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AGGREGATIONS PRESERVING CLASSES OF FUZZY RELATIONS[1]

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We consider aggregations of fuzzy relations using means in [0,1] (especially: minimum, maximum and quasilinear mean). After recalling fundamental properties of fuzzy relations we examine means, which preserve reflexivity, symmetry, connectedness and transitivity of fuzzy relations. Conversely, some properties of aggregated relations can be inferred from properties of aggregation results. Results of the paper are completed by suitable examples and counter-examples, which is summarized in a special table at the end of the paper.

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1. INTRODUCTION

Aggregations of relations are important in group choice theory (cf. [11]) and multiple-criteria decision making (cf. [19]). Formally, instead of crisp relations we aggregate their characteristic functions. However, the aggregated result appears to be a fuzzy relation. Therefore, the most fruitful approach begins with fuzzy relations (cf. [12, 15] or [17]). In particular, internal unary and binary operations in classes of fuzzy relations were considered (cf. [7]).

We consider fundamental properties of fuzzy relations during aggregations of finite families of these relations. As aggregation functions we use classical means. More general aggregations were considered in [4]. Firstly, we recall basic definitions and examples of means (Section 2). Next, we complete the definitions of commonly used properties of fuzzy relations in the strongest and the weakest form (Section 3). Finally, we examine families of these properties such as: reflexivity (Section 4), symmetry (Section 5), connectedness (Section 6), and transitivity (Sections 7 and 8).

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2. AGGREGATIONS OF FUZZY RELATIONS

Means are considered as sequences of functions (cf. Kolmogorov [14]). We examine here the basic model of aggregation by means. After Cauchy [5], means $M : \mathbb{R}^n \to \mathbb{R}$ are characterized by the inequalities

$$\mathop{\forall}_{t_1,\ldots,t_n \in \mathbb{R}} \min(t_1, \ldots, t_n) \leq M(t_1, \ldots, t_n) \leq \max(t_1, \ldots, t_n). \tag{1}$$

In particular, such operations are idempotent, i. e.

$$\mathop{\forall}_{u \in \mathbb{R}} M(u, \ldots, u) = u. \tag{2}$$

Conversely, we have

Lemma 1. (cf. [12], Proposition 5.1) If operation $M : \mathbb{R}^n \to \mathbb{R}$ is increasing, i. e. fulfils

$$\mathop{\forall}_{s,t \in \mathbb{R}^n} (s \leqslant t \Rightarrow M(s) \leqslant M(t)), \tag{3}$$

then fulfils (1), iff it is idempotent.

We restrict ourselves to increasing aggregations, so we use the following

Definition 1. Let $n \geqslant 2$. An operation $M : \mathbb{R}^n \to \mathbb{R}$ is a mean, if it is idempotent and increasing (cf. (2), (3)).

From the property (1) we can see that means can be restricted to any interval. Our domain of interest is the interval $[0, 1]$. The bounds in (1) give simple examples of means: $M = \min$ and $M = \max$. We quote here other important examples of means.

Example 1. Let $\varphi : [0, 1] \to [0, 1]$ be an increasing bijection, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$. We consider weights $w_1, \ldots, w_n \in [0, 1]$ and index set K, where

$$\sum_{k=1}^{n} w_k = 1, \quad K = \{k : w_k > 0\}. \tag{4}$$

The quasilinear mean (cf. Aczél [1], p. 287) has the form

$$M(t_1, \ldots, t_n) = \varphi^{-1}\left(\sum_{k=1}^{n} w_k \varphi(t_k)\right) = \varphi^{-1}\left(\sum_{k \in K} w_k \varphi(t_k)\right). \tag{5}$$

As a particular case we get the quasi-arithmetic mean (cf. Aczél [1], p. 281)

$$M(t_1, \ldots, t_n) = \varphi^{-1}\left(\frac{1}{n} \sum_{k=1}^{n} \varphi(t_k)\right). \tag{6}$$

In the case $\varphi = id$ we get the weighted average in (5):

$$M(t_1, \ldots, t_n) = \sum_{k=1}^{n} w_k t_k = \sum_{k \in K} w_k t_k, \tag{7}$$

and the arithmetic mean in (6):

$$M(t_1, \ldots, t_n) = \frac{1}{n} \sum_{k=1}^{n} t_k. \tag{8}$$

Definition 2. (Zadeh [25]) Let $X \neq \emptyset$. A binary fuzzy relation in X is an arbitrary function $R : X \times X \to [0, 1]$. The family of all binary fuzzy relations in X is denoted by $FR(X)$.

Since in the sequel all considered fuzzy relations are binary, then further we shall omit this specification.

Definition 3. (Ovchinnikov [16]) Let $n \in \mathbb{N}$, $M : [0, 1]^n \to [0, 1]$ be an arbitrary function and $R_1, \ldots, R_n \in FR(X)$. A fuzzy relation $R \in FR(X)$ defined by

$$R(x, y) = M(R_1(x, y), \ldots, R_n(x, y)),\ x, y \in X \tag{9}$$

is called an aggregation of given fuzzy relations $R_1, \ldots, R_n$ (pointwise aggregation).

We shall examine the properties of the relation R defined by (9) under suitable assumptions of the aggregation function M and the involved fuzzy relations $R_1, \ldots, R_n$. We shall concentrate on the case, where the aggregation M is a mean, especially: minimum, maximum and the quasilinear mean (5) with special cases listed in (6), (7) and (8).

Definition 4. Let P denote a property of fuzzy relations. We say that an aggregation function M preserves the property P, if the relation R defined by (9) has the property P for arbitrary $R_1, \ldots, R_n \in FR(X)$ fulfilling the property P.

For example, any projection function

$$P_k(t_1, \ldots, t_n) = t_k, \quad t_1, \ldots, t_n \in [0, 1],\ k = 1, \ldots, n \tag{10}$$

preserves arbitrary properties of fuzzy relations $R_1, \ldots, R_n \in FR(X)$, because in the formula (9) we get $R = R_k$ for $M = P_k$, $k = 1, \ldots, n$.

We will consider the question which aggregation functions preserve fixed properties of the underlying fuzzy relations $R_1, \ldots, R_n$ during the aggregation process (sufficiency condition). On the other hand, we will also ask which properties of the relation R defined by (9) are necessary for the relations $R_1, \ldots, R_n$ depending on the involved aggregation function (necessity condition).

This second problem have not been considered in previous papers. Partial answers to the first question can be found, for example in [7, 8, 19] and [22]. However, our results take into account new properties of fuzzy relations (weak ones) and the methods of proof are unified.

3. CLASSES OF FUZZY RELATIONS

Now, we recall fuzzy versions of known relation properties.

Definition 5. (cf. Drewniak [6]) A fuzzy relation $R \in FR(X)$ is called

- reflexive, if $\forall_{x \in X} R(x,x) = 1,$ (11)
- weakly reflexive, if $\forall_{x \in X} R(x,x) > 0,$ (12)
- irreflexive, if $\forall_{x \in X} R(x,x) = 0,$ (13)
- weakly irreflexive, if $\forall_{x \in X} R(x,x) < 1,$ (14)
- symmetric, if $\forall_{x,y \in X} R(y,x) = R(x,y),$ (15)
- weakly symmetric, if $\forall_{x,y \in X} R(x,y) = 1 \Rightarrow R(y,x) = 1,$ (16)
- semi-symmetric, if $\forall_{x,y \in X} R(x,y) > 0 \Rightarrow R(y,x) > 0,$ (17)
- asymmetric, if $\forall_{x,y \in X} R(x,y) > 0 \Rightarrow R(y,x) = 0,$ (18)
- weakly asymmetric, if $\forall_{x,y \in X} R(x,y) = 1 \Rightarrow R(y,x) < 1,$ (19)
- antisymmetric, if $\forall_{x,y \in X, x \neq y} R(x,y) > 0 \Rightarrow R(y,x) = 0,$ (20)
- weakly antisymmetric, if $\forall_{x,y \in X, x \neq y} R(x,y) = 1 \Rightarrow R(y,x) < 1,$ (21)
- connected, if $\forall_{x,y \in X, x \neq y} R(x,y) < 1 \Rightarrow R(y,x) = 1,$ (22)
- weakly connected, if $\forall_{x,y \in X, x \neq y} R(x,y) = 0 \Rightarrow R(y,x) > 0,$ (23)
- totally connected, if $\forall_{x,y \in X} R(x,y) < 1 \Rightarrow R(y,x) = 1,$ (24)
- weakly totally connected, if $\forall_{x,y \in X} R(x,y) = 0 \Rightarrow R(y,x) > 0,$ (25)
- transitive, if $\forall_{x,y,z \in X} R(x,z) \geq \min(R(x,y), R(y,z)),$ (26)
- weakly transitive, if $\forall_{x,y,z \in X} \min(R(x,y), R(y,z)) > 0 \Rightarrow R(x,z) > 0,$ (27)
- semi-transitive, if $\forall_{x,y,z \in X} \min(R(x,y), R(y,z)) = 1 \Rightarrow R(x,z) = 1.$ (28)

The listed properties can be combined together in order to obtain new classes of fuzzy relations (cf. [3], p. 49). The commonly used notions of relation properties are based on monographs Roubens, Vincke [21] and Schreider [23]. However, the expression "complete relation" is changed for "connected relation", because "complete" also has another meaning in order structures (cf. [2], Chapter V).

It is evident that the characteristic function of a crisp binary relation with suitable property fulfils the corresponding condition (then weak and strict properties coincide). In order to compare the above properties with respective ones of crisp relations we consider the greatest relation and the least relation connected with the fuzzy one.

Following Zadeh [25], we put

Definition 6. A support of a fuzzy relation R is the crisp relation

$$\operatorname{Supp} R = \{(x, y) : R(x, y) > 0\} \subset X \times X.$$

A core of a fuzzy relation R is the crisp relation

$$\operatorname{Core} R = \{(x, y) : R(x, y) = 1\} \subset X \times X.$$

Directly from the above definitions we get the following characterization

Theorem 1. A fuzzy relation R is reflexive, weakly irreflexive, weakly symmetric, weakly asymmetric, weakly antisymmetric, connected, totally connected or semi-transitive, respectively, iff the relation $\operatorname{Core} R$ is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, connected, totally connected or transitive, respectively.

A fuzzy relation R is weakly reflexive, irreflexive, semi-symmetric, asymmetric, antisymmetric, weakly connected, weakly totally connected or weakly transitive, respectively, iff the relation $\operatorname{Supp} R$ is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, connected, totally connected or transitive, respectively.

P r o o f. In order to avoid a repetition of crisp relations classifications we consider as an example only the case of weak transitivity (see (27)). Let $R \in FR(X)$. Relation $\operatorname{Supp} R$ is transitive, iff

$$\begin{aligned} & \forall_{x,y,z \in X} \ (x, y), (y, z) \in \operatorname{Supp} R \Rightarrow (x, z) \in \operatorname{Supp} R \\ \Leftrightarrow \quad & \forall_{x,y,z \in X} \ R(x, y) > 0, R(y, z) > 0 \Rightarrow R(x, z) > 0 \\ \Leftrightarrow \quad & \forall_{x,y,z \in X} \ \min(R(x, y), R(y, z)) > 0 \Rightarrow R(x, z) > 0, \end{aligned}$$

which means that R is weakly transitive. The remaining cases can be proven in a similar way. □

As we can see, the above-defined weak properties of fuzzy relations are the weakest ones. Other weak properties of fuzzy relations were considered in [19, 20]. These properties were rather "middle" (dependent on the membership value 0.5).

The way of the above characterization fails in the case of symmetry and transitivity of fuzzy relations. It can be checked by a simple example (cf. however [6], Theorems 5.1 and 5.2).

Example 2. Let card $X = 3$. We describe fuzzy relations and respective crisp relations by 3×3 square matrices ($R = [r_{i,k}]$). We have

$$R = \begin{bmatrix} 1 & 1 & 0.2 \\ 1 & 1 & 0.6 \\ 0.6 & 0.2 & 1 \end{bmatrix}, \qquad \operatorname{Supp} R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \operatorname{Core} R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It can be directly checked that $\operatorname{Supp} R$ and $\operatorname{Core} R$ are characteristic matrices of symmetric and transitive relations, while the fuzzy relation R is neither symmetric nor transitive, because

$$r_{1,3} = 0.2 \neq 0.6 = r_{3,1}, \quad r_{1,3} = 0.2 < 0.6 = \min(r_{1,2}, r_{2,3}).$$

4. REFLEXIVITY PROPERTIES

First, we examine the properties related to reflexivity (cf. (11) – (14)) of the relation R defined by (9).

Theorem 2. Every mean preserves reflexivity, weak reflexivity, irreflexivity and weak irreflexivity of fuzzy relations.

Proof. Let M be an arbitrary mean of n variables (cf. Definition 1). If $R_1, \ldots, R_n$ are reflexive (cf. (11)), then we can conclude by (2), (9) that

$$R(x,x) = M(R_1(x,x), \ldots, R_n(x,x)) = M(1, \ldots, 1) = 1, \ x \in X,$$

expressing that also R is reflexive.

If $R_1, \ldots, R_n$ are weakly reflexive (cf. (12)), then we can conclude by (1), (9) that

$$R(x,x) = M(R_1(x,x), \ldots, R_n(x,x)) \geq \min_{1 \leqslant k \leqslant n} R_k(x,x) > 0, \ x \in X,$$

so R is also weakly reflexive.

If $R_1, \ldots, R_n$ are irreflexive (cf. (13)), then we can conclude by (2) that

$$R(x,x) = M(R_1(x,x), \ldots, R_n(x,x)) = M(0, \ldots, 0) = 0, \ x \in X,$$

thus R is also irreflexive.

If $R_1, \ldots, R_n$ are weakly irreflexive (cf. (14)), then we can conclude by (1) that

$$R(x,x) = M(R_1(x,x), \ldots, R_n(x,x)) \leq \max_{1 \leq k \leq n} R_k(x,x) < 1, \ x \in X,$$

and R is weakly irreflexive, which finishes the proof. □

In general, the reflexivity properties of R do not affect on the reflexivity properties of the aggregated relations. To show it, we provide the following example (even for crisp relations).

Example 3. Let $n = 2$ and card $X = 2$. We have

$$R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$

$$T = \max(R, S) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad U = \min(R, S) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where relations R and S have no properties

$$P \in \{\text{reflexive, weakly reflexive, irreflexive, weakly irreflexive}\},$$

while aggregation relation T is reflexive (weakly reflexive), and relation U is irreflexive (weakly irreflexive).

We shall obtain the converse results under additional assumptions. At first, we prove useful interdependences.

Lemma 2. If $\varphi : [0,1] \to [0,1]$ is an increasing bijection, then for every $s \in [0,1]$

$$\varphi(s) = 0 \Leftrightarrow s = 0, \qquad \varphi(s) = 1 \Leftrightarrow s = 1, \tag{29}$$

$$\varphi(s) > 0 \Leftrightarrow s > 0, \qquad \varphi(s) < 1 \Leftrightarrow s < 1. \tag{30}$$

If M is a quasilinear mean, then necessarily $K \neq \emptyset$ and for all $t_1, \dots, t_n \in [0,1]$

$$M(t_1, \dots, t_n) = 0 \Leftrightarrow \underset{k \in K}{\forall} t_k = 0, \qquad M(t_1, \dots, t_n) = 1 \Leftrightarrow \underset{k \in K}{\forall} t_k = 1, \tag{31}$$

$$M(t_1, \dots, t_n) > 0 \Leftrightarrow \underset{k \in K}{\exists} t_k > 0, \qquad M(t_1, \dots, t_n) < 1 \Leftrightarrow \underset{k \in K}{\exists} t_k < 1. \tag{32}$$

Proof. Let $s \in [0,1]$. Conditions (29), (30) are immediate consequences of inequalities

$$0 \le \varphi(s) \le 1, \qquad 0 \le \varphi^{-1}(s) \le 1.$$

Now, let w fulfil (4). By (29) we get

$$\begin{aligned}
M(t_1, \dots, t_n) = 0 \quad &\Leftrightarrow \quad \sum_{k=1}^{n} w_k \varphi(t_k) = 0 \Leftrightarrow \sum_{k \in K} w_k \varphi(t_k) = 0 \\
&\Leftrightarrow \quad \underset{k \in K}{\forall} \varphi(t_k) = 0 \Leftrightarrow \underset{k \in K}{\forall} t_k = 0, \\
M(t_1, \dots, t_n) = 1 \quad &\Leftrightarrow \quad \sum_{k=1}^{n} w_k \varphi(t_k) = 1 \Leftrightarrow \sum_{k=1}^{n} w_k - \sum_{k=1}^{n} w_k \varphi(t_k) = 0 \\
&\Leftrightarrow \quad \sum_{k=1}^{n} w_k \left(1 - \varphi(t_k)\right) = 0 \Leftrightarrow \underset{k \in K}{\forall} \varphi(t_k) = 1 \Leftrightarrow \underset{k \in K}{\forall} t_k = 1,
\end{aligned}$$

which proves (31). Similarly, by (30) we get (32). □

Theorem 3. If the fuzzy relation R defined by (9) through some quasilinear mean M is reflexive (irreflexive), then fuzzy relations R_k for $k \in K$ are reflexive (irreflexive).

Proof. Let M be an arbitrary quasilinear mean (cf. (4), (5)), $x \in X$, and $t_k = R_k(x, x)$ for $k = 1, \ldots, n$. We shall apply Lemma 2. First, observe that in virtue of (29), (30) we can omit φ and φ^{-1} in our considerations.

If the relation R defined by (9) is reflexive, then directly from (31) we see that $R_k(x, x) = 1$ for $k \in K$, which proves that relations R_k are reflexive. In the same way we get the proof in the case of irreflexivity. □

Weak properties are less useful for the determination of converse dependency.

Example 4. Let $n = 2$ and card $X = 2$. We have

$$R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad T = \frac{R+S}{2} = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix},$$

where relation T is weakly reflexive and weakly irreflexive, while these properties fail in relations R and S.

Remark 1. Let $R_1, \ldots, R_n \in FR(X)$. Fuzzy relations greater than the reflexive one are also reflexive. Similarly, fuzzy relations less than the irreflexive one are also irreflexive. Therefore, if the fuzzy relation $\min(R_1, \ldots, R_n)$ is reflexive (weakly reflexive), then relations $R_1, \ldots, R_n$ are reflexive (weakly reflexive), too. If the fuzzy relation $\max(R_1, \ldots, R_n)$ is irreflexive (weakly irreflexive), then relations $R_1, \ldots, R_n$ are irreflexive (weakly irreflexive), too.

Examples 3 and 4 show means without similar results.

5. SYMMETRY PROPERTIES

Now, we examine the properties (15) – (21) related to the symmetry, asymmetry and antisymmetry of aggregated relations.

Theorem 4. Every mean preserves symmetry of fuzzy relations.

Proof. Let M be an arbitrary mean (cf. Definition 1) and $x, y \in X$. If $R_1, \ldots, R_n$ are symmetric (cf. (15)), then

$$R(x, y) = M(R_1(x, y), \ldots, R_n(x, y)) = M(R_1(y, x), \ldots, R_n(y, x)) = R(y, x),$$

and relation R is also symmetric, which finishes the proof. □

Theorem 5. Every quasilinear mean preserves weak symmetry, semi-symmetry, weak asymmetry, and weak antisymmetry of fuzzy relations.

Proof. Let M be a quasilinear mean, $R_1, \dots, R_n \in FR(X)$, and R be the fuzzy relation defined by (9). Using Lemma 2 we can omit φ and φ^{-1} in our considerations. Let $x,\ y \in X$. We denote $t_k = R_k(x,y)$, $u_k = R_k(y,x)$ for $k = 1, \dots, n$ using dependences from Lemma 2.

If $R_1, \dots, R_n$ are weakly symmetric (cf. (16)), then we get by (31) that

$$R(x,y) = M(t_1, \dots, t_n) = 1 \Leftrightarrow \underset{k\in K}{\forall} R_k(x,y) = t_k = 1 \Rightarrow \underset{k\in K}{\forall} u_k = R_k(y,x) = 1$$
$$\Leftrightarrow M(u_1, \dots, u_n) = R(y,x) = 1,$$

i. e. R is weakly symmetric.

If $R_1, \dots, R_n$ are semi-symmetric (cf. (17)), then we obtain by (32) that

$$\begin{aligned} R(x,y) = M(t_1, \dots, t_n) > 0 \quad &\Leftrightarrow \quad \underset{k\in K}{\exists} R_k(x,y) = t_k > 0 \Rightarrow \underset{k\in K}{\exists} u_k = R_k(y,x) > 0 \\ &\Leftrightarrow \quad M(u_1, \dots, u_n) = R(y,x) > 0, \end{aligned}$$

so R is semi-symmetric.

If $R_1, \dots, R_n$ are weakly asymmetric (cf. (19)), then we get by (31), (32) that

$$\begin{aligned} R(x,y) = M(t_1, \dots, t_n) = 1 \quad &\Leftrightarrow \quad \underset{k\in K}{\forall} R_k(x,y) = t_k = 1 \\ &\Rightarrow \underset{k\in K}{\forall} u_k = R_k(y,x) < 1 \Leftrightarrow \underset{k\in K}{\exists} u_k < 1 \\ &\Leftrightarrow \quad M(u_1, \dots, u_n) = R(y,x) < 1, \end{aligned}$$

which proves that R is weakly asymmetric and the proof for weak antisymmetry is equivalent for $x \neq y$, which finishes the proof. □

Theorem 6. Means $M = \min$ and $M = \max$ preserve weak symmetry and semi-symmetry of fuzzy relations.

Proof. We consider the case of weak symmetry. Let $R_1, \dots, R_n$ be weakly symmetric (cf. (16)) and $x, y \in X$. If $R(x,y) = \min(R_1(x,y), \dots, R_n(x,y)) = 1$, then $R_k(x,y) = 1, k = 1, \dots, n$. Thus $R_k(y,x) = 1, k = 1, \dots, n$, which means that

$$R(y,x) = \min(R_1(y,x), \dots, R_n(y,x)) = 1,$$

i. e. R is weakly symmetric.

If $R(x,y) = \max(R_1(x,y), \dots, R_n(x,y)) = 1$, then there exists an index k for which $R_k(x,y) = 1$. So $R_k(y,x) = 1, k = 1, \dots, n$ and we get

$$R(y,x) = \max(R_1(y,x), \dots, R_n(y,x)) = 1,$$

i. e. R is weakly symmetric. Therefore, both these means preserve weak symmetry of fuzzy relations. In the case of semi-symmetry the proof is similar. □

There exist means which do not preserve weak symmetry or semi-symmetry.

Example 5. Let $n = 2$ and card $X = 2$. We consider means of the form

$$M(x,y) = \begin{cases} \min(x,y), & 2y < 1-x \\ \max(x,y), & 2y \geqslant 1-x, \end{cases} \quad N(x,y) = \begin{cases} \min(x,y), & y < 2(1-x) \\ \max(x,y), & y \geqslant 2(1-x) \end{cases}$$

for $x, y \in [0,1]$, and matrices of fuzzy relations

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0.6 \\ 0.3 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$U = M(R,S) = \begin{bmatrix} 1 & 0.6 \\ 0 & 1 \end{bmatrix}, \quad V = N(S,T) = \begin{bmatrix} 0 & 1 \\ 0.3 & 0 \end{bmatrix}.$$

Relations R, S, T are weakly symmetric and semi-symmetric, while fuzzy relation U is not semi-symmetric and relation V is not weakly symmetric.

Converse results to the above theorems are not possible in the case of symmetry, weak symmetry and semi-symmetry. If the aggregation result has one of these properties, then we cannot infer such properties of aggregated relations.

Example 6. Let $n = 2$, card $X = 2$. Fuzzy relations

$$R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

have no properties

$$P \in \{\text{symmetry, weak symmetry, semi-symmetry}\},$$

although their averages

$$\max(R,S) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \min(R,S) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \frac{R+S}{2} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

have such properties.

Remark 2. Fuzzy relations less than the asymmetric (antisymmetric) one are also asymmetric (antisymmetric). Thus, if one of the fuzzy relations $R_1, \ldots, R_n$ is asymmetric, weakly asymmetric, antisymmetric, weakly antisymmetric then the relation $R = \min(R_1, \ldots, R_n)$ has also the respective property (cf. [7] in the case $n = 2$). If the fuzzy relation $\max(R_1, \ldots, R_n)$ is asymmetric, weakly asymmetric, antisymmetric, weakly antisymmetric then relations $R_1, \ldots, R_n$ have the respective property.

The cases omitted above are illustrated by examples.

Example 7. Putting $n = 2$ and card $X = 2$ we consider matrices of fuzzy relations R and S:

$$R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Relations R and S are asymmetric, weakly asymmetric, antisymmetric and weakly antisymmetric. However for their aggregations

$$T = \max(R, S) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad U = \frac{R+S}{2} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix},$$

relation T is not asymmetric (weakly asymmetric, antisymmetric, weakly antisymmetric) and relation U is not asymmetric (antisymmetric).

Conversely, let $n = 2$, card $X = 3$ and

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \min(P, Q) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Relation $\min(P, Q)$ is asymmetric, weakly asymmetric, antisymmetric and weakly antisymmetric, while relations P and Q do not have such properties.

Theorem 7. If the fuzzy relation R defined by (9) through some quasilinear mean M is asymmetric (antisymmetric), then fuzzy relations R_k, $k \in K$ are asymmetric (antisymmetric).

P r o o f. Let M be a quasilinear mean (cf. (4), (5)), x, $y \in X$, and $t_k = R_k(x, y)$, $u_k = R_k(y, x)$, $k = 1, \ldots, n$. As previously we shall apply Lemma 2. Let $k \in K$. If $t_k > 0$, then we can conclude by (32) that

$$R(x, y) = M(t_1, \ldots, t_n) > 0 \Rightarrow M(u_1, \ldots, u_n) = R(y, x) = 0.$$

Now, by (31) $u_k = R_k(y, x) = 0$, which proves that relation R_k is asymmetric. In the case of antisymmetric relations the proof is similar. □

Similar result is not possible for weak properties.

Example 8. Let $n = 2$, card $X = 2$,

$$R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad T = \frac{R+S}{2} = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0.5 \end{bmatrix}.$$

Relation T is weakly asymmetric and weakly antisymmetric, while relation S does not have such properties.

6. CONNECTEDNESS PROPERTIES

Next, we examine the properties (22) – (25) of the relation R defined by (9).

Theorem 8. Every quasilinear mean preserves weak connectedness and weak total connectedness of fuzzy relations.

Proof. Let M be a quasilinear mean, $R_1, \ldots, R_n \in FR(X)$, relation R be defined by (9), and $x, y \in X$, $x \neq y$. Putting $t_k = R_k(x, y)$, $u_k = R_k(y, x)$, $k = 1, \ldots, n$ we use dependences from Lemma 2. If $R_1, \ldots, R_n$ are weakly connected (cf. (23)), then we get by (31), (32) that

$$M(t_1, \ldots, t_n) = R(x, y) = 0 \Leftrightarrow \underset{k\in K}{\forall} R_k(x, y) = t_k = 0 \Rightarrow \underset{k\in K}{\forall} u_k = R_k(y, x) > 0$$

$$\Rightarrow \underset{k\in K}{\exists} u_k > 0 \Leftrightarrow R(y, x) = M(u_1, \ldots, u_n) > 0,$$

and therefore R is weakly connected. In the same way we get the result for weak total connectedness, which finishes the proof. □

Remark 3. Fuzzy relations greater than the connected one are also connected. If one of the fuzzy relations $R_1, \ldots, R_n$ is connected, weakly connected, totally connected or weakly totally connected, then the fuzzy relation $R = \max(R_1, \ldots, R_n)$ has also the respective property (cf. [7] in the case $n = 2$). If the fuzzy relation $\min(R_1, \ldots, R_n)$ is connected, weakly connected, totally connected or weakly totally connected, then relations $R_1, \ldots, R_n$ have the respective property.

Certain methods of aggregation lose the connectedness property of aggregated relations.

Example 9. Let $n = 2$ and card $X = 2$,

$$R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \min(R, S) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where relations R, S are connected, weakly connected, totally connected and weakly totally connected, while $\min(R, S)$ does not have such properties.

Conversely, let $n = 2$ and card $X = 3$, and

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \max(P, Q) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Relation $\max(P, Q)$ is connected, weakly connected, totally connected and weakly totally connected, while relations P and Q do not have such properties.

Theorem 9. If the fuzzy relation defined by (9) through some quasilinear mean M is connected (totally connected), then fuzzy relations R_k for $k \in K$ are connected (totally connected).

Proof. Let M be a quasilinear mean (cf. (4), (5)), $x,\ y \in X,\ x \neq y$, and $t_k = R_k(x,y)$, $u_k = R_k(y,x)$, $k = 1,\dots,n$. As previously we shall apply Lemma 2. Let $k \in K$. If $t_k < 1$, then by (32),

$$R(x,y) = M(t_1,\dots,t_n) < 1 \Rightarrow M(u_1,\dots,u_n) = R(y,x) = 1.$$

Now, by (31) $u_k = R_k(y,x) = 1$, which proves that relation R_k is connected. In the case of totally connected relations the proof is similar. □

Example 10. For fuzzy relations from Example 9 we have

$$T = \frac{R+S}{2} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \quad U = \frac{P+Q}{2} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix}.$$

Relations R and S from Example 8 are connected and totally connected, while fuzzy relation T does not have such properties. Conversely, fuzzy relation U is weakly connected (weakly totally connected), while relations P and Q from Example 9 do not have such properties.

7. TRANSITIVITY PROPERTIES

Finally, we examine the properties (26) – (28) of the relation R defined by (9). In the case of transitivity property even the arithmetic mean does not preserve it.

Example 11. Let $n = 2$ and card $X = 2$. We have

$$R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \max(R,S) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \frac{R+S}{2} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix},$$

where relations R and S are transitive, weakly transitive and semi-transitive, while fuzzy relation $\max(R,S)$ does not have such properties and fuzzy relation $\frac{R+S}{2}$ is not transitive and weakly transitive.

Theorem 10. Every quasilinear mean preserves semi-transitivity of fuzzy relations.

Proof. Let M be a quasilinear mean (cf. (4), (5)), $R_1,\dots,R_n \in FR(X)$, R be defined by (9) and $x,y,z \in X$. Putting $t_k = R_k(x,y)$, $u_k = R_k(y,z)$, $v_k = R_k(x,z)$, $k = 1,\dots,n$ we use dependences from Lemma 2. If $R_1,\dots,R_n$ are semi-transitive (cf. (28)), then we get by (31) that

$$\begin{aligned}
&\min(R(x,y),R(y,z))=1\\
\Rightarrow\quad &(M(t_1,\dots,t_n)=R(x,y)=1,\ M(u_1,\dots,u_n)=R(y,z)=1)\\
\Rightarrow\quad &\underset{k\in K}{\forall}(R_k(x,y)=t_k=1,\ R_k(y,z)=u_k=1)\\
\Rightarrow\quad &\underset{k\in K}{\forall}\min(R_k(x,y),\ R_k(y,z))=1\\
\Rightarrow\quad &\underset{k\in K}{\forall} v_k=R_k(x,z)=1\\
\Rightarrow\quad &R(x,z)=M(v_1,\dots,v_n)=1.
\end{aligned}$$

This proves that fuzzy relation R is semi-transitive. □

Similarly as in [7] (case $n=2$) or in [12], Theorem 7.2 (transitivity) we obtain

Theorem 11. The mean $M=\min$ preserves transitivity, weak transitivity, and semi-transitivity.

Proof. We shall consider the case of semi-transitivity, and the proof in the remaining cases is similar. Let us consider semi-transitive fuzzy relations $R_1,\dots,R_n$, the relation $R=\min(R_1,\dots,R_n)$, and $x,y,z\in X$. If $\min(R(x,y),R(y,z))=1$, then $\min(R_k(x,y),R_k(y,z))=1$ and we get by (28) that $R_k(x,z)=1$, for $k=1,\dots,n$. This gives $R(x,z)=\min(R_1(x,z),\dots,R_n(x,z))=1$, i.e. relation R is semi-transitive. □

The above theorems complete the cases omitted in Example 11. We have no positive results of the converse problem for considered means. The result of aggregation may have greater collection of relation properties than aggregated relations.

Example 12. Let $n=2$, card $X=3$. For fuzzy relations described by matrices

$$R=\begin{bmatrix}0&1&1\\1&1&0\\0&0&1\end{bmatrix},\qquad S=\begin{bmatrix}1&0&0\\0&1&1\\1&1&0\end{bmatrix}$$

we have the following aggregations

$$\min(R,S)=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\ \max(R,S)=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix},\ \frac{R+S}{2}=\begin{bmatrix}0.5&0.5&0.5\\0.5&1&0.5\\0.5&0.5&0.5\end{bmatrix},$$

which are transitive, weakly transitive and semi-transitive, while relations R and S do not have such properties.

From the above consideration we can conclude that the transitivity property is too strong. Therefore, modified transitivity conditions are considered (a generalized transitivity).

8. GENERALIZED TRANSITIVITY

Diverse transitivity concepts were compared in [24]. We apply only one way of generalization by modifications of operation minimum in condition (26) (cf. Goguen [13] or Fodor, Roubens [12], Definition 2.12).

Definition 7. Let $\star : [0,1]^2 \to [0,1]$ be an arbitrary binary operation. A fuzzy relation R is called $\star$-transitive or weakly $\star$-transitive, if it respectively fulfils

$$\underset{x,y,z \in X}{\forall} R(x,z) \geq R(x,y) \star R(y,z), \tag{33}$$

$$\underset{x,y,z \in X}{\forall} R(x,y) \star R(y,z) > 0 \Rightarrow R(x,z) > 0. \tag{34}$$

Example 13. For constant binary operation $\star = 1$ there is exactly one $\star$-transitive fuzzy relation $R = 1$. Conversely, if $\star = 0$, then any fuzzy relation R is $\star$-transitive. Thus such modification provides a wide spectrum of $\star$-transitive relation families.

Directly from the above definition we see that

Lemma 3. If binary operations $\star$ and $\circ$ are comparable, $\star \leqslant \circ$, and a fuzzy relation R is $\circ$-transitive (weakly $\circ$-transitive), then it is $\star$-transitive (weakly $\star$-transitive).

Corollary 1. If a relation R is transitive (weakly transitive), then it is $\star$-transitive (weakly $\star$-transitive) for any binary operation $\star \leqslant \min$.

Roughly speaking, the situation with aggregation of $\star$-transitive relations is very similar to the case $\star = \min$. First of all, Theorem 11 can be generalized to the case of $\star$-transitivity.

Theorem 12. Let $\star : [0,1]^2 \to [0,1]$ be an increasing binary operation. The mean $M = \min$ preserves $\star$-transitivity and weak $\star$-transitivity.

Proof. Let $R_1, \ldots, R_n \in FR(X)$, $R = \min(R_1, \ldots, R_n)$, and $x, y, z \in X$. If $R_1, \ldots, R_n$ are $\star$-transitive (cf. (34)), then we obtain by monotonicity of operation $\star$ that

$$\min_{1 \leqslant k \leqslant n} R_k(x,y) \star \min_{1 \leqslant k \leqslant n} R_k(y,z) \leqslant R_i(x,y) \star R_i(y,z) \leqslant R_i(x,z), \quad i = 1, \ldots, n.$$

Thus

$$R(x,y) \star R(y,z) \leqslant \min_{1 \leqslant i \leqslant n} R_i(x,z) = R(x,z),$$

which proves $\star$-transitivity of R.

Similarly, if $R_1, \ldots, R_n$ are weakly $\star$-transitive (cf. (34)), and $R(x,y) \star R(y,z) > 0$, then we get by monotonicity of operation $\star$ that

$$0 < \min_{1 \leqslant k \leqslant n} R_k(x,y) \star \min_{1 \leqslant k \leqslant n} R_k(y,z) \leqslant R_i(x,y) \star R_i(y,z), \quad i = 1, \ldots, n,$$

and

$$\underset{1 \leqslant i \leqslant n}{\forall} R_i(x, z) > 0.$$

Therefore, $R(x,z) = \min(R_1(x,z), \ldots, R_n(x,z)) > 0$, which proves that relation R is weakly $\star$-transitive. □

Example 14. Let us consider fuzzy relations R, S from Example 11. If operation $\star$ fulfils the conjunction binary truth table:

$$0 \star 0 = 0 \star 1 = 1 \star 0 = 0, 1 \star 1 = 1, \tag{35}$$

then R and S are $\star$-transitive (weakly $\star$-transitive), but $\max(R, S)$ is not $\star$-transitive (weakly $\star$-transitive). If moreover $0.5 \star 0.5 > 0$, then $\frac{R+S}{2}$ is not $\star$-transitive (weakly $\star$-transitive).

Example 15. Let us consider fuzzy relations R, S from Example 12. If operation $\star$ fulfils (35), then $\max(R, S)$ and $\min(R, S)$ are $\star$-transitive (weakly $\star$-transitive), while R and S are not $\star$-transitive (weakly $\star$-transitive). If moreover $\star \leqslant \min$, then also $\frac{R+S}{2}$ is $\star$-transitive (weakly $\star$-transitive).

Now, we see that better results can be obtained for increasing 'conjunctive' binary operations $\star$ with zero divisors (in particular $0.5 \star 0.5 = 0$). The most important operation among them is denoted by T_L and called the Łukasiewicz multivalued conjunction:

$$T_L(x, y) = \max(0, x + y - 1), \ x, y \in [0, 1]. \tag{36}$$

At first, we need some auxiliary properties of maximum. By a direct verification we obtain

Lemma 4. For a, b, $c \in \mathbb{R}$ we have

$$\max(a, b) + \max(a, c) \geq \max(a, b + c). \tag{37}$$

Then, we obtain (cf. Peneva, Popchev [18], Saminger et al. [22] for T_L-transitivity)

Theorem 13. The weighted average (7) preserves T_L-transitivity and weak T_L-transitivity.

Proof. Let M be a weighted average (cf. (4), (7)), $R_1, \ldots, R_n \in FR(X)$, R be defined by (9) and $x, y, z \in X$. The case of T_L-transitivity was proved in [18] (cf. Section 2.1, Proposition 1). We consider here the case of weak transitivity. If $R_1, \ldots, R_n$ are weakly T_L-transitive, i. e.

$$\max(0, R_k(x, y) + R_k(y, z) - 1) > 0 \Rightarrow R_k(x, z) > 0, k = 1, \ldots, n,$$

then by using (32) (case $\varphi = Id$) and (37) we have

$$\begin{aligned}
& \max(0, R(x,y) + R(y,z) - 1) > 0 \\
\Leftrightarrow\quad & \max\left(0, \sum_{k=1}^{n} w_k R_k(x,y) + \sum_{k=1}^{n} w_k R_k(y,z) - \sum_{k=1}^{n} w_k\right) > 0 \\
\Leftrightarrow\quad & \max\left(0, \sum_{k=1}^{n} w_k (R_k(x,y) + R_k(y,z) - 1)\right) > 0 \\
\Leftrightarrow\quad & \sum_{k=1}^{n} w_k (R_k(x,y) + R_k(y,z) - 1) > 0 \\
\Leftrightarrow\quad & \underset{k\in K}{\exists}\ R_k(x,y) + R_k(y,z) - 1 > 0 \\
\Rightarrow\quad & \underset{k\in K}{\exists}\ R_k(x,z) > 0 \\
\Leftrightarrow\quad & R(x,z) = \sum_{k=1}^{n} w_k R_k(x,z) > 0,
\end{aligned}$$

so R is weakly T_L-transitive. □

As a direct consequence of the above theorem and Lemma 3 we obtain

Corollary 2. If $R_1, \ldots, R_n$ are $\star$-transitive (weakly $\star$-transitive), and $\star \geqslant T_L$, then for every weighted average M the aggregated relation defined by (9) is T_L-transitive (weakly T_L-transitive).

Similar results are not possible for the quasilinear mean (5) or for the quasi-arithmetic mean (6) with a bijection $\varphi \neq Id$.

Example 16. Let $n = 2$, card $X = 3$, $\varphi(x) = x^2$, $x \in [0,1]$. Using the operation T_L, relations S, T with matrices

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \tag{38}$$

are T_L-transitive (and weakly T_L-transitive). However, fuzzy relation $R = [r_{ik}]$,

$$r_{ik} = \sqrt{\frac{s_{ik}^2 + t_{ik}^2}{2}}, \quad i,k = 1,2,3, \quad R = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

is not T_L-transitive and even not weakly T_L-transitive, because

$$0 = r_{12} < \max(0, r_{13} + r_{32} - 1) = \sqrt{2} - 1.$$

The result of Theorem 13 is not valid for other binary operations $\star$.

Example 17. Let us observe that relations (38) are product-transitive (and weakly product-transitive) with ordinary product in $[0,1]$. However, fuzzy relation $R = \frac{S+T}{2}$ is not product-transitive and even weakly product-transitive, because

$$R = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad 0 = r_{12} < r_{13} r_{32} = \frac{1}{4}.$$

9. CONCLUDING REMARKS

As a summary of presented results we include the following Table.

Table. Comparison of means preserving fuzzy relations properties.

Class \ Mean		Arbitrary	Minimum	Arithmetic	Quasilinear	Maximum
Reflexive	$\Rightarrow$	Theorem 2				
	$\Leftarrow$		Remark 1		Theorem 3	Example 3
Weakly	$\Rightarrow$	Theorem 2				
reflexive	$\Leftarrow$		Remark 1	Example 4		Example 3
Irreflexive	$\Rightarrow$	Theorem 2				
	$\Leftarrow$		Example 3		Theorem 3	Remark 1
Weakly	$\Rightarrow$	Theorem 2				
irreflexive	$\Leftarrow$		Example 3	Example 4		Remark 1
Symmetric	$\Rightarrow$	Theorem 4				
	$\Leftarrow$		Example 6	Example 6		Example 6
Weakly	$\Rightarrow$	Example 5	Theorem 6		Theorem 5	Theorem 6
symmetric	$\Leftarrow$		Example 6	Example 6		Example 6
Semi-	$\Rightarrow$	Example 5	Theorem 6		Theorem 5	Theorem 6
symmetric	$\Leftarrow$		Example 6	Example 6		Example 6
Asymmetric	$\Rightarrow$		Remark 2	Example 7		Example 7
	$\Leftarrow$		Example 7		Theorem 7	Remark 2
Weakly	$\Rightarrow$		Remark 2		Theorem 5	Example 7
asymmetric	$\Leftarrow$		Example 7	Example 8		Remark 2
Anti	$\Rightarrow$		Remark 2	Example 7		Example 7
-symmetric	$\Leftarrow$		Example 7		Theorem 7	Remark 2
Weakly anti	$\Rightarrow$		Remark 2		Theorem 5	Example 7
-symmetric	$\Leftarrow$		Example 7	Example 8		Remark 2
Connected	$\Rightarrow$		Example 9	Example 10		Remark 3
	$\Leftarrow$		Remark 3		Theorem 12	Example 9
Weakly	$\Rightarrow$		Example 9		Theorem 8	Remark 3
connected	$\Leftarrow$		Remark 3	Example 10		Example 9
Totally	$\Rightarrow$		Example 9	Example 10		Remark 3
connected	$\Leftarrow$		Remark 3		Theorem 12	Example 9
Weakly	$\Rightarrow$		Example 9		Theorem 8	Remark 3
totally con.	$\Leftarrow$		Remark 3	Example 10		Example 9
Transitive	$\Rightarrow$		Theorem 11	Example 11		Example 11
	$\Leftarrow$		Example 12	Example 12		Example 12
Weakly	$\Rightarrow$		Theorem 11	Example 11		Example 11
transitive	$\Leftarrow$		Example 12	Example 12		Example 12
Semi-	$\Rightarrow$		Theorem 11		Theorem 10	Example 11
transitive	$\Leftarrow$		Example 12	Example 12		Example 12
$\star$-transitive	$\Rightarrow$		Theorem 12	Example 14		Example 14
	$\Leftarrow$		Example 15	Example 15		Example 15
Weakly	$\Rightarrow$		Theorem 12	Example 14		Example 14
$\star$-transitive	$\Leftarrow$		Example 15	Example 15		Example 15

In the table positive answers are represented by references to theorems and remarks. Negative answers refer to examples. The empty fields are covered by a suitable theorem or example in the same line. For example, asymmetric relations have positive answer for quasilinear means (Theorem 7), which covers the field of arithmetic mean. Similarly, symmetric relations have a negative answer for arithmetic mean (Example 6), which covers the field of quasilinear means.

In this paper we confine ourselves to the examination of single relation properties from Definition 5. However, the crucial meaning in applications have aggregations preserving $\star$-transitivity from Definition 7. Many recent results concerning such aggregations can be found in Saminger et al. [22]. In our further considerations we deal with such aggregations (cf. [9, 10]).

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