument. On vertices of $A_{n-1}$ both functions attain nonnegative values. Indeed, let $v = (v_1, v_2, \ldots, v_{n-1})$ be any vertex of $A_{n-1}$. Then $f_*(v) = f(v_1, v_2, \ldots, v_{n-1}, a_n^l)$ is a value of $f$ at one vertex of $A_n$ which is by assumption nonnegative. Analogically for $f^*$.

Thus $f_*$ and $f^*$ are nonnegative on $A_{n-1}$ by assumption. Particularly,

$$f_*(x_1, x_2, \ldots, x_{n-1}) = f(x_*) \geq 0,$$
$$f^*(x_1, x_2, \ldots, x_{n-1}) = f(x^*) \geq 0.$$

By assumptions, the function $g(y) = f(x_1, \ldots, x_{n-1}, y)$ is linear on $[a_n^l, a_n^r]$ and

$$g(a_n^l) = f(x_*) \geq 0,$$
$$g(x_n) = f(x),$$
$$g(a_n^r) = f(x^*) \geq 0.$$

Thus $f(x) = g(x_n) \geq 0$. \hfill \Box

**Proposition 3.** $T^F_T \gg T_L$ for each $\lambda \in ]0,1[ \cup ]1,\infty[.$

**Proof.** We have to show that any $x, y, u, v \in [0,1]$ satisfy the inequality

$$T^F_T(T_L(x,y), T_L(u,v)) \geq T_L(T^F_T(x,u), T^F_T(y,v)). \quad (3)$$

Let us consider two mutually exclusive cases. First that the left-hand side of (3) equals zero and the second that it is positive:
(i) Since for \( \lambda \in [0,1] \cup [1,\infty[ \) \( T^F_\lambda \) is strict, the left-hand side of (3) can be zero iff at least one of the Łukasiewicz t-norms involved attains the value 0. Without loss of generality assume \( T^L(x,y) = 0 \) which is equivalent to \( x + y - 1 \leq 0 \). It suffices to show that

\[
T^L(T^F_\lambda (x,u), T^F_\lambda (y,v)) = \max(0, T^F_\lambda (x,u) + T^F_\lambda (y,v) - 1) = 0
\]

or simply \( T^F_\lambda (x,u) + T^F_\lambda (y,v) - 1 \leq 0 \). But from the nondecreasingness of \( T^F_\lambda \) and from the neutrality of 1 it follows

\[
T^F_\lambda (x,u) + T^F_\lambda (y,v) - 1 \leq T^F_\lambda (x,1) + T^F_\lambda (y,1) - 1 = x + y - 1 \leq 0.
\]

(ii) Assume that the left-hand side of (3) is positive, so that \( x + y - 1 > 0 \) as well as \( u + v - 1 > 0 \) holds. Inequality (3) can be rewritten in the form

\[
T^F_\lambda (x + y - 1, u + v - 1) \geq \max(0, T^F_\lambda (x,u) + T^F_\lambda (y,v) - 1)
\]

which is further equivalent to

\[
T^F_\lambda (x + y - 1, u + v - 1) \geq T^F_\lambda (x,u) + T^F_\lambda (y,v) - 1
\]

since the left-hand side is positive. After expansion of the definitions of \( T^F_\lambda \) the inequality can be rewritten as

\[
\log_\lambda \left[ \frac{(\lambda^x\lambda^y - 1)(\lambda^u\lambda^v - 1)}{\lambda - 1} + 1 \right] \geq \log_\lambda \left[ \frac{(\lambda^x - 1)(\lambda^u - 1)}{\lambda - 1} + 1 \right] \frac{(\lambda^y - 1)(\lambda^v - 1)}{\lambda - 1} + 1
\]

and by further de-logarithmation we end up with

\[
\text{sgn}(\lambda - 1) \left[ \frac{(\lambda^x\lambda^y - 1)(\lambda^u\lambda^v - 1)}{\lambda - 1} + 1 - \frac{(\lambda^x - 1)(\lambda^u - 1)}{\lambda - 1} \frac{(\lambda^y - 1)(\lambda^v - 1)}{\lambda - 1} + 1 \right] \geq 0.
\]

Note that the multiplicative constant \( \text{sgn}(\lambda - 1) \) prevents the reversion of the order after de-logarithmation whenever \( \lambda \in [0,1] \).

The expression on the left-hand side is nonnegative for any \( x, y, u, v \in [0,1] \). Indeed, by substitution \( \lambda^x = X, \lambda^y = Y, \lambda^u = U \) and \( \lambda^v = V \) where \( X, Y, U, V \in [\min(1, \lambda), \max(1, \lambda)] \) we obtain

\[
\text{sgn}(\lambda - 1) \left[ \frac{(X^Y - 1)(U^V - 1)}{\lambda - 1} + 1 - \frac{(X - 1)(U - 1)}{\lambda - 1} \frac{(Y - 1)(V - 1)}{\lambda - 1} + 1 \right] \geq 0. \tag{4}
\]

Let us define the function \( G: [\min(1, \lambda), \max(1, \lambda)]^4 \rightarrow \mathbb{R} \) in variables \( X, Y, U, V \) to be the value of the expression on the left-hand side of (4). One can easily see that \( G \)}
is linear in each argument. A very simple computation reveals that \( G \) attains zero value at all vertices of \([\min(1, \lambda), \max(1, \lambda)]^4\) up to the following seven exceptions

\[
G(1, 1, 1, 1) = \frac{\text{sgn}(\lambda - 1)(\lambda^2 - 1)}{\lambda^2} \geq 0,
\]

\[
G(\lambda, 1, 1, 1) = G(1, \lambda, 1, 1) = \frac{\text{sgn}(\lambda - 1)(\lambda - 1)}{\lambda} \geq 0,
\]

\[
G(1, 1, \lambda, 1) = G(1, 1, 1, \lambda) = \frac{\text{sgn}(\lambda - 1)(\lambda - 1)}{\lambda} \geq 0,
\]

\[
G(1, \lambda, \lambda, 1) = G(\lambda, 1, \lambda, 1) = \frac{\text{sgn}(\lambda - 1)(\lambda - 1)}{\lambda} \geq 0.
\]

which all are nonnegative values. Thus the function \( G \) satisfies all assumptions of Lemma 2 by which \( G \) is nonnegative which proves inequality (4).

Proposition 3 together with \( T_M \gg T_L \) and \( T_P \gg T_L \) show that any Frank t-norm dominates \( T_L \). Further we discuss the mutual domination of nonextremal Frank t-norms.

**Lemma 4.** Let \( f: \mathbb{R} \to \mathbb{R} \) be \( n \)-times differentiable in \( 0 \), \( f^{(i)}(0) = 0 \) for all \( i = 0, 1, \ldots, n - 1 \) and \( f^{(n)}(0) < 0 \). There exists \( \delta > 0 \) such that \( f(x) < 0 \) for each \( x \in ]0, \delta[ \).

**Proof.** The claim of the lemma is a well-known result of real analysis.

**Proposition 5.** There does not exist \( \lambda_1, \lambda_2 \in ]0, \infty[ \) such that \( \lambda_1 < \lambda_2 \) and \( T_{\lambda_1}^F \gg T_{\lambda_2}^F \).

**Proof.** Suppose arbitrary \( \lambda_1, \lambda_2 \in ]0, \infty[ \) with \( \lambda_1 < \lambda_2 \). We shall show that there exists some \( x \in ]0, 1[ \) such that

\[
T_{\lambda_1}(T_{\lambda_2}(x, x), T_{\lambda_2}(x, x)) < T_{\lambda_2}(T_{\lambda_1}(x, x), T_{\lambda_1}(x, x))
\]

so that the defining inequality for domination (1) is violated. Let us define the function \( \delta_{\lambda_2}^F: [0, 1] \to [0, 1] \) to be the diagonal of a Frank t-norm so that \( \delta_{\lambda_2}^F(x) = T_{\lambda_2}^F(x, x) \) for any \( x \in [0, 1] \). Due to the strictness of \( T_{\lambda_2}^F \) we know that \( \delta_{\lambda_2}^F \) is an order isomorphism of the interval \([0, 1] \). Inequality (5) can be rewritten into the form

\[
\delta_{\lambda_2}^F(\delta_{\lambda_2}^F(x)) < \delta_{\lambda_1}^F(\delta_{\lambda_1}^F(x)).
\]

Further define the function \( f(\lambda_1, \lambda_2): [0, 1] \to \mathbb{R} \) by expression

\[
f(\lambda_1, \lambda_2)(x) = \delta_{\lambda_1}^F(\delta_{\lambda_2}^F(x)) - \delta_{\lambda_2}^F(\delta_{\lambda_1}^F(x)),
\]

Now another alternative reformulation of (5) is that there exists some \( x > 0 \) such that \( f_{\lambda_1, \lambda_2}(x) < 0 \). We prove this claim by means of Lemma 4.
Let us compute $\delta^F_\lambda$ as well as its first and second derivatives which we will use later:

$$
\delta^F_\lambda(x) = \begin{cases} 
\log_\lambda \left( \frac{(\lambda^x-1)^2}{\lambda-1} + 1 \right) & \lambda \neq 1 \\
x^2 & \lambda = 1,
\end{cases}
$$

$$
\delta^{(1)}_\lambda(x) = \begin{cases} 
\frac{2(\lambda^x-1)x^x}{(\lambda^x-1)^2+\lambda-1} & \lambda \neq 1 \\
2x & \lambda = 1,
\end{cases}
$$

$$
\delta^{(2)}_\lambda(x) = \begin{cases} 
\frac{2\lambda^x \ln(\lambda)((2\lambda^x-1)(\lambda-1)-(\lambda^x-1)^2)}{((\lambda^x-1)^2+\lambda-1)^2} & \lambda \neq 1 \\
\frac{2\ln(\lambda)}{\lambda-1} & \lambda = 1.
\end{cases}
$$

Their values at point 0 are

$$
\delta^F_\lambda(0) = 0 \quad \delta^{(1)}_\lambda(0) = 0 \quad \delta^{(2)}_\lambda(0) = \begin{cases} 
\frac{2\ln(\lambda)}{\lambda-1} & \lambda \neq 1 \\
\frac{2\ln(\lambda)}{2} & \lambda = 1.
\end{cases}
$$

(7)

so that the first nonzero derivative of $\delta^{(2)}_\lambda$ at point 0 is the second derivative. Thereout the first nonzero derivative of $f_{(\lambda_1,\lambda_2)}$, according to its definition, is the fourth derivative for which we have

$$
f^{(4)}_{(\lambda_1,\lambda_2)}(0) = 3\delta^{(2)}_{\lambda_1}(0) \left( \delta^{(2)}_{\lambda_2}(0) \right)^2 - 3\delta^{(2)}_{\lambda_2}(0) \left( \delta^{(2)}_{\lambda_1}(0) \right)^2.
$$

(8)

Now we can compute the value of this derivative for all feasible combinations of $\lambda_1$ and $\lambda_2$. Let us distinguish three mutually exclusive cases — the first that $\lambda_2 = 1$, then $\lambda_1 = 1$ and finally, $\lambda_1 \neq 1 \neq \lambda_2$.

(i) Let us consider $\lambda_1 < \lambda_2 = 1$. Combining (7) and (8) we obtain the expression

$$
f^{(4)}_{(\lambda_1,1)}(0) = -24 \frac{\ln(\lambda_1)}{\lambda_1-1} \left( \frac{\ln(\lambda_1)}{\lambda_1-1} - 1 \right).
$$

The sign of this derivative is determined by the sign of the expression in parenthesis. Under the assumption $\lambda_1 < 1$, the expression in parenthesis is positive because the expression $\ln(\lambda)/(\lambda - 1)$ is decreasing, continuous on $]0,1[\cup]1,\infty[$ and

$$
\lim_{\lambda \to 1} \frac{\ln(\lambda)}{\lambda - 1} = 1.
$$

Thus the first nonzero derivative of $f_{(\lambda_1,1)}$ is negative at point 0.

(ii) Let us consider $1 = \lambda_1 < \lambda_2$. Combining (7) and (8) we obtain the expression

$$
f^{(4)}_{(1,\lambda_2)}(0) = 24 \frac{\ln(\lambda_2)}{\lambda_2-1} \left( \frac{\ln(\lambda_2)}{\lambda_2-1} - 1 \right).
$$

Following the considerations from (i) we find out that $f^{(4)}_{(1,\lambda_2)}(0)$ is negative.
(iii) Let us consider $\lambda_1 \neq 1 \neq \lambda_2$. Combining (7) and (8) gives us the expression

$$f^{(4)}_{(\lambda_1, \lambda_2)}(0) = -24 \frac{\ln(\lambda_1) \ln(\lambda_2)}{(\lambda_1 - 1)(\lambda_2 - 1)} \left( \frac{\ln(\lambda_1)}{\lambda_1 - 1} - \frac{\ln(\lambda_2)}{\lambda_2 - 1} \right).$$

The sign of the derivative is determined by the sign of expression in ellipses. From the decreasingness of this expression and from $\lambda_1 < \lambda_2$ it follows that $f^{(4)}_{(\lambda_1, \lambda_2)}(0) < 0$.

We distinguished all possible cases and regardless of the values of $\lambda_1$ and $\lambda_2$ the value of $f^{(4)}_{(\lambda_1, \lambda_2)}(0)$ is negative. In addition, all lower-order derivatives of $f^{(4)}_{(\lambda_1, \lambda_2)}$ vanish at point 0. By Lemma 4 there exists some $x \in ]0, 1[$ such that $f(x) < 0$. 

**Corollary 6.** Any case of domination within the family of Frank t-norms is one of these

$$T^F_\lambda \gg T^F_\alpha,$$ $T_M \gg T^F_\alpha,$ $T^F_\alpha \gg T_L$$

for arbitrary $\lambda \in [0, \infty]$. Moreover, domination is transitive within this family so that it is partially ordered by $\gg$.

### 3. HAMACHER t-NORMS

Hamacher t-norms form another one-parametric family of t-norms. It has been proved in [6, 7] that members of this family are the only ones to be expressed as quotient of two polynomials in two variables. The family of Hamacher t-norms is parameterized by $\lambda \in [0, \infty]$

$$T^H_\lambda(x, y) = \begin{cases} T_D(x, y) & \lambda = \infty \\ 0 & \lambda = x = y = 0 \\ \frac{x+y}{\lambda+(1-\lambda)(x+y-x-y)} & \text{otherwise} \end{cases} \quad (9)$$

The Hamacher family is strictly decreasing in $\lambda$ which means that $T^H_{\lambda_1} > T^H_{\lambda_2}$ iff $\lambda_1 < \lambda_2$. The drastic t-norm $T_D = T^H_\infty$ is the minimal element and the t-norm $T^H_0$ is the maximal element of the family.

In this section we answer the question for which $\lambda_1, \lambda_2 \in [0, \infty]$ the relation $T^H_{\lambda_1} \gg T^H_{\lambda_2}$ is satisfied. Recall that for $\lambda_2 = \infty$ the question is trivial as $T^H_\infty = T_D$ is dominated by any t-norm. Moreover, $T^H_{\lambda_1} \gg T^H_{\lambda_2}$ cannot be satisfied for $\lambda_1 > \lambda_2$ due to decreasingness within the family of Hamacher t-norms. That is why we will only deal with $\lambda_1 < \lambda_2$ in the sequel.

**Proposition 7.** For each $\lambda \in [0, \infty]$ it holds that $T^H_0 \gg T^H_\lambda$.

**Proof.** We divide the proof into two parts. We first show that $T^H_0 \gg T_P$ and then we prove the claim of proposition by virtue of $\varphi$-transform.
(i) We show that $T^H_0(xy, uv) \geq T^H_0(x, u)T^H_0(y, v)$ holds for any $x, y, u, v \in [0, 1]$. This inequality is trivially fulfilled whenever at least one variable equals 0. Therefore assume $xyuv > 0$. After expansion of the definitions we have

$$\frac{xyuv}{xy + uv - xyuv} \geq \frac{xu}{x + u - xu} \frac{yv}{y + v - yv}$$

or equivalently, by inversion

$$\frac{xy + uv - xyuv}{xyuv} \leq \frac{(x + u - xu)(y + v - yv)}{xyuv}.$$

As the denominators of both fractions are equal and positive, we can drop them, and by further manipulation we obtain the third equivalent inequality

$$0 \leq (x + u - xu)(y + v - yv) - xy - uv + xyuv$$

or

$$0 \leq xv(1-u)(1-y) + uy(1-v)(1-x)$$

where the expression on the right-hand side is evidently nonnegative.

(ii) Now, let $\varphi_\lambda$ be the multiplicative generator of the nonextremal Hamacher t-norm $T^H_\lambda$. So that for $\lambda \in ]0, \infty[$, $\varphi_\lambda$ and its inverse are given by

$$\varphi_\lambda(x) = \frac{x}{\lambda + (1-\lambda)x}, \quad \varphi^{-1}_\lambda(x) = \frac{\lambda x}{1 + (1-\lambda)x}. $$

Let us apply the $\varphi$-transform to both $T^H_0$ and $T_P$. Since $T^H_0$ dominates $T_P$, the corresponding $\varphi$-transforms do as well.

The $\varphi_\lambda$-transform of $T_P$ is $T^H_\lambda$ by the definition of multiplicative generator. Now we shall show that $\varphi_\lambda$-transform of $T^H_0$ is again $T^H_0$, i.e., the strongest Hamacher t-norm is stable under the $\varphi_\lambda$-transform whenever $\varphi_\lambda$ is a multiplicative generator of a nonextremal Hamacher t-norm. The equality

$$\varphi^{-1}_\lambda(T^H_0(\varphi_\lambda(x), \varphi_\lambda(y))) = T^H_0(x, y)$$

is trivially fulfilled whenever $xy = 0$. Now assume $xy > 0$. Then we have

$$\varphi^{-1}_\lambda(T^H_0(\varphi_\lambda(x), \varphi_\lambda(y))) = \varphi^{-1}_\lambda \left( \frac{\varphi_\lambda(x)\varphi_\lambda(y)}{\varphi_\lambda(x) + \varphi_\lambda(y) - \varphi_\lambda(x)\varphi_\lambda(y)} \right)$$

$$= \varphi^{-1}_\lambda \left( \frac{xy}{\lambda(x + y) + (1 - 2\lambda)xy} \right)$$

$$= \frac{xy}{x + y - xy}$$

$$= T^H_0(x, y).$$

Since $T^H_0 \gg T_P$, by virtue of $\varphi_\lambda$-transform we have that $T^H_0 \gg T^H_\lambda$ which is our claim. $\square$
Proposition 8. There does not exist $\lambda_1, \lambda_2 \in [0, \infty]$ such that $\lambda_1 < \lambda_2$ and $T_{\lambda_1}^{\mathbf{H}} \succ T_{\lambda_2}^{\mathbf{H}}$.

Proof. Let $\lambda_1$ and $\lambda_2$ satisfy assumptions of the proposition. We shall show that there exists $x \in [0,1]$ such that

$$T_{\lambda_1}^{\mathbf{H}}(T_{\lambda_2}^{\mathbf{H}}(x,x), T_{\lambda_2}^{\mathbf{H}}(x,x)) < T_{\lambda_1}^{\mathbf{H}}(T_{\lambda_1}^{\mathbf{H}}(x,x), T_{\lambda_1}^{\mathbf{H}}(x,x))$$

(10)

so that the defining inequality for domination (1) is violated. Let us define the function $\delta^{\mathbf{H}}_\lambda : [0,1] \to [0,1]$ to be the diagonal of a Hamacher t-norm so that $\delta^{\mathbf{H}}_\lambda(x) = T^{\mathbf{H}}_{\lambda}(x,x)$ for any $x \in [0,1]$. The inequality (10) can be rewritten as

$$\delta^{\mathbf{H}}_{\lambda_1}(\delta^{\mathbf{H}}_{\lambda_2}(x)) < \delta^{\mathbf{H}}_{\lambda_2}(\delta^{\mathbf{H}}_{\lambda_1}(x)).$$

(11)

In order to show that (11) is satisfied for some $x \in [0,1]$ it suffices to show that this $x$ satisfies

$$\frac{x^4}{\delta^{\mathbf{H}}_{\lambda_1}(\delta^{\mathbf{H}}_{\lambda_2}(x))} > \frac{x^4}{\delta^{\mathbf{H}}_{\lambda_2}(\delta^{\mathbf{H}}_{\lambda_1}(x))}$$

(12)

since we consider $x \neq 0$ and both compositions of the diagonals are positive whenever $x \in [0,1]$. The diagonal of a Hamacher t-norm $T^{\mathbf{H}}_{\lambda}$ is given by the expression

$$T^{\mathbf{H}}_{\lambda}(x,x) = \frac{x^2}{\lambda + (1-\lambda)(2-x)x}$$

by which

$$\delta^{\mathbf{H}}_{\lambda_1}(\delta^{\mathbf{H}}_{\lambda_2}(x)) = \frac{x^4}{(\lambda_1 + (1-\lambda_1)(2-x)x)^2} \left[2 - \frac{x^2}{\lambda_1 + (1-\lambda_1)(2-x)x} \frac{x^2}{\lambda_2 + (1-\lambda_2)(2-x)x}\right]$$

$$= \frac{x^4}{\lambda_1(\lambda_2(x-1)-2x)^2(x-1)^2 + x^2(2\lambda_2(x-1)^2 + (4-3x)x)}$$

and

$$\delta^{\mathbf{H}}_{\lambda_2}(\delta^{\mathbf{H}}_{\lambda_1}(x)) = \frac{x^4}{(\lambda_2 + (1-\lambda_2)(2-x)x)^2} \left[2 - \frac{x^2}{\lambda_2 + (1-\lambda_2)(2-x)x} \frac{x^2}{\lambda_1 + (1-\lambda_1)(2-x)x}\right]$$

$$= \frac{x^4}{\lambda_2(\lambda_1(x-1)-2x)^2(x-1)^2 + x^2(2\lambda_1(x-1)^2 + (4-3x)x)}.$$}

According to these expressions, (12) can be rewritten in the form

$$\lambda_1(\lambda_2(x-1)-2x)^2(x-1)^2 + x^2(2\lambda_2(x-1)^2 + (4-3x)x) > \lambda_2(\lambda_1(x-1)-2x)^2(x-1)^2 + x^2(2\lambda_1(x-1)^2 + (4-3x)x)$$

which is further equivalent to

$$\begin{align*}
(\lambda_2 - \lambda_1)(x-1)^2 & \frac{(\lambda_1\lambda_2(x-1)^2 - 2x^2)}{(\lambda_1\lambda_2(x-1)^2 - 2x^2)} > 0.
\end{align*}$$

(13)

The expression on the left-hand side of (13) is polynomial in $x$ which is a continuous function. Moreover, the value of this expression at 0 is $(\lambda_2 - \lambda_1)\lambda_1\lambda_2$ which is strictly positive under assumption $\lambda_2 > \lambda_1 > 0$. From continuity and strict positivity at 0, it follows that there exists $x \in [0,1]$ which satisfies (13).  

\[\square\]
Corollary 9. Any case of domination within the family of Hamacher t-norms is one of these

\[ T^H_\lambda \succ T^H_\lambda \]

\[ T^H_0 \succ T^H_\lambda \]

\[ T^H_\lambda \succ T_D \]

for arbitrary \( \lambda \in [0, \infty) \). Moreover, domination is transitive within this family so that it is partially ordered by \( \succ \).

4. CONCLUDING REMARKS

Posets \( \{T^F_\lambda \mid \lambda \in [0, \infty]\}, \succ \) and \( \{T^H_\lambda \mid \lambda \in [0, \infty]\}, \succ \) are order isomorphical since \( T^F_{\lambda_1} \succ T^F_{\lambda_2} \) holds iff \( T^H_{\lambda_1} \succ T^H_{\lambda_2} \) does so. Results of this paper can be transformed to other families of t-norms by means of \( \varphi \)-transforms.

In Introduction we have mentioned that \( T_1 \geq T_2 \) is not satisfactory for \( T_1 \succ T_2 \). This claim is exemplified by any pair of nonextremal Frank (Hamacher) t-norms.

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