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DOMINATION IN THE FAMILIES OF FRANK AND HAMACHER t-NORMS

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Domination is a relation between general operations defined on a poset. The old open problem is whether domination is transitive on the set of all t-norms. In this paper we contribute partially by inspection of domination in the family of Frank and Hamacher t-norms. We show that between two different t-norms from the same family, the domination occurs iff at least one of the t-norms involved is a maximal or minimal member of the family. The immediate consequence of this observation is the transitivity of domination on both inspected families of t-norms.

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1. INTRODUCTION

The concept of domination has been introduced within the framework of probabilistic metric spaces for triangle functions and for building cartesian products of probabilistic metric spaces [12]. Afterwards the domination of t-norms was studied in connection with construction of fuzzy equivalence relations [2, 3, 13] and construction of fuzzy orderings [1]. Recently, the concept of domination was extended to the much general class of aggregation operators [9]. The domination of aggregation operators emerges when investigating which aggregation procedures applied to the system of T-transitive fuzzy relations yield a T-transitive fuzzy relation again [9] or when seeking aggregation operators which preserves the extensionality of fuzzy sets with respect to given T-equivalence relations [10]. The most general definition of domination considered so far demands the operations to be defined on arbitrary poset [4].

Definition 1. Let $(P, \geq)$ be a poset and let $A\colon P^m \to P$, $B\colon P^n \to P$ be two operations defined on P with arity m and n, respectively. Then we say that A dominates B ($A \gg B$ in symbols) if each matrix $(x_{i,j})$ of type $m \times n$ over P satisfies

$$\begin{aligned} & A(B(x_{1,1}, x_{1,2}, \ldots, x_{1,n}), \ldots, B(x_{m,1}, x_{m,2}, \ldots, x_{m,n})) \\ \geq\ & B(A(x_{1,1}, x_{2,1}, \ldots, x_{m,1}), \ldots, A(x_{1,n}, x_{2,n}, \ldots, x_{m,n})). \end{aligned}$$

Let us recall that a t-norm [12, 8] is a monotone, associative and commutative binary operation $T\colon [0,1]^2 \to [0,1]$ with neutral element 1. Important examples of t-norms are: the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$, the Łukasiewicz t-norm $T_{\mathbf{L}}$ and the drastic t-norm $T_{\mathbf{D}}$ given by

$$\begin{aligned} T_{\mathbf{M}}(x,y) &= \min(x,y), \\ T_{\mathbf{P}}(x,y) &= xy, \\ T_{\mathbf{L}}(x,y) &= \max(0, x+y-1), \\ T_{\mathbf{D}}(x,y) &= \begin{cases} xy & \max(x,y) = 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We say that a t-norm T_1 is stronger than a t-norm T_2($T_1 \geq T_2$ in symbols) if any $x, y \in [0,1]$ satisfy $T_1(x,y) \geq T_2(x,y)$. We use the notation $T_1 > T_2$ whenever simultaneously $T_1 \geq T_2$ and $T_1 \neq T_2$ hold. One can easily show that each t-norm is weaker than $T_{\mathbf{M}}$ and stronger than $T_{\mathbf{D}}$. Particularly, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$ satisfy $T_{\mathbf{M}} > T_{\mathbf{P}} > T_{\mathbf{L}} > T_{\mathbf{D}}$. It is obvious that $\geq$ is a partial order on the set of all t-norms, i. e., the reflexive, antisymmetric and transitive relation.

By Definition 1 we have that two t-norms T_1 and T_2 satisfy $T_1 \gg T_2$ iff for each x, y, u, $v \in [0,1]$

$$T_1(T_2(x,y), T_2(u,v)) \geq T_2(T_1(x,u), T_1(y,v)). \tag{1}$$

It is easy to show that each t-norm T satisfies $T_{\mathbf{M}} \gg T$, $T \gg T_{\mathbf{D}}$ and $T \gg T$. Moreover, by [8, 11], the representative t-norms $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$ satisfy $T_{\mathbf{P}} \gg T_{\mathbf{L}}$. If $T_1 \gg T_2$ then by inequality (1), the neutrality of 1 and the commutativity of t-norms we have that any $y, u \in [0,1]$ satisfy

$$\begin{aligned} T_1(y,u) &= T_1(T_2(1,y), T_2(u,1)) \\ &\geq T_2(T_1(1,u), T_1(y,1)) = T_2(u,y) = T_2(y,u) \end{aligned}$$

so that $T_1 \geq T_2$, see [8]. This means that satisfaction of $T_1 \geq T_2$ is a necessary condition for $T_1 \gg T_2$ or, in other words, that domination is a subrelation of $\geq$. The converse implication does not hold as it is demonstrated by results of this paper. Domination of t-norms is moreover an antisymmetric relation which is a consequence of the fact that it is a subrelation of the antisymmetric relation $\geq$. The old open problem [12, Problem 12.11.3] is whether domination is transitive on the set of all t-norms. If it were true domination would be a partial order.

When inspecting domination, the tool of φ-transform can be helpful. Let φ be an order isomorphism of the interval $[0,1]$ and let T be an arbitrary t-norm. Define $T_\varphi\colon [0,1]^2 \to [0,1]$ by

$$T_\varphi(x,y) = \varphi^{-1}\left(T(\varphi(x), \varphi(y))\right)$$

to be the φ-transform of T. It is easy to show that T_φ is again a t-norm [8]. Moreover, for arbitrary t-norms T_1 and T_2 and for arbitrary order isomorphism φ the satisfaction of $T_1 \gg T_2$ is equivalent to $(T_1)_\varphi \gg (T_2)_\varphi$ so that φ-transforms preserve domination [9]. Let us recall that a t-norm is strict (nilpotent) iff there exists φ such that $T = (T_{\mathbf{P}})_\varphi$ ($T = (T_{\mathbf{L}})_\varphi$) [8]. Moreover, it is clear that each φ-transform of a strict (nilpotent) t-norm is again strict (nilpotent). Thus in order to characterize pairs of dominating strict (nilpotent) t-norms it suffices to characterize strict (nilpotent) t-norms dominating $T_{\mathbf{P}}$ ($T_{\mathbf{L}}$).

The following result relates domination and powers of additive generators [8]. Let T be a continuous Archimedean t-norm with additive generator f and let $\lambda \in]0,\infty[$ be a positive number. Define $T^{(\lambda)}$ to be a t-norm with additive generator $f^\lambda(x)$, i.e., the λ-power of f. It is known that for each $\lambda > \mu$ is $T^{(\lambda)} \gg T^{(\mu)}$. This construction of dominating t-norms gives rise to many parametrical families of t-norms such as the Aczél–Alsina or the Dombi family.

Although the structure of domination on the set of all t-norms is still unknown, it is possible to inspect it on particular families of t-norms. One of the oldest results of this type is due to Sherwood [11] who solved the structure of domination on the family of Schweizer–Sklar t-norms. Another result of this type is the above mentioned solution of domination in the Aczél–Alsina or the Dombi family. In the next two sections we inspect another two important families – the Frank and Hamacher t-norms.

2. FRANK t-NORMS

Frank t-norms $T^{\mathbf{F}}_\lambda$ are given as

$$T^{\mathbf{F}}_\lambda(x,y) = \begin{cases} T_{\mathbf{M}}(x,y) & \lambda = 0 \\ T_{\mathbf{P}}(x,y) & \lambda = 1 \\ T_{\mathbf{L}}(x,y) & \lambda = \infty \\ \log_\lambda\left(\frac{(\lambda^x-1)(\lambda^y-1)}{\lambda-1}+1\right) & \text{otherwise} \end{cases} \tag{2}$$

where $\lambda \in [0,\infty]$ is the characterizing parameter of the Frank t-norm. Note that the family of Frank t-norms is strictly decreasing in λ which means that $T^{\mathbf{F}}_{\lambda_1} > T^{\mathbf{F}}_{\lambda_2}$ iff $\lambda_1 < \lambda_2$. In [5] M. J. Frank solved the problem of characterization of all continuous t-norms T such that the function $F\colon [0,1]^2 \to [0,1]$ given by

$$F(x,y) = x + y - T(x,y)$$

is associative. Each $T^{\mathbf{F}}_\lambda$ solves this problem.

In what follows we find out which $\lambda_1, \lambda_2 \in [0,\infty]$ satisfy $T^{\mathbf{F}}_{\lambda_1} \gg T^{\mathbf{F}}_{\lambda_2}$. Recall that for $\lambda_1 = 0$ the question is trivial as $T^{\mathbf{F}}_0 = T_{\mathbf{M}}$ dominates any t-norm. Particulary, for $\lambda_1 = 1$ and $\lambda_2 = \infty$ the question is solved as well since $T^{\mathbf{F}}_1 = T_{\mathbf{P}} \gg T_{\mathbf{L}} = T^{\mathbf{F}}_\infty$, see, for example, the already mentioned work of Sherwood [11]. Finally $T^{\mathbf{F}}_{\lambda_1} \gg T^{\mathbf{F}}_{\lambda_2}$ cannot be satisfied for $\lambda_1 > \lambda_2$ due to the decreasingness of the Frank family. That's why we consider $\lambda_1 < \lambda_2$ in the following.

Lemma 2. Let $A_n = [a^l_1, a^r_1] \times [a^l_2, a^r_2] \times \cdots \times [a^l_n, a^r_n]$, $a^l_i < a^r_i$, $i = 1,2,\dots,n$, be an n-dimensional interval. Let $f\colon A_n \to \mathbb{R}$ be a real function, linear in each argument.

Moreover, let the value of f be nonnegative in each vertex of A_n, i.e., at each point with coordinates $(b_1, b_2, \dots, b_n)$, $b_i \in \{a_i^l, a_i^r\}$. Then f is nonnegative on whole A_n.

Proof. By induction with respect to the dimension n. The statement is obvious for $n = 1$.

Let us assume that the claim of the lemma is true for all intervals of dimension $n-1$ and that A_n and f fulfill all assumptions of the lemma. Consider arbitrary $x = (x_1, x_2, \dots, x_n) \in A_n$. Define points

$$x_\star = (x_1, x_2, \dots, x_{n-1}, a_n^l),$$
$$x^\star = (x_1, x_2, \dots, x_{n-1}, a_n^r)$$

to be the left and right projections of the point x along the last coordinate. Further define functions $f_\star$ and $f^\star$ by expressions

$$f_\star(x_1, x_2, \dots, x_{n-1}) = f(x_1, x_2, \dots, x_{n-1}, a_n^l),$$
$$f^\star(x_1, x_2, \dots, x_{n-1}) = f(x_1, x_2, \dots, x_{n-1}, a_n^r).$$

Both functions $f_\star$ and $f^\star$ are defined on $(n-1)$-dimensional interval

$$A_{n-1} = [a_1^l, a_1^r] \times [a_2^l, a_2^r] \times \dots \times [a_{n-1}^l, a_{n-1}^r]$$

and both functions are linear in each argument. On vertices of A_{n-1} both functions attain nonnegative values. Indeed, let $v = (v_1, v_2, \dots, v_{n-1})$ be any vertex of A_{n-1}. Then $f_\star(v) = f(v_1, v_2, \dots, v_{n-1}, a_n^l)$ is a value of f at one vertex of A_n which is by assumption nonnegative. Analogically for $f^\star$.

Thus $f_\star$ and $f^\star$ are nonnegative on A_{n-1} by assumption. Particularly,

$$f_\star(x_1, x_2, \dots, x_{n-1}) = f(x_\star) \geq 0,$$
$$f^\star(x_1, x_2, \dots, x_{n-1}) = f(x^\star) \geq 0.$$

By assumptions, the function $g(y) = f(x_1, \dots, x_{n-1}, y)$ is linear on $[a_n^l, a_n^r]$ and

$$g(a_n^l) = f(x_\star) \geq 0,$$
$$g(x_n) = f(x),$$
$$g(a_n^r) = f(x^\star) \geq 0.$$

Thus $f(x) = g(x_n) \geq 0$. □

Proposition 3. $T_\lambda^{\mathbf{F}} \gg T_{\mathbf{L}}$ for each $\lambda \in]0,1[\cup]1,\infty[$.

Proof. We have to show that any x, y, u, $v \in [0,1]$ satisfy the inequality

$$T_\lambda^{\mathbf{F}}(T_{\mathbf{L}}(x,y), T_{\mathbf{L}}(u,v)) \geq T_{\mathbf{L}}(T_\lambda^{\mathbf{F}}(x,u), T_\lambda^{\mathbf{F}}(y,v)). \tag{3}$$

Let us consider two mutually exclusive cases. First that the left-hand side of (3) equals zero and the second that it is positive:

(i) Since for $\lambda \in]0,1[\,\cup\,]1,\infty[$ $T^{\mathbf{F}}_\lambda$ is strict, the left-hand side of (3) can be zero iff at least one of the Lukasiewitz t-norms involved attains the value 0. Without loss of generality assume $T_{\mathbf{L}}(x,y)=0$ which is equivalent to $x+y-1\leq 0$. It suffices to show that

$$T_{\mathbf{L}}(T^{\mathbf{F}}_\lambda(x,u),T^{\mathbf{F}}_\lambda(y,v)) = \max(0,T^{\mathbf{F}}_\lambda(x,u)+T^{\mathbf{F}}_\lambda(y,v)-1)=0$$

or simply $T^{\mathbf{F}}_\lambda(x,u)+T^{\mathbf{F}}_\lambda(y,v)-1\leq 0$. But from the nondecreasingness of $T^{\mathbf{F}}_\lambda$ and from the neutrality of 1 it follows

$$T^{\mathbf{F}}_\lambda(x,u)+T^{\mathbf{F}}_\lambda(y,v)-1\leq T^{\mathbf{F}}_\lambda(x,1)+T^{\mathbf{F}}_\lambda(y,1)-1=x+y-1\leq 0.$$

(ii) Assume that the left-hand side of (3) is positive, so that $x+y-1>0$ as well as $u+v-1>0$ holds. Inequality (3) can be rewritten in the form

$$T^{\mathbf{F}}_\lambda(x+y-1,u+v-1)\geq \max(0,T^{\mathbf{F}}_\lambda(x,u)+T^{\mathbf{F}}_\lambda(y,v)-1)$$

which is further equivalent to

$$T^{\mathbf{F}}_\lambda(x+y-1,u+v-1)\geq T^{\mathbf{F}}_\lambda(x,u)+T^{\mathbf{F}}_\lambda(y,v)-1$$

since the left-hand side is positive. After expansion of the definitions of $T^{\mathbf{F}}_\lambda$ the inequality can be rewritten as

$$\log_\lambda\left[\frac{(\frac{\lambda^x\lambda^y}{\lambda}-1)(\frac{\lambda^u\lambda^v}{\lambda}-1)}{\lambda-1}+1\right]\geq \log_\lambda\frac{\left[\frac{(\lambda^x-1)(\lambda^u-1)}{\lambda-1}+1\right]\left[\frac{(\lambda^y-1)(\lambda^v-1)}{\lambda-1}+1\right]}{\lambda}$$

and by further de-logarithmation we end up with

$$\operatorname{sgn}(\lambda-1)\left[\frac{(\frac{\lambda^x\lambda^y}{\lambda}-1)(\frac{\lambda^u\lambda^v}{\lambda}-1)}{\lambda-1}+1-\frac{\left[\frac{(\lambda^x-1)(\lambda^u-1)}{\lambda-1}+1\right]\left[\frac{(\lambda^y-1)(\lambda^v-1)}{\lambda-1}+1\right]}{\lambda}\right]\geq 0.$$

Note that the multiplicative constant $\operatorname{sgn}(\lambda-1)$ prevents the reversion of the order after de-logarithmation whenever $\lambda\in\,]0,1[$.

The expression on the left-hand side is nonnegative for any x, y, u, $v\in[0,1]$. Indeed, by substitution $\lambda^x=X$, $\lambda^y=Y$, $\lambda^u=U$ and $\lambda^v=V$ where X, Y, U, $V\in[\min(1,\lambda),\max(1,\lambda)]$ we obtain

$$\operatorname{sgn}(\lambda-1)\left[\frac{(\frac{XY}{\lambda}-1)(\frac{UV}{\lambda}-1)}{\lambda-1}+1-\frac{\left[\frac{(X-1)(U-1)}{\lambda-1}+1\right]\left[\frac{(Y-1)(V-1)}{\lambda-1}+1\right]}{\lambda}\right]\geq 0. \quad (4)$$

Let us define the function G: $[\min(1,\lambda),\max(1,\lambda)]^4\to\mathbb{R}$ in variables X, Y, U, V to be the value of the expression on the left-hand side of (4). One can easily see that G

is linear in each argument. A very simple computation reveals that G attains zero value at all vertices of $[\min(1,\lambda),\max(1,\lambda)]^4$ up to the following seven exceptions

$$\begin{aligned} G(1,1,1,1) &= \frac{\operatorname{sgn}(\lambda-1)(\lambda^2-1)}{\lambda^2} \geq 0,\\ G(\lambda,1,1,1)=G(1,\lambda,1,1) &= \frac{\operatorname{sgn}(\lambda-1)(\lambda-1)}{\lambda} \geq 0,\\ G(1,1,\lambda,1)=G(1,1,1,\lambda) &= \frac{\operatorname{sgn}(\lambda-1)(\lambda-1)}{\lambda} \geq 0,\\ G(1,\lambda,\lambda,1)=G(\lambda,1,1,\lambda) &= \frac{\operatorname{sgn}(\lambda-1)(\lambda-1)}{\lambda} \geq 0. \end{aligned}$$

which all are nonnegative values. Thus the function G satisfies all assumptions of Lemma 2 by which G is nonnegative which proves inequality (4). □

Proposition 3 together with $T_{\mathbf{M}} \gg T_{\mathbf{L}}$ and $T_{\mathbf{P}} \gg T_{\mathbf{L}}$ show that any Frank t-norm dominates $T_{\mathbf{L}}$. Further we discuss the mutual domination of nonextremal Frank t-norms.

Lemma 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be n-times differentiable in 0, $f^{(i)}(0) = 0$ for all $i = 0,1,\dots n-1$ and $f^{(n)}(0) < 0$. There exists $\delta > 0$ such that $f(x) < 0$ for each $x \in]0,\delta[$.

Proof. The claim of the lemma is a well-known result of real analysis. □

Proposition 5. There does not exist $\lambda_1, \lambda_2 \in]0,\infty[$ such that $\lambda_1 < \lambda_2$ and $T^{\mathbf{F}}_{\lambda_1} \gg T^{\mathbf{F}}_{\lambda_2}$.

Proof. Suppose arbitrary $\lambda_1, \lambda_2 \in]0,\infty[$ with $\lambda_1 < \lambda_2$. We shall show that there exists some $x \in]0,1[$ such that

$$T^{\mathbf{F}}_{\lambda_1}(T^{\mathbf{F}}_{\lambda_2}(x,x), T^{\mathbf{F}}_{\lambda_2}(x,x)) < T^{\mathbf{F}}_{\lambda_2}(T^{\mathbf{F}}_{\lambda_1}(x,x), T^{\mathbf{F}}_{\lambda_1}(x,x)) \tag{5}$$

so that the defining inequality for domination (1) is violated. Let us define the function $\delta^{\mathbf{F}}_{\lambda}\colon [0,1] \to [0,1]$ to be the diagonal of a Frank t-norm so that $\delta^{\mathbf{F}}_{\lambda}(x) = T^{\mathbf{F}}_{\lambda}(x,x)$ for any $x \in [0,1]$. Due to the strictness of $T^{\mathbf{F}}_{\lambda}$ we know that $\delta^{\mathbf{F}}_{\lambda}$ is an order isomorphism of the interval $[0,1]$. Inequality (5) can be rewritten into the form

$$\delta^{\mathbf{F}}_{\lambda_1}(\delta^{\mathbf{F}}_{\lambda_2}(x)) < \delta^{\mathbf{F}}_{\lambda_2}(\delta^{\mathbf{F}}_{\lambda_1}(x)). \tag{6}$$

Further define the function $f_{(\lambda_1,\lambda_2)}\colon [0,1] \to \mathbb{R}$ by expression

$$f_{(\lambda_1,\lambda_2)}(x) = \delta^{\mathbf{F}}_{\lambda_1}(\delta^{\mathbf{F}}_{\lambda_2}(x)) - \delta^{\mathbf{F}}_{\lambda_2}(\delta^{\mathbf{F}}_{\lambda_1}(x)),$$

Now another alternative reformulation of (5) is that there exists some $x > 0$ such that $f_{\lambda_1,\lambda_2}(x) < 0$. We prove this claim by means of Lemma 4.

Let us compute $\delta_\lambda^{\mathbf{F}}$ as well as its first and second derivatives which we will use later:

$$\delta_\lambda^{\mathbf{F}}(x) = \begin{cases} \log_\lambda\left(\frac{(\lambda^x-1)^2}{\lambda-1}+1\right) & \lambda \neq 1 \\ x^2 & \lambda = 1, \end{cases}$$

$$\delta_\lambda^{\mathbf{F}(1)}(x) = \begin{cases} \frac{2(\lambda^x-1)\lambda^x}{(\lambda^x-1)^2+\lambda-1} & \lambda \neq 1 \\ 2x & \lambda = 1, \end{cases}$$

$$\delta_\lambda^{\mathbf{F}(2)}(x) = \begin{cases} \frac{2\lambda^x \ln(\lambda)\left((2\lambda^x-1)(\lambda-1)-(\lambda^x-1)^2\right)}{((\lambda^x-1)^2+\lambda-1)^2} & \lambda \neq 1 \\ 2 & \lambda = 1. \end{cases}$$

Their values at point 0 are

$$\delta_\lambda^{\mathbf{F}}(0) = 0 \qquad \delta_\lambda^{\mathbf{F}(1)}(0) = 0 \qquad \delta_\lambda^{\mathbf{F}(2)}(0) = \begin{cases} \frac{2\ln(\lambda)}{\lambda-1} & \lambda \neq 1 \\ 2 & \lambda = 1 \end{cases} \tag{7}$$

so that the first nonzero derivative of $\delta_\lambda^{\mathbf{F}(2)}$ at point 0 is the second derivative. Thereout the first nonzero derivative of $f_{(\lambda_1,\lambda_2)}$, according to its definition, is the fourth derivative for which we have

$$f^{(4)}_{(\lambda_1,\lambda_2)}(0) = 3\delta_{\lambda_1}^{\mathbf{F}\,(2)}(0)\left(\delta_{\lambda_2}^{\mathbf{F}\,(2)}(0)\right)^2 - 3\delta_{\lambda_2}^{\mathbf{F}\,(2)}(0)\left(\delta_{\lambda_1}^{\mathbf{F}\,(2)}(0)\right)^2. \tag{8}$$

Now we can compute the value of this derivative for all feasible combinations of λ_1 and λ_2. Let us distinguish three mutually exclusive cases – the first that $\lambda_2 = 1$, then $\lambda_1 = 1$ and finally, $\lambda_1 \neq 1 \neq \lambda_2$.

(*i*) Let us consider $\lambda_1 < \lambda_2 = 1$. Combining (7) and (8) we obtain the expression

$$f^{(4)}_{(\lambda_1,1)}(0) = -24\frac{\ln(\lambda_1)}{\lambda_1 - 1}\left(\frac{\ln(\lambda_1)}{\lambda_1 - 1} - 1\right)$$

The sign of this derivative is determined by the sign of the expression in parenthesis. Under the assumption $\lambda_1 < 1$, the expression in parenthesis is positive because the expression $\ln(\lambda)/(\lambda-1)$ is decreasing, continuous on $]0,1[\,\cup\,]1,\infty[$ and

$$\lim_{\lambda \to 1} \frac{\ln(\lambda)}{\lambda - 1} = 1.$$

Thus the first nonzero derivative of $f_{(\lambda_1,1)}$ is negative at point 0.

(*ii*) Let us consider $1 = \lambda_1 < \lambda_2$. Combining (7) and (8) we obtain the expression

$$f^{(4)}_{(1,\lambda_2)}(0) = 24\frac{\ln(\lambda_2)}{\lambda_2 - 1}\left(\frac{\ln(\lambda_2)}{\lambda_2 - 1} - 1\right).$$

Following the considerations from (*i*) we find out that $f^{(4)}_{(1,\lambda_2)}(0)$ is negative.

(*iii*) Let us consider $\lambda_1 \neq 1 \neq \lambda_2$. Combining (7) and (8) gives us the expression

$$f^{(4)}_{(\lambda_1,\lambda_2)}(0) = -24\frac{\ln(\lambda_1)\ln(\lambda_2)}{(\lambda_1-1)(\lambda_2-1)}\left(\frac{\ln(\lambda_1)}{\lambda_1-1} - \frac{\ln(\lambda_2)}{\lambda_2-1}\right).$$

The sign of the derivative is determined by the sign of expression in ellipses. From the decreasingness of this expression and from $\lambda_1 < \lambda_2$ it follows that $f^{(4)}_{(\lambda_1,\lambda_2)}(0) < 0$.

We distinguished all possible cases and regardless of the values of λ_1 and λ_2 the value of $f^{(4)}_{(\lambda_1,\lambda_2)}(0)$ is negative. In addition, all lower-order derivatives of $f_{(\lambda_1,\lambda_2)}$ vanish at point 0. By Lemma 4 there exists some $x \in]0,1[$ such that $f(x) < 0$. □

Corollary 6. Any case of domination within the family of Frank t-norms is one of these

$$\begin{aligned} T^{\mathbf{F}}_{\lambda} &\gg T^{\mathbf{F}}_{\lambda} \\ T_{\mathbf{M}} &\gg T^{\mathbf{F}}_{\lambda} \\ T^{\mathbf{F}}_{\lambda} &\gg T_{\mathbf{L}} \end{aligned}$$

for arbitrary $\lambda \in [0,\infty]$. Moreover, domination is transitive within this family so that it is partially ordered by $\gg$.

3. HAMACHER t-NORMS

Hamacher t-norms form another one-parametric family of t-norms. It has been proved in [6, 7] that members of this family are the only ones to be expressed as quotient of two polynomials in two variables. The family of Hamacher t-norms is parameterized by $\lambda \in [0,\infty]$

$$T^{\mathbf{H}}_{\lambda}(x,y) = \begin{cases} T_{\mathbf{D}}(x,y) & \lambda = \infty \\ 0 & \lambda = x = y = 0 \\ \frac{xy}{\lambda + (1-\lambda)(x+y-xy)} & \text{otherwise.} \end{cases} \tag{9}$$

The Hamacher family is strictly decreasing in λ which means that $T^{\mathbf{H}}_{\lambda_1} > T^{\mathbf{H}}_{\lambda_2}$ iff $\lambda_1 < \lambda_2$. The drastic t-norm $T_{\mathbf{D}} = T^{\mathbf{H}}_{\infty}$ is the minimal element and the t-norm $T^{\mathbf{H}}_{0}$ is the maximal element of the family.

In this section we answer the question for which $\lambda_1, \lambda_2 \in [0,\infty]$ the relation $T^{\mathbf{H}}_{\lambda_1} \gg T^{\mathbf{H}}_{\lambda_2}$ is satisfied. Recall that for $\lambda_2 = \infty$ the question is trivial as $T^{\mathbf{H}}_{\infty} = T_{\mathbf{D}}$ is dominated by any t-norm. Moreover, $T^{\mathbf{H}}_{\lambda_1} \gg T^{\mathbf{H}}_{\lambda_2}$ cannot be satisfied for $\lambda_1 > \lambda_2$ due to decreasingness within the family of Hamacher t-norms. That is why we will only deal with $\lambda_1 < \lambda_2$ in the sequel.

Proposition 7. For each $\lambda \in]0,\infty]$ it holds that $T^{\mathbf{H}}_{0} \gg T^{\mathbf{H}}_{\lambda}$.

Proof. We divide the proof into two parts. We first show that $T^{\mathbf{H}}_{0} \gg T_{\mathbf{P}}$ and then we prove the claim of proposition by virtue of φ-transform.

(*i*) We show that $T_0^{\mathbf{H}}(xy, uv) \geq T_0^{\mathbf{H}}(x,u)T_0^{\mathbf{H}}(y,v)$ holds for any x, y, u, $v \in [0,1]$. This inequality is trivially fulfilled whenever at least one variable equals 0. Therefore assume $xyuv > 0$. After expansion of the definitions we have

$$\frac{xyuv}{xy + uv - xyuv} \geq \frac{xu}{x + u - xu}\,\frac{yv}{y + v - yv}$$

or equivalently, by inversion

$$\frac{xy + uv - xyuv}{xyuv} \leq \frac{(x + u - xu)(y + v - yv)}{xyuv}.$$

As the denominators of both fractions are equal and positive, we can drop them, and by further manipulation we obtain the third equivalent inequality

$$0 \leq (x + u - xu)(y + v - yv) - xy - uv + xyuv$$

or

$$0 \leq xv(1-u)(1-y) + uy(1-v)(1-x)$$

where the expression on the right-hand side is evidently nonnegative.

(*ii*) Now, let φ_λ be the multiplicative generator of the nonextremal Hamacher t-norm $T_\lambda^{\mathbf{H}}$. So that for $\lambda \in\,]0,\infty[$, φ_λ and its inverse are given by

$$\varphi_\lambda(x) = \frac{x}{\lambda + (1-\lambda)x}, \qquad \varphi_\lambda^{-1}(x) = \frac{\lambda x}{1 + (1-\lambda)x}.$$

Let us apply the φ-transform to both $T_0^{\mathbf{H}}$ and $T_{\mathbf{P}}$. Since $T_0^{\mathbf{H}}$ dominates $T_{\mathbf{P}}$, the corresponding φ-transforms do as well.

The φ_λ-transform of $T_{\mathbf{P}}$ is $T_\lambda^{\mathbf{H}}$ by the definition of multiplicative generator. Now we shall show that φ_λ-transform of $T_0^{\mathbf{H}}$ is again $T_0^{\mathbf{H}}$, i. e., the strongest Hamacher t-norm is stable under the φ_λ-transform whenever φ_λ is a multiplicative generator of a nonextremal Hamacher t-norm. The equality

$$\varphi_\lambda^{-1}(T_0^{\mathbf{H}}(\varphi_\lambda(x), \varphi_\lambda(y))) = T_0^{\mathbf{H}}(x,y)$$

is trivially fulfilled whenever $xy = 0$. Now assume $xy > 0$. Then we have

$$\begin{aligned}
\varphi_\lambda^{-1}(T_0^{\mathbf{H}}(\varphi_\lambda(x), \varphi_\lambda(y))) &= \varphi_\lambda^{-1}\left(\frac{\varphi_\lambda(x)\varphi_\lambda(y)}{\varphi_\lambda(x) + \varphi_\lambda(y) - \varphi_\lambda(x)\varphi_\lambda(y)}\right) \\
&= \varphi_\lambda^{-1}\left(\frac{xy}{\lambda(x+y) + (1-2\lambda)xy}\right) \\
&= \frac{xy}{x + y - xy} \\
&= T_0^{\mathbf{H}}(x,y).
\end{aligned}$$

Since $T_0^{\mathbf{H}} \gg T_{\mathbf{P}}$, by virtue of φ_λ-transform we have that $T_0^{\mathbf{H}} \gg T_\lambda^{\mathbf{H}}$ which is our claim. □

Proposition 8. There does not exist $\lambda_1, \lambda_2 \in]0, \infty[$ such that $\lambda_1 < \lambda_2$ and $T^{\mathbf{H}}_{\lambda_1} \gg T^{\mathbf{H}}_{\lambda_2}$.

Proof. Let λ_1 and λ_2 satisfy assumptions of the proposition. We shall show that there exists $x \in]0,1[$ such that

$$T^{\mathbf{H}}_{\lambda_1}(T^{\mathbf{H}}_{\lambda_2}(x,x), T^{\mathbf{H}}_{\lambda_2}(x,x)) < T^{\mathbf{H}}_{\lambda_2}(T^{\mathbf{H}}_{\lambda_1}(x,x), T^{\mathbf{H}}_{\lambda_1}(x,x)) \tag{10}$$

so that the defining inequality for domination (1) is violated. Let us define the function $\delta^{\mathbf{H}}_{\lambda}\colon [0,1] \to [0,1]$ to be the diagonal of a Hamacher t-norm so that $\delta^{\mathbf{H}}_{\lambda}(x) = T^{\mathbf{H}}_{\lambda}(x,x)$ for any $x \in [0,1]$. The inequality (10) can be rewritten as

$$\delta^{\mathbf{H}}_{\lambda_1}(\delta^{\mathbf{H}}_{\lambda_2}(x)) < \delta^{\mathbf{H}}_{\lambda_2}(\delta^{\mathbf{H}}_{\lambda_1}(x)). \tag{11}$$

In order to show that (11) is satisfied for some $x \in]0,1[$ it suffices to show that this x satisfies

$$\frac{x^4}{\delta^{\mathbf{H}}_{\lambda_1}(\delta^{\mathbf{H}}_{\lambda_2}(x))} > \frac{x^4}{\delta^{\mathbf{H}}_{\lambda_2}(\delta^{\mathbf{H}}_{\lambda_1}(x))} \tag{12}$$

since we consider $x \neq 0$ and both compositions of the diagonals are positive whenever $x \in]0,1[$. The diagonal of a Hamacher t-norm $T^{\mathbf{H}}_{\lambda}$ is given by the expression

$$T^{\mathbf{H}}_{\lambda}(x,x) = \frac{x^2}{\lambda + (1-\lambda)(2-x)x}$$

by which

$$\begin{aligned} \delta^{\mathbf{H}}_{\lambda_1}(\delta^{\mathbf{H}}_{\lambda_2}(x)) &= \frac{\frac{x^4}{(\lambda_2+(1-\lambda_2)(2-x)x)^2}}{\lambda_1 + (1-\lambda_1)\left[2 - \frac{x^2}{\lambda_2+(1-\lambda_2)(2-x)x}\right]\frac{x^2}{\lambda_2+(1-\lambda_2)(2-x)x}} \\ &= \frac{x^4}{\lambda_1(\lambda_2(x-1)-2x)^2(x-1)^2+x^2(2\lambda_2(x-1)^2+(4-3x)x)} \end{aligned}$$

and

$$\begin{aligned} \delta^{\mathbf{H}}_{\lambda_2}(\delta^{\mathbf{H}}_{\lambda_1}(x)) &= \frac{\frac{x^4}{(\lambda_1+(1-\lambda_1)(2-x)x)^2}}{\lambda_2 + (1-\lambda_2)\left[2 - \frac{x^2}{\lambda_1+(1-\lambda_1)(2-x)x}\right]\frac{x^2}{\lambda_1+(1-\lambda_1)(2-x)x}} \\ &= \frac{x^4}{\lambda_2(\lambda_1(x-1)-2x)^2(x-1)^2+x^2(2\lambda_1(x-1)^2+(4-3x)x)}. \end{aligned}$$

According to these expressions, (12) can be rewritten in the form

$$\begin{aligned} &\lambda_1(\lambda_2(x-1)-2x)^2(x-1)^2+x^2(2\lambda_2(x-1)^2+(4-3x)x) \\ > \; &\lambda_2(\lambda_1(x-1)-2x)^2(x-1)^2+x^2(2\lambda_1(x-1)^2+(4-3x)x) \end{aligned}$$

which is further equivalent to

$$(\lambda_2 - \lambda_1)(x-1)^2\left(\lambda_1\lambda_2(x-1)^2 - 2x^2\right) > 0. \tag{13}$$

The expression on the left-hand side of (13) is polynomial in x which is a continuous function. Moreover, the value of this expression at 0 is $(\lambda_2-\lambda_1)\lambda_1\lambda_2$ which is strictly positive under assumption $\lambda_2 > \lambda_1 > 0$. From continuity and strict positivity at 0, it follows that there exists $x \in]0,1[$ which satisfies (13). □

Corollary 9. Any case of domination within the family of Hamacher t-norms is one of these

$$\begin{aligned} T_\lambda^{\mathbf{H}} &\gg T_\lambda^{\mathbf{H}} \\ T_0^{\mathbf{H}} &\gg T_\lambda^{\mathbf{H}} \\ T_\lambda^{\mathbf{H}} &\gg T_{\mathbf{D}} \end{aligned}$$

for arbitrary $\lambda \in [0, \infty]$. Moreover, domination is transitive within this family so that it is partially ordered by $\gg$.

4. CONCLUDING REMARKS

Posets $(\{T_\lambda^{\mathbf{F}} \mid \lambda \in [0,\infty]\}, \gg)$ and $(\{T_\lambda^{\mathbf{H}} \mid \lambda \in [0,\infty]\}, \gg)$ are order isomorphical since $T_{\lambda_1}^{\mathbf{F}} \gg T_{\lambda_2}^{\mathbf{F}}$ holds iff $T_{\lambda_1}^{\mathbf{H}} \gg T_{\lambda_2}^{\mathbf{H}}$ does so. Results of this paper can be transformed to other families of t-norms by means of φ-transforms.

In Introduction we have mentioned that $T_1 \geq T_2$ is not satisfactory for $T_1 \gg T_2$. This claim is exemplified by any pair of nonextremal Frank (Hamacher) t-norms.

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