KYBERNETIKA — VOLUME 41 (2005), NUMBER 3, PAGES 285 - 296

THE CANCELLATION LAW FOR PSEUDO-CONVOLUTION[1]

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Cancellation law for pseudo-convolutions based on triangular norms is discussed. In more details, the cases of extremal t-norms T_M and T_D, of continuous Archimedean t-norms, and of general continuous t-norms are investigated. Several examples are included.

Keywords: cancellation law, t-norm, pseudo-convolution

AMS Subject Classification: 03E72, 28E10

1. INTRODUCTION

In algebraic structures, a commutative binary operation $*$ is said to be cancellative if for all elements $g,\ h,\ v$ it holds

$$g * v = h * v \ \Rightarrow g = h.$$

The cancellation law ensures for example the uniqueness of solution of equation $x * v = u$ (if a solution exists).

The aim of this paper is investigation of the cancellativity of pseudo-convolutions introduced in [16]. Recall that the standard probabilistic convolution of distribution functions is cancellative.

The paper is organized as follows. In the next section, pseudo-convolutions are introduced. In Section 3, cancellation law for pseudo-convolutions based on boundary t-norms is discussed. Section 4 and Section 5 are devoted to the study of cancellation law in the case of continuous Archimedean t-norms and more general continuous t-norms based pseudo-convolutions.

2. PSEUDO-CONVOLUTIONS

2.1. Pseudo-convolution of real functions

Let $[a, b]$ be a closed subinterval of the extended real line (sometimes also semiclosed subintervals are taken into account).

[1]Presented at the 7th FSTA international conference held in Liptovský Mikuláš, Slovakia, on January 26–30, 2004.

Definition 1. A binary operation $\oplus$ on $[a,b]$ is called a *pseudo-addition* on $[a,b]$ if it is commutative, nondecreasing, associative, continuous (possibly up to the points (a,b), (b,a)) and with a neutral element, denoted by $\mathbf{0}$, i.e., for each $x \in [a,b]$ $\mathbf{0} \oplus x = x$ holds.

So, $\oplus$ is either a t-norm, or a t-conorm or a uni-norm on $[a,b]$, see [4]. Because of the duality, it is sufficient to deal with t-conorms and uninorms only. Denote $[a,b]_+ = \{x;\ x \in [a,b], x \geq \mathbf{0}\}$.

Definition 2. A binary operation $\otimes$ on $[a,b]$ is called a *pseudo-multiplication* with respect to $\oplus$ if it is commutative, associative, distributive with respect to $\oplus$, positively nondecreasing (i.e., $x \leq y \Rightarrow x \otimes z \leq y \otimes z$ if $z \in [a,b]_+$) with a unit element, denoted by $\mathbf{1}$, (i.e., for each $x \in [a,b]$ $\mathbf{1} \otimes x = x$ holds). We suppose, further, $\mathbf{0} \otimes x = \mathbf{0}$, i.e., $\mathbf{0}$ is annihilator.

The structure $([a,b], \oplus, \otimes)$ is called *a semiring*, see e.g., [2].

Let $([a,b], \oplus, \otimes)$ be a semiring with continuous operations (possibly up to the continuity of $\otimes$ in points $(\mathbf{0},a)$, $(\mathbf{0},b)$, $(a,\mathbf{0})$ and $(b,\mathbf{0})$). The standart building up of an integral with respect to $\oplus$-decomposable measures based on the pseudo-addition and pseudo-multiplication leads to the definition of a pseudo-integral [12]. *The pseudo-convolution* of the functions defined on $[0,\infty[$ with values in $[a,b]$ was introduced in [16], see also [12, 14], by means of the corresponding pseudo-integral,

$$g * h(z) = \int_{[0,z]} g(z-x) \otimes h(x)\mathrm{d}x. \tag{1}$$

In our paper we will deal with the special semiring only, so we will not describe some details here. (It is possible to find them in [14, 16].)

2.2. Pseudo-convolution with respect to the semiring $([0,1], \vee, T)$

One of typical examples of a semiring is $([0,1], \vee, T)$, where $\vee = \sup$ and T is a t-norm, see [4]. This is the semiring with $\mathbf{0} = 0$ and $\mathbf{1} = 1$. In this case the formula for convolution (1) can be rewritten to

$$g * h(z) = \sup_{x \in [0,z]} T(g(z-x), h(x)), \tag{2}$$

where T is a t-norm.

Observe that the pseudo-convolution $*$ is commutative due to the commutativity of T, however, it need not be associative, in general. Nevertheless, for t-norms continuous on $[0,1[^2$, $*$ is also associative.

Note that the kernel of a function $g : [0,\infty[\to [0,1]$ is defined as

$$\ker(g) = \{x \in [0,\infty[; g(x) = 1\}.$$

Denote by $\mathcal{D}$ the class of all continuous distribution functions on $[0,\infty[$ and by $\mathcal{S}$ the subclass of $\mathcal{D}$ such that the restriction of g on $]a_g, b_g[:= \operatorname{supp} g \setminus \ker(g)$ (if $\ker(g) = \emptyset$ then $b_g = \infty$) is strictly increasing, i.e.,

$$\mathcal{S} = \left\{ g : [0,\infty[\to [0,1];\ g(0) = 0,\ g|_{]a_g,b_g[} \to]0,1[\text{ is increasing bijection} \right\}.$$

Lemma 1. Let $\ker(v) \neq \emptyset$ for a function $v \in \mathcal{D}$. Then for all $g,\ h \in \mathcal{S}$ the following implication holds:

$$g * v = h * v \Rightarrow \ker(g) = \ker(h), \quad \text{i.e., } b_g = b_h. \tag{3}$$

Proof. Let $\ker(v) \neq \emptyset$. We can get the formula (3) from the property $\ker(g*v) = \ker(g) + \ker(v)$. First we suppose that $b_g < \infty$.

- Let $z \geq b_g + b_v$. Then

$$\begin{aligned} g * v(z) &= \sup_{x\in[0,z]} T(g(z-x), v(x)) \\ &= \max\left\{ \sup_{x\in[0,z-b_g[} T(g(z-x), v(x)),\ \sup_{x\in[z-b_g,z]} T(g(z-x), v(x)) \right\}, \end{aligned}$$

  – if $0 \leq x < \underbrace{z - b_g}_{\geq b_v}$, i.e., $z - x > b_g$

$$g * v(z) = \sup_{x\in[0,z-b_g[} T(1, v(x)) = \sup_{x\in[0,z-b_g[} v(x) = 1,$$

  – if $b_v \leq z - b_g \leq x \leq z$, i.e., $z - x \leq b_g$

$$g * v(z) = \sup_{x\in]z-b_g,z]} T(g(z-x), 1) = \sup_{x\in]z-b_g,z]} g(z-x) = g(b_g) = 1.$$

- Let $0 \leq z < b_g + b_v$. Then

$$g * v(z) = \max\left\{ \sup_{x\in[0,z-b_g]} T(g(z-x), v(x)),\ \sup_{x\in]z-b_g,b_v[} T(g(z-x), v(x)), \right.$$
$$\left. \sup_{x\in[b_v,z]} T(g(z-x), v(x)) \right\}$$

  – if $0 \leq x \leq z - b_g$, i.e., $z - x \geq b_g$

$$g * v(z) = \sup_{x\in[0,z-b_g]} T(1, v(x)) = \sup_{x\in[0,z-b_g]} v(x) = v(\underbrace{z - b_g}_{<b_v}) < 1,$$

– if $z - b_g < x < b_v$, i. e., $z - x < b_g$

$$g * v(z) = \sup_{x \in]z-b_g, b_v[} T(g(z-x), v(x)) < 1,$$

– if $b_v \leq x \leq z$

$$g * v(z) = \sup_{x \in [b_v, z]} T(g(z-x), 1) = \sup_{x \in [b_v, z]} g(z-x) = g(\underbrace{z - b_v}_{< b_g}) < 1.$$

It is easy to see that if $b_v < \infty$ then $b_g = \infty$ if and only if $b_{g*v} = \infty$, i. e., if $b_g = \infty$ then $b_h = \infty$ too. □

3. THE CANCELLATION LAW FOR PSEUDO-CONVOLUTION BASED ON THE SEMIRING $([0,1], \vee, T_M)$ AND $([0,1], \vee, T_D)$, RESPECTIVELY

Recall that T is a t-norm if it is associative, commutative, non-decreasing binary operation on $[0,1]$ with neutral element 1. For more details we recommend [4]. For any t-norm T it holds $T_D \leq T \leq T_M$, where the strongest t-norm $T_M = \min$ and the weakest t-norm T_D (the drastic product) is given by

$$T_D(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Theorem 1. Consider the strongest t-norm T_M. Let g, h, $v \in \mathcal{D}$. Then the cancellation law holds,i. e.,

$$g * v = h * v \;\Rightarrow g = h.$$

Proof. We denote $g^{(c)}$ the c-cut of function g, i. e., $g^{(c)} = \{x; g(x) \geq c\}$ for $c \in]0,1]$. Then for convolution based on the T_M it holds

$$(g * h)^{(c)} = g^{(c)} + h^{(c)} \text{ for any } c \in]0,1].$$

An arbitrary c-cut of function g from $\mathcal{D}$ is interval $[a_g^{(c)}, \infty[$. Suppose $g * v = h * v$. Then

$$\begin{aligned} & [a_g^{(c)}, \infty[+ [a_v^{(c)}, \infty[= [a_h^{(c)}, \infty[+ [a_v^{(c)}, \infty[\text{ for all } c \in]0,1]. \text{ Thus } [a_g^{(c)} + a_v^{(c)}, \infty[\\ = \; & [a_h^{(c)} + a_v^{(c)}, \infty[\Rightarrow a_g^{(c)} = a_h^{(c)} \Rightarrow g^{(c)} = h^{(c)} \text{ for all } c \in]0,1] \Rightarrow g = h, \end{aligned}$$

i. e., the cancellation law holds. □

Remark 1. The cancellation law with respect to T_M fails if $\sup v < 1$ or $\inf v > 0$ or if we deal with non-monotone functions. See Example 1.

Example 1. Consider the t-norm T_M. Let $v(x) = \begin{cases} x, & x \in [0,1] \\ 1, & x \in]1,\infty[, \end{cases}$

$$g(x) = \begin{cases} x, & x \in [0,\frac{1}{2}] \\ \frac{1}{2}, & x \in]\frac{1}{2},2] \\ x-\frac{3}{2}, & x \in]2,\frac{5}{2}] \\ 1, & x \in]\frac{5}{2},\infty[\end{cases} \quad \text{and} \quad h(x) = \begin{cases} x, & x \in [0,\frac{1}{2}] \\ \frac{1}{2}, & x \in]\frac{1}{2},1] \\ -x+\frac{3}{2}, & x \in]1,\frac{5}{4}] \\ x-1, & x \in]\frac{5}{4},\frac{3}{2}] \\ \frac{1}{2}, & x \in]\frac{3}{2},2] \\ x-\frac{3}{2}, & x \in]2,\frac{5}{2}] \\ 1, & x \in]\frac{5}{2},\infty[. \end{cases}$$

The function h is not a monotone function. Then pseudo-convolutions of functions g, v and h, v based on semiring $([0,1],\vee,T_M)$ are the same, i. e.

$$g * v(x) = h * v(x) = \begin{cases} \frac{1}{2}x, & x \in [0,1] \\ \frac{1}{2}, & x \in]1,\frac{5}{2}] \\ \frac{x}{2}-\frac{3}{4}, & x \in]\frac{5}{2},\frac{7}{2}] \\ 1, & x \in]\frac{7}{2},\infty[. \end{cases}$$

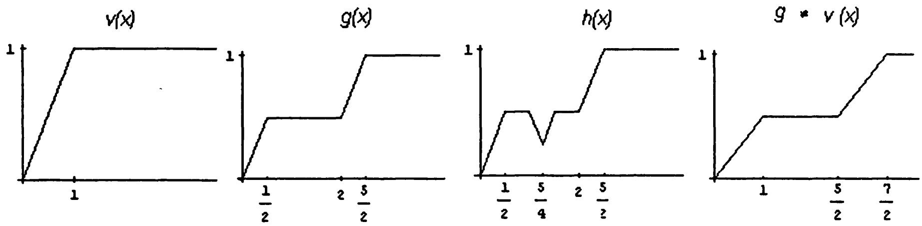

Fig. 1.

On the other hand, consider the weakest t-norm T_D. Then the cancellation law holds only in special cases.

Theorem 2. Consider the pseudo-convolution based on the T_D. Let g, h, $v \in \mathcal{D}$. Moreover, let

$$v(b_v - x) \le \min\left(g(b_g - x), h(b_h - x)\right) \quad \text{for all } x \in [0,b],$$

where $b := \min\{b_v, b_g, b_h\}$. Then $g * v = h * v \Leftrightarrow g = h$.

Proof. Applying the formula for sum of fuzzy quantities based on the drastic product from [9], we get

$$g * v(x) = \max\{g(/x - b_v/), v(/x - b_g/)\}$$

for all $x \in [0, \infty[$, where $/x/ = \min\{\max\{0, x\}, 1\}$. Now we can easily get condition for cancellativity. □

4. THE CANCELLATION LAW FOR PSEUDO–CONVOLUTION BASED ON $([0,1], \vee, T)$, WHERE T IS AN ARCHIMEDEAN CONTINUOUS t-NORM

In this section at first we describe some Zagrodny's results [20]. Further we will apply them for investigation of validity of cancellation law for pseudo-convolution of functions based on a strict t-norm. Finally, the case of nilpotent t-norms will be discussed.

4.1. The cancellation law for inf-convolution – Zagrodny's results

Definition 3. Let g, $h : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$. The inf-convolution of g and h at $x \in \mathbb{R}$ is defined by

$$g \,\Box\, h(x) := \inf_{y+z=x} (g(y) + h(z)).$$

Definition 4. Let $h : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$. The function h is said to be uniformly convex if for $\forall \varepsilon \geq 0 \;\exists \delta > 0$

$$|a-b| \geq \varepsilon \;\Rightarrow h\left(\frac{a+b}{2}\right) \leq \frac{h(a)+h(b)-\delta|a-b|}{2}, \;\forall\, a, b \in \text{dom } h.$$

Note that the domain of functions g, h can be restricted to some intervals. Zagrodny in [20] deal with more general functions on Banach space.

Theorem 3. Let X be a reflexive Banach space. If q, g, $h : X \to \mathbb{R} \cup \{\infty\}$ are proper lower semicontinuous convex functions such that h is strictly convex and $\lim_{\|x\| \to \infty} \frac{h(x)}{\|x\|} = \infty$ then $q \,\Box\, h = g \,\Box\, h$ implies $q = g$.

Theorem 4. Let X be a Banach space and q, g, $h : X \to \mathbb{R} \cup \{\infty\}$ be proper lower semicontinuous convex functions. Moreover, suppose h is uniformly convex. Then $q \,\Box\, h = g \,\Box\, h$ implies $q = g$.

4.2. The cancellation law for pseudo-convolution based on a strict t-norm

Recall that the pseudo-convolution of functions based on semiring $([0,1], \vee, T)$ with some Archimedean continuous t-norm T can be expressed by

$$g * h(x) = f^{[-1]}\left(\inf_{y+z=x} (f(g(y)) + f(h(z))) = f^{[-1]}(f \circ g \,\Box\, f \circ h\ (x)\right), \;x \in [0, \infty[,$$

where f is additive generator of t-norm T, i. e., $f : [0,1] \to [0,\infty]$ is continuous strictly decreasing mapping verifying $f(1) = 0$, and pseudo-inverse $f^{[-1]} : [0,\infty] \to [0,1]$ of f is defined by

$$f^{[-1]}(x) = f^{-1}(\min(f(0), x)).$$

Archimedean continuous t-norms can be divided into two classes: strict and nilpotent. An additive generator of a strict t-norm is unbounded, and then $f^{[-1]} = f^{-1}$.

Theorem 5. Consider a strict t-norm T with an additive generator f. Let $g, h, v \in \mathcal{S}$ such that $f \circ g$ and $f \circ h$ are convex on $[a_g, b_g[$ and $[a_h, b_h[$, respectively and $f \circ v$ is either

(i) uniformly convex on $[a_v, b_v[$ or

(ii) strictly convex on $[a_v, b_v[$

and if $b_v = \infty$ then $\lim\limits_{x\to\infty} \frac{f \circ v(x)}{x} = \infty$.

Then $g * v = h * v$ implies $g = v$.

Proof. Assume $f \circ g$, $f \circ h$ and $f \circ v$ verify conditions from theorem. Let $g * v = h * v$. This imply $f \circ g \,\Box\, f \circ v = f \circ h \,\Box\, f \circ v$ and by Zagrodny's results $f \circ g = f \circ h \Rightarrow g = h$, i. e., the cancellativity is valid. □

4.3. The cancellation law for pseudo-convolution based on a nilpotent t-norm

The case of nilpotent t-norm is more complicated. Conditions from Theorem 5 are deficient. See Example 2.

Example 2. Consider the Lukasiewicz t-norm T_L with additive generator

$$f(x) = 1 - x \qquad \text{and functions} \qquad g(x) = \begin{cases} x, & x \in [0,1] \\ 1, & x \in]1,\infty[, \end{cases}$$

$$h(x) = \begin{cases} 0, & x \in [0, 0.1] \\ 2x - 0.2, & x \in]0.1, 0.2] \\ x, & x \in]0.2, 1] \\ 1, & x \in]1,\infty[\end{cases} \quad \text{and } v(x) = \begin{cases} 1-(x-1)^2, & x \in [0,1] \\ 1, & x \in]1,\infty[. \end{cases}$$

The interval $[a_v, b_v[= [0,1[$ and $f \circ v$ is given by formula $f \circ v(x) = 1 - (x-1)^2$ (i. e., strictly convex function).

The interval $[a_g, b_g[= [0,1[$ too and $f \circ g(x) = 1 - x$ on $[0,1[$ (i. e., convex function). Finally, $[a_h, b_h[= [0.1, 1[$ and

$$f \circ h = \begin{cases} 1.2 - 2x, & x \in [0.1, 0.2[\\ 1 - x, & x \in [0.2, 1[, \end{cases}$$

(i. e., convex function).

However, the pseudo-convolution based on $([0,1],\vee,T_L)$ of functions v and g is the same as pseudo-convolution (based on the same semiring) of functions v and h.

$$g*v(x)=h*v(x)=\begin{cases}0, & x\in[0,\frac{3}{4}]\\ x-\frac{3}{4}, & x\in]\frac{3}{4},\frac{3}{2}]\\ 1-(x-2)^2, & x\in]\frac{3}{2},2]\\ 1, & x\in]2,\infty[.\end{cases}$$

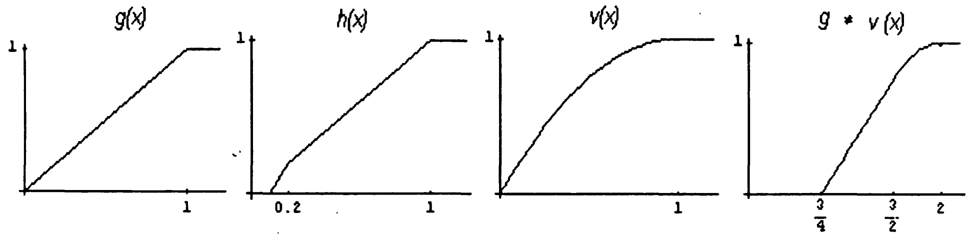

Fig. 2.

Thus Theorem 5 is not valid in the case when T is a nilpotent t-norm, in general. For nilpotent t-norms, we have only the following special cancellation theorems.

Theorem 6. Consider a nilpotent t-norm T with normed additive generator f. Let g, h, $v\in\mathcal{S}$, such that $f\circ g$, $f\circ h$ and $f\circ v$ are concave on the interval $[a_g,b_g[$, $[a_h,b_h[$ and $[a_v,b_v[$ respectively. Moreover,

$$v(b_v-x)\le\min\left(g(b_g-x),h(b_h-x)\right)\qquad\text{for all } x\in[0,b],$$

where $b:=\min\{b_v,\, b_g,\, b_h\}$. Then $g*v=h*v\ \Leftrightarrow\ g=h$.

The proof follows from the fact that under requirements of the theorem, the pseudo-convolution of function based on semiring $([0,1],\vee,T)$ with some nilpotent t-norm T behaves as the pseudo-convolution of function based on semiring $([0,1],\vee,T_D)$, see [7, 9]. Note that the same claim is true also for strict t-norms. However then $b_g=b_h=b_v=\infty$.

Consider $(a,b)\in\mathbb{R}^2$, $a\ne b$, then $\phi_{(a,b)}$ is the linear transformation defined by

$$\phi_{(a,b)}(x)=\frac{x-a}{b-a}.$$

Note that the inverse mapping $\phi^{-1}_{(a,b)}$ of $\phi_{(a,b)}$ is given by $\phi^{-1}_{(a,b)}(x)=a+(b-a)x$.□

Theorem 7. Consider a nilpotent t-norm T with normed additive generator f. Let g, h, $v \in \mathcal{S}$, such that b_g, b_h, $b_v < \infty$ and $f \circ v \circ \phi_v^{-1}(x) = f \circ g \circ \phi_g^{-1}(x) = f \circ h \circ \phi_h^{-1}(x) = 1 - (1-x)^p$ on the interval $(0,1)$ for some $p \in (1, \infty)$, where $\phi_v = \phi_{(a_v, b_v)}$ and similarly for functions g, h. Then $g * v = h * v \Rightarrow g = h$.

Proof. Following [8], under requirements of the theorem,

$$f \circ (g * v) \circ \phi_{g*v}^{-1}(x) = 1 - (1-x)^p,$$

where $b_{g*v} = b_g + b_v$ and

$$(b_{g*v} - a_{g*v})^{\frac{1}{p}-1} = (b_g - a_g)^{\frac{1}{p}-1} + (b_v - a_v)^{\frac{1}{p}-1}.$$

Similarly,

$$f \circ (h * v) \circ \phi_{h*v}^{-1}(x) = 1 - (1-x)^p,$$

where $b_{h*v} = b_h + b_v$ and

$$(b_{h*v} - a_{h*v})^{\frac{1}{p}-1} = (b_h - a_h)^{\frac{1}{p}-1} + (b_v - a_v)^{\frac{1}{p}-1}.$$

Now, it is evident that $g * v = h * v$ if and only if $a_g = a_h$, $b_g = b_h$, i.e., $g = h$.□

5. THE CANCELLATION LAW FOR PSEUDO-CONVOLUTION BASED ON $([0,1], \vee, T)$, WHERE T IS A CONTINUOUS t-NORM

5.1. Ordinal sums of t-norms

Definition 5. Consider a family $(T_k)_{k \in K}$ of t-norms and a family $(]\alpha_k, \beta_k[)_{k \in K}$ of pairwise disjoint open non-degenerate subintervals of $[0,1]$. The $[0,1]^2 \to [0,1]$ mapping T defined by

$$T(x,y) = \begin{cases} \phi_k^{-1}\left(T_k\left(\phi_k(x), \phi_k(y)\right)\right), & \text{if } (x,y) \in [\alpha_k, \beta_k]^2 \\ T_M(x,y), & \text{elsewhere,} \end{cases}$$

where $\phi_k = \phi_{(\alpha_k, \beta_k)}$, is a t-norm. T is called the ordinal sum of the summands $\langle \alpha_k, \beta_k, T_k \rangle$, and is denoted by $T \equiv (\langle \alpha_k, \beta_k, T_k \rangle \mid k \in K)$.

Note that in the foregoing proposition the case of an empty index set is also allowed, and obviously leads to the minimum operator T_M. The notion 'ordinal sum' has led to the following important characterization of continuous t-norms.

Theorem 8. A $[0,1]^2 \to [0,1]$ mapping T is a continuous t-norm if and only if it is an ordinal sum of continuous Archimedean t-norms.

5.2. Cancellation law for pseudo-convolution

Theorem 8 and the results from [1] allow to transform the cancellation law for pseudo-convolution based on a continuous t-norm T to the cases discussed in the previous sections.

Definition 6. Consider a real function g and $(a,b) \in [0,1]^2$, $a < b$.

(i) The function $g^{[a,b]}$ is defined as

$$g^{[a,b]} = /\phi_{(a,b)} \circ g/\,,$$

i. e. $g^{[a,b]}(x) = \Big/\frac{g(x)-a}{b-a}\Big/$, where $/x/ = \min\{\max\{0,x\},1\}$

(ii) The function $g_{[a,b]}$ is defined by

$$g_{[a,b]}(x) = \begin{cases} \phi^{-1}_{(a,b)}(g(x)), & \text{if } g(x) > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Theorem 9. Consider an ordinal sum $T \equiv (\langle a_i, b_i, T_i\rangle \mid i \in I)$ written in such a way that $\bigcup_{i\in I}[a_i,b_i] = [0,1]$, and functions $g,\ h \in \mathcal{S}$, then the pseudo-convolution based on the semiring $([0,1],\vee,T)$ is given by

$$g * h(x) = \sup_{i\in I}\left(g^{[a_i,b_i]} *_{T_i} h^{[a_i,b_i]}\right)_{[a_i,b_i]}(x),$$

where $*_{T_i}$ is pseudo-convolution based on semiring $([0,1],\vee,T_i)$.

Theorem 10. Let T be a continuous t-norm represented as an ordinal sum of Archimedean continuous t-norms, $T \equiv (\langle a_i, b_i, T_i\rangle \mid i \in I)$ and let $g,\ h,\ v \in \mathcal{S}$. Then cancellation law for pseudo-convolution based on the semiring $([0,1],\vee,T)$ is valid iff for $\forall\, i \in I$ holds

$$g^{[a_i,b_i]} *_{T_i} v^{[a_i,b_i]} = h^{[a_i,b_i]} *_{T_i} v^{[a_i,b_i]} \Rightarrow g^{[a_i,b_i]} = h^{[a_i,b_i]}.$$

Example 3. Consider the continuous t-norm $T = \{\langle 0, \frac{1}{2}\rangle, T_P\}$ and $g,\ h,\ v \in \mathcal{S}$. Let $-\ln v(x)$ be strictly convex on the interval $[a_v, v^{-1}(\frac{1}{2})]$ and $-\ln g(x)$ and $-\ln h(x)$ be convex on $[a_g, g^{-1}(\frac{1}{2})]$ and $[a_h, h^{-1}(\frac{1}{2})]$ respectively. Then $g * v = h * v \Rightarrow g = h$.

6. CONCLUSION

We have discussed the cancellation law for pseudo-convolutions based on triangular norms. While for the case of T_M the cancellation law is valid without special requirements, in all other cases it holds only under special restrictions. Note that T-based pseudo-convolutions acting on (continuous) distribution functions are special triangle functions, see e. g. [4, Chapter 9], and thus our results provide a partial answer to an open problem of V. Höhle posed in [3, Problem 13]. As a continuation of our work, we aim to discuss the cancellation law for another types of triangle functions.

ACKNOWLEDGEMENT

The work on this contribution was supported by Science and Technology Assistance Agency under the contract No. APVT-20-046402 and by the grant VEGA 1/0273/03.

(Received June 15, 2004.)

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