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GENERALIZED HOMOGENEOUS, PRELATTICE AND MV–EFFECT ALGEBRAS

ZDENKA RIEČANOVÁ AND IVICA MARINOVÁ

We study unbounded versions of effect algebras. We show a necessary and sufficient condition, when lattice operations of a such generalized effect algebra P are inherited under its embeding as a proper ideal with a special property and closed under the effect sum into an effect algebra. Further we introduce conditions for a generalized homogeneous, prelattice or MV-effect effect algebras. We prove that every prelattice generalized effect algebra P is a union of generalized MV-effect algebras and every generalized homogeneous effect algebra is a union of its maximal sub-generalized effect algebras with hereditary Riesz decomposition property (blocks). Properties of sharp elements, the center and center of compatibility of P are shown. We prove that on every generalized MV-effect algebra there is a bounded orthogonally additive measure.

Keywords: effect algebra, generalized effect algebra, generalized MV-effect algebra, prelattice and homogeneous generalized effect algebra

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1. BASIC DEFINITIONS AND FACTS

In 1994, Foulis and Bennett [3] have introduced a new algebraic structure, called an effect algebra. Effects represent unsharp measurements or observations on the quantum mechanical system. For modelling unsharp measurements in Hilbert space, the set of all effects is the set of all selfadjoint operators T on a Hilbert space H with $0 \leq T \leq 1$. In a general algebraic form an effect algebra is defined as follows:

Definition 1.1. A partial algebra $(E; \oplus, 0, 1)$ is called an *effect-algebra* if $0, 1$ are two distinguished elements and $\oplus$ is a partially defined binary operation on P which satisfies the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$, if $a \oplus b$ is defined,

(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$, if one side is defined,

(Eiii) for every $a \in P$ there exists a unique $b \in P$ such that $a \oplus b = 1$,

(Eiv) if $1 \oplus a$ is defined then $a = 0$.

From axioms of an effect algebra it follows immediately:

Cancellation law. In an effect algebra $(E;\oplus,0,1)$ for all $a,b,c\in E$ with defined $a\oplus b$ and $a\oplus c$ it holds

(CL) $a\oplus b=a\oplus c$ implies $b=c$.

Cancellation law guarantees that in every effect algebra E the partial binary operation $\ominus$ and the relation $\leq$ can be defined by

(ED) $a\leq c$ and $c\ominus a=b$ iff $a\oplus b$ is defined and $a\oplus b=c$.

Under the partial order defined by (ED), 0 is the least and 1 the greatest element of E. Hence every effect algebra is a bounded poset. Moreover, $a\oplus b$ is defined iff $a\leq b'$.

Unbounded versions (mutually equivalent) of effect algebras were studied by Foulis and Bennett (cones), Kalmbach and Riečanová (abelian RI-semigroups), Hedlíková and Pulmannová (cancellative positive partial abelian semigroups). Their common definition is the following:

Definition 1.2. A partial algebra $(E;\oplus,0)$ is called a generalized effect algebra if $0\in E$ is a distinguished element and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $a,b,c\in E$:

(GEi) $a\oplus b=b\oplus a$, if one side is defined,

(GEii) $(a\oplus b)\oplus c=a\oplus(b\oplus c)$, if one side is defined,

(GEiii) $a\oplus 0=a$ for all $a\in E$,

(GEiv) $a\oplus b=a\oplus c$ implies $b=c$ (cancellation law),

(GEv) $a\oplus b=0$ implies $a=b=0$.

We often denote a generalized effect algebra briefly by E. In every generalized effect algebra E the partial binary operation $\ominus$ and relation $\leq$ can be defined by (ED). Then $\leq$ is a partial order on E under which 0 is the least element of E. The following proposition with a trivial verification indicates the relation between effect algebras and generalized effect algebras.

Definition 1.3. [14] Let $(P;\oplus,0)$ be a generalized effect algebra. If $Q\subseteq P$ is such that $0\in Q$ and for all $a,b,c\in Q$ with $a\oplus b=c$ if at least two of a,b,c are in Q then $a,b,c\in Q$, then Q is called a *sub-generalized effect algebra* of P.

Note that every sub-generalized effect algebra Q of a generalized effect algebra P is a generalized effect algebra in its own right.

Proposition 1.1. If $(E;\oplus,0)$ is a generalized effect algebra and there is $1 \in E$ such that for all $a \in E$ there is $b \in E$ with $a \oplus b = 1$ then $(E;\oplus,0,1)$ is an effect algebra. Conversely, if $(E;\oplus,0,1)$ is an effect algebra then partial operation $\oplus$ satisfies axioms (GEi) – (GEv) of a generalized effect algebra.

Let us note that if for an effect algebra $(E;\oplus,0,1)$ we define on E a partial operation $\ominus$ and a partial order $\leq$ by condition (ED) then $(E;\leq,\ominus,0,1)$ is a D-poset introduced by Kôpka and Chovanec [12] and $(E;\ominus,0,1)$ is a D-algebra defined by Gudder [4]. Moreover, using condition (ED) for generalized effect algebra $(E;\oplus,0)$ we obtain generalized D-poset $(E;\ominus,0)$ studied by Hedlíková and Pulmannová [6]. Conversely, for mentioned partial algebras with operation $\ominus$ we can derive a partial operation $\oplus$ using the same condition (ED) and we obtain effect algebras $(E;\oplus,0,1)$ or generalized effect algebras $(E;\oplus,0)$.

It has been shown in [18] and an alternative result for generalized D-poset in [6] that every generalized effect algebra is an order ideal with special properties of an effect algebra. These results extend similar results which have been obtained for particular structures, namely that: (i) generalized orthomodular lattices are order ideals of orthomodular lattices (proved by Janowitz [7]), (ii) (weak) generalized orthomodular posets are order ideals of orthomodular posets (proved by Mayet–Ippolito [13]).

Recall that a nonvoid subset I of a partially ordered set L is an *order ideal* if $a \in L$, $b \in I$ and $a \leq b$ then $a \in I$.

Let $(P;\oplus,0)$ be a generalized effect algebra. Let P^* be a set disjoint from P with the same cardinality. Consider a bijection $a \mapsto a^*$ from P onto P^* and let us denote $P\dot{\cup}P^*$ by E. Define a partial binary operation $\oplus^*$ on E by the following rules. For $a, b \in P$

(i) $a \oplus^* b$ is defined if and only if $a \oplus b$ is defined, and $a \oplus^* b = a \oplus b$

(ii) $b^* \oplus^* a$ and $a \oplus^* b^*$ are defined iff $b \ominus a$ is defined and then $b^* \oplus^* a = (b \ominus a)^* = a \oplus^* b^*$.

Theorem 1.1. [2, p. 18] For every generalized effect algebra P and $E = P\dot{\cup}P^*$ the structure $(E;\oplus^*,0,0^*)$ is an effect algebra. Moreover, P is a proper order ideal in E closed under $\oplus^*$ and the partial order induced by $\oplus^*$, when restricted to P, coincides with the partial order induced by $\oplus$. P is a sub-generalized effect algebra of E and for every $a \in P$, $a \oplus a^* = 0^*$.

Since the definition of $\oplus^*$ on $E = P\dot{\cup}P^*$ coincides with $\oplus$- operation on P, it will cause no confusion if from now on we will use the notation $\oplus$ also for its extension on E.

2. PRELATTICE GENERALIZED EFFECT ALGEBRAS AND SHARP ELEMENTS

Assume that P is a generalized effect algebra. For the effect algebra $E = P\dot{\cup}P^*$ constructed in Theorem 1.1 the partial order on E, when restricted to P, coincides with the original partial order on P. In spite of this fact, the lattice operations join and meet of elements $a, b \in P$ ($a \vee_P b$, $a \wedge_P b$) need not be preserved for E, if they exist in P.

Example 2.1. Let $P = \{0, a, b, a \oplus b, b \oplus b\}$ be a generalized effect algebra and let $E = P\dot{\cup}P^*$ be an effect algebra constructed in Theorem 1.1. Obviously, $a \oplus b = a \vee_P b$, while $a \vee_E b$ does not exist.

The following theorem establishes a necessary and sufficient condition for the inheritance of $a \vee_P b$, for $a, b \in P$, by the effect algebra $E = P\dot{\cup}P^*$.

Theorem 2.1. Let P be a generalized effect algebra and $E = P\dot{\cup}P^*$. For every $a, b \in P$ with $a \vee_P b$ existing in P the following conditions are equivalent

(i) $a \vee_E b$ exists and $a \vee_E b = a \vee_P b$.

(ii) For every $c \in P$ the existence of $a \oplus c$ and $b \oplus c$ implies the existence of $(a \vee_P b) \oplus c$.

Proof. (i) $\Rightarrow$ (ii): If $a \oplus c$ and $b \oplus c$ exists in P then $a \leq c^*$ and $b \leq c^*$ which gives $a \vee_E b \leq c^*$ and hence $(a \vee_E b) \oplus c = (a \vee_P b) \oplus c$ exists in P.

(ii) $\Rightarrow$ (i): By the assumptions for all $c, d \in P$ we have: $a, b \leq d$ implies $a \vee_P b \leq d$, as well as $a, b \leq c^*$ implies $a \vee_P b \leq c^*$, since $(a \vee_P b) \oplus c$ exists in P. Thus $a \vee_E b = a \vee_P b$. □

In this section, it is of our interest to answer a question for which generalized effect algebra P the effect algebra $E = P\dot{\cup}P^*$ from Theorem 1.1 is lattice ordered, under which all joins and meets existing in P are preserved for E. We will call such generalized effect algebra a *prelattice generalized effect algebra.*

It is rather surprising that a prelattice effect algebra P need not be lattice ordered. *In general, a prelattice generalized effect algebra P need not be a sublattice of the lattice effect algebra $E = P\dot{\cup}P^*$.*

Example 2.2. Let $P = \{0, a, b, a \oplus a, b \oplus b\}$ be a generalized effect algebra. It is easy to check that $E = P\dot{\cup}P^*$ is a lattice effect algebra in spite of the fact that P is not a lattice, as, e.g., $a \vee_P b$ does not exist.

Theorem 2.2. Let P be a generalized effect algebra. Then $E = P\dot{\cup}P^*$ is a lattice effect algebra preserving joins and meets existing in P if and only if the following conditions are satisfied for all $a, b \in P$:

(i) $a \wedge_P b$ exists.

(ii) If there is $d \in P$ such that $a, b \leq d$ then $a \vee_P b$ exists.

(iii) For all $c \in P$ the existence of $a \vee_P b$, $a \oplus c$ and $b \oplus c$ implies the existence of $(a \vee_P b) \oplus c$.

(iv) Either $a \vee_p b$ exists or $\bigvee\{c \in P \mid a \oplus c \text{ and } b \oplus c \text{ are defined}\}$ exists in P.

(v) $\bigvee\{c \in P \mid c \leq b \text{ and } a \oplus c \text{ is defined}\}$ exists in P.

Proof. Let $a, b \in P$. If $c \in E$ and $c \leq a, b$ then $c \in P$ and hence $a \wedge_E b$ exists iff $a \wedge_P b$ exists, in which case $a \wedge_E b = a \wedge_P b$.

Let $d \in P$ such that $a, b \leq d$. Then the existence of $a \vee_E b$ implies $a \vee_E b \leq d$, which gives $a \vee_E b \in P$ and hence there is $a \vee_P b$ and $a \vee_P b = a \vee_E b$. Conversely, by (iii), the existence of $a \vee_P b$ implies that for all $c \in P$ with $a, b \leq c^*$ we have $a \vee_P b \leq c^*$ and hence there is $a \vee_E b$ and $a \vee_E b = a \vee_P b$.

If there is no $d \in P$ with $a, b \leq d$ then $a \vee_P b$ does not exist and then $a \vee_E b$ exists iff there is $x \in P$ such that $x = \bigvee\{c \in P \mid a \oplus c \text{ and } b \oplus c \text{ are defined}\}$, in which case $a \vee_E b = x^*$. Hence $a \vee_E b$ exists by (iv).

Finally, $a^* \wedge_E b$ exists iff there is $y \in P$ such that $y = \bigvee\{c \in P \mid c \leq b \text{ and } a \oplus c \text{ is defined}\}$. In this case $a^* \wedge_E b = y$. Thus, using d'Morgan laws, we obtain that E is a lattice effect algebra iff (i) – (v) are satisfied for every pair $a, b \in P$. □

Note that if for $a, b \in P$ in Theorem 2.2 the element $a \vee_P b$ exists then the the existence of $\bigvee\{c \in P \mid a \oplus c \text{ and } b \oplus c \text{ are defined}\}$ in P is not necessary to obtain $E = P\dot{\cup}P^*$ lattice ordered. For instance, this occurs when $P = [0, \infty)$ with usual addition.

Definition 2.1. A generalized effect algebra P satisfying conditions (i) – (v) of Theorem 2.2 is called a *prelattice generalized effect algebra.*

Theorem 2.3. Let P be an effect algebra and let $E = P\dot{\cup}P^*$. Then

(i) 1^* is an atom of E.

(ii) $a \oplus 1^* = (a')^*$, for every $a \in P$.

(iii) $a \oplus b^* = (b \ominus a)^*$, for all $a, b \in P$ with $a \leq b$.

(iv) $E \cong P \times \{0, 1^*\}$.

(v) E is a lattice effect algebra iff P is a lattice effect algebra, in which case P is a sublattice of E.

(vi) E is a distributive or modular effect algebra or MV-effect algebra iff P has these properties.

Proof. (i) Since for $a \in P$ the existence of $a \oplus 1$ implies $a = 0$, we obtain that the condition $a \leq 1^*$ implies $a = 0$. Moreover, for all $a \in P$ we have $1^* \leq a^*$ and hence 1^* is an atom of E.

(ii) Let $a \in P$. Then $a \leq 1 = (1^*)^*$, hence $a \oplus 1^*$ exists in E and $a \oplus 1^* = (1 \ominus a)^* = (a')^*$.

(iii) If $a, b \in P$ with $a \leq b$ then $a \leq (b^*)^*$ which gives the existence of $a \oplus b^*$. Further, $(a \oplus b^*)^* = 0^* \ominus (b^* \oplus a) = (0^* \ominus b^*) \ominus a = b \ominus a$, hence $a \oplus b^* = (b \ominus a)^*$.

(iv) Let us define a map $\varphi : E \to P \times \{0, 1^*\}$ as follows: for $a \in P$ let $\varphi(a) = (a, 0)$ and $\varphi(a^*) = (a', 1^*)$. Evidently φ is a bijection of E onto $P \times \{0, 1^*\}$. Further, if $a, b \in P$ and $a \oplus b$ is defined in P then $\varphi(a \oplus b) = (a \oplus b, 0) = (a, 0) \oplus (b, 0) = \varphi(a) \oplus \varphi(b)$. If $a \in P$, $b^* \in P^*$ and $a \oplus b^*$ is defined in E then by (iii) we have $\varphi(a \oplus b^*) = \varphi((b \ominus a)^*) = ((b \ominus a)', 1^*) = (b' \oplus a, 1^*) = (b', 1^*) \oplus (a, 0) = \varphi(a) \oplus \varphi(b^*)$. If $a^*, b^* \in P^*$ then $a^* \oplus b^*$ does not exist. This proves that φ is an isomorphism.

(v) Evidently, a lattice effect algebra P satisfies conditions (i) – (v) of Theorem 2.2, which implies that $E = P \dot{\cup} P^*$ is a lattice effect algebra in which P is a sublattice. Conversely, if E is a lattice then P is a sublattice of E, since P is a bounded poset.

(vi) This follows by the fact that E is a direct product of P and the Boolean algebra $\{0, 1^*\}$.

Recall that an element z of an effect algebra E is *sharp* if $z \wedge z' = 0$. It has been proved in [9] that in every lattice effect algebra E the set $S(E) = \{z \in E \mid z \wedge z' = 0\}$ is an orthomodular lattice. Moreover, $S(E)$ is a sub-lattice and a sub-effect algebra of E. □

Definition 2.2. An element z of a generalized effect algebra P is called a *sharp element* if for all $e \in P$ the conditions $e \leq z$ and $z \oplus e$ is defined imply that $e = 0$. Let $S(P) = \{z \in P \mid z \text{ is sharp element of } P\}$.

For the definition of a (weak) generalized orthomodular poset (introduced by A. Mayet–Ippolito [13]) we refer the reader to [2, p. 39]. Relation between generalized effect algebra and (weak) generalized orthomodular poset are shown in Theorem 1.5.13, Lemma 1.5.14 and Theorem 1.5.17 in [2].

Theorem 2.4. Let P be a prelattice generalized effect algebra and let $S(P) = \{z \in P \mid z \text{ is a sharp element of } P\}$. Let $E = P \dot{\cup} P^*$. Then

(i) $S(P) = S(E) \cap P$.

(ii) If $z_1, z_2 \in S(P)$ and $z_1 \oplus z_2$ is defined in E then $z_1 \oplus z_2 \in S(P)$.

(iii) $S(P)$ is a prelattice generalized effect algebra and $S(E) = S(P) \dot{\cup} (S(P))^*$, when $S(E)$ is considered as lattice effect algebra and $(S(P))^* = \{z^* \mid z \in S(P)\}$.

(iv) $S(P)$ is a generalized orthomodular poset being a proper ideal in the orthomodular lattice $S(E)$, closed under orthogonal joins and for every $z \in S(E)$ either $z \in S(P)$ or $0^* \ominus z \in S(P)$.

Proof. (i) Since E is a lattice effect algebra, for $e, z \in P$ we have $e \leq z \wedge z^*$ iff $e \leq z$ and $z \oplus e$ is defined. It follows that $z \wedge z^* = 0$ iff for all $e \in P$ the conditions $e \leq z$ and $z \oplus e$ is defined imply $e = 0$. It follows that $z \in S(P)$ iff $z \in S(E) \cap P$.

(ii) If $z_1, z_2 \in S(P)$ and $z_1 \oplus z_2$ is defined in E then $z_1 \leq z_2^*$ which gives that $z_1 \wedge z_2 \leq z_2^* \wedge z_2 = 0$ and hence $z_1 \oplus z_2 = z_1 \vee z_2 \in S(E) \cap P$ because $S(P) \subseteq P$, P is closed under $\oplus$ and $S(E)$ is a sublattice of E.

(iii) Since $S(E)$ is a sublattice and a sub-effect algebra of a lattice effect algebra E we may consider $S(E)$ as a lattice effect algebra in its own right. Further, by (i) and (ii), $S(P)$ is a proper order ideal in $S(E)$ closed under $\oplus$. If we set $(S(P))^* = \{z^* \mid z \in S(P)\}$ then $S(E) = S(P)\dot{\cup}(S(P))^*$ and by [2, Proposition 1.2.7] we obtain that the effect algebra $S(P)\dot{\cup}(S(P))^*$ coincides with $S(E)$.

(iv) This follows by (ii) and (iii) and the facts that for all $z_1, z_2 \in S(E)$ with $z_1 \le z_2^*$ we have $z_1 \wedge z_2 = 0$ and $z_1 \oplus z_2 = z_1 \vee z_2$, under which $S(E) = S(P)\dot{\cup}(S(P))^*$. □

3. GENERALIZED MV–EFFECT ALGEBRAS AND BLOCKS OF PRELATTICE GENERALIZED EFFECT ALGEBRAS

In [11] compatibility of elements a, b of a D-poset (effect algebra) E was introduced (denoted by $a \leftrightarrow b$). In [12] it has been proved that in a D-lattice (lattice effect algebra) $a \leftrightarrow b$ iff $(a \vee b) \ominus b = a \ominus (a \wedge b))$.

Lemma 3.1. Let E be a lattice effect algebra and let $a, b \in E$. The following conditions are equivalent:

(i) $a \leftrightarrow b$.

(ii) $(a \ominus (a \wedge b)) \oplus (b \ominus (a \wedge b))$ is defined.

Proof. Since E is a lattice effect algebra, for all $a, b \in E$ we have $0 = (a \wedge b) \ominus (a \wedge b) = (a \ominus (a \wedge b)) \wedge (b \ominus (a \wedge b))$, see [2, p. 70].

Assume that $a \leftrightarrow b$. Then $a \vee b = b \oplus (a \ominus (a \wedge b)) = (a \wedge b) \oplus (b \ominus (a \wedge b)) \oplus (a \ominus (a \wedge b))$ which implies (ii). Conversely (ii) implies $(a \ominus (a \wedge b)) \oplus (b \ominus (a \wedge b)) = (a \ominus (a \wedge b)) \vee (b \ominus (a \wedge b)) \le (a \wedge b)'$ which gives that $(a \wedge b) \oplus (a \ominus (a \wedge b)) \oplus (b \ominus (a \wedge b)) = [(a \ominus (a \wedge b)) \oplus (a \wedge b)] \vee [(b \ominus (a \wedge b)) \oplus (a \wedge b)] = a \vee b$ which implies that $a \vee b = b \oplus (a \ominus (a \wedge b))$.

If P is a prelattice generalized effect algebra then elements $a, b \in P$ are compatible in the lattice effect algebra $E = P\dot{\cup}P^*$ iff $(a \ominus (a \wedge b)) \oplus (b \ominus (a \wedge b))$ exists. Since in this case we have $(a \ominus (a \wedge b)) \oplus (b \ominus (a \wedge b)) \in P$, it make sense to call elements a, b compatible in P. In this case $a \vee b \in P$ since $a \vee b = b \oplus (a \ominus (a \wedge b))$. □

Definition 3.1. Elements a, b of a prelattice generalized effect algebra P are called *compatible* if $(a \ominus (a \wedge b)) \oplus (b \ominus (a \wedge b))$ is defined.

A lattice effect algebra E is called an *MV-effect algebra* if every pair of elements $a, b \in E$ is compatible. In [16] has been shown that every lattice effect algebra E is a set-theoretical union of maximal subsets of pairwise compatible elements, called *blocks*. Moreover, blocks of E are maximal sub MV-effect algebras as well as sublattices of E. It is worth noting that every MV-effect algebra M can be organized into an MV-algebra (extending the partial binary operation $\oplus$ onto total operation $\overline{\oplus}$ by $a \overline{\oplus} b = a \oplus (a' \wedge b)$ for all $a, b \in M$ ([1, 5, 12]). Thus every lattice effect algebra E is a union of MV-algebras. If that E is an orthomodular lattice (see [10]) then every block of E is a Boolean algebra.

In what follows for $Q \subseteq P$ we will denote by Q^* the set $\{y^* \in P^* \mid y \in Q\}$.

Theorem 3.1. Let P be a prelattice generalized effect algebra and $E = P\dot{\cup}P^*$. Let $M \subseteq E$ be a block of E. Then

(i) $M \cap P^* = (M \cap P)^*$.

(ii) $M \cap P$ is a maximal pairwise compatible subset of P and a sublattice of E. Conversely, if Q is a maximal subset of pairwise compatible elements of P then $Q\dot{\cup}Q^*$ is a block of E.

(iii) $M \cap P$ is a prelattice generalized effect algebra and $(M \cap P)\dot{\cup}(M \cap P)^* = M$.

Proof. (i) For $x, y \in E$ we have $x \leftrightarrow y$ iff $x \leftrightarrow y^*$ (see [16]) which gives $y \in M$ iff $y^* \in M$ and therefore $y \in M \cap P$ iff $y^* \in M \cap P^*$. It follows that $M \cap P^* = (M \cap P)^*$.

(ii) Let $x \in P$ and $x \leftrightarrow y$ for all $y \in M \cap P$. Then $x \leftrightarrow y^*$ for all $y \in M \cap P^*$ and thus $x \leftrightarrow y$ for all $y \in M$ which gives $x \in M$ by maximality of M. Further, by [16], if $x \leftrightarrow y$ and $x \leftrightarrow z$ then $x \leftrightarrow y \vee z$ and $x \leftrightarrow y \wedge z$, therefore $M \cap P$ is a sublattice of E, because $y, z \in P$ and $y \leftrightarrow z$ implies $y \vee_E z \in P$. If Q is a maximal subset of pairwise compatible elements of P then $Q\dot{\cup}Q^*$ is a maximal subset of pairwise compatible elements of E, hence $Q\dot{\cup}Q^*$ is a block of E (see [16]).

(iii) $M \cap P$ is a generalized effect algebra since both M and P are generalized effect algebras. Further, P satisfies conditions (i) – (v) of Theorem 2.2 by the assumption that P is prelattice and M satisfies these conditions, since M is a lattice. Therefore $M \cap P$ is a prelattice generalized effect algebra. Obviously $M = (M \cap P)\dot{\cup}(M \cap P)^*$ as $M \cap P^* = (M \cap P)^*$ by (i). □

Corollary 3.1. Every prelattice generalized effect algebra P is a union of maximal subsets of pairwise compatible elements of P.

Proof. If P is a prelattice effect algebra then $E = P\dot{\cup}P^*$ is a lattice effect algebra and by [16] we have $E = \bigcup\{M \subseteq E \mid M \text{ is a block of } E\}$. Therefore $P = \bigcup\{M \cap P \mid M$ is a block of $E\}$. The rest follows by Theorem 3.1, (ii). □

Definition 3.2. A maximal subset of pairwise compatible elements of a prelattice generalized effect algebra P is called a *block* of P. A prelattice generalized effect algebra with a unique block is called a *generalized MV-effect algebra.*

Theorem 3.2. For a generalized effect algebra P the following conditions are equivalent:

(i) P is a generalized MV-effect algebra.

(ii) $E = P\dot{\cup}P^*$ is an MV-effect algebra.

(iii) P is a prelattice generalized effect algebra and for all $a, b \in P$ the sum $(a \ominus (a \wedge b)) \oplus (b \ominus (a \wedge b))$ exists in P.

The proof is straightforward.

Theorem 3.3. A generalized effect algebra P is a generalized MV-effect algebra iff the following conditions are satisfied

(i) P is a lattice.

(ii) For all $a, b, c \in P$ the existence of $a \oplus c$ and $b \oplus c$ implies the existence of $(a \vee_P b) \oplus c$.

(iii) $\bigvee\{c \in P \mid a \oplus c$ exists and $c \leq b\}$ exists in P, for all $a, b \in P$.

(iv) $(a \ominus (a \wedge b)) \oplus (b \ominus (a \wedge b))$ exists for all $a, b \in P$.

Proof. Obviously conditions (i) – (iii) imply conditions (i) – (v) of Theorem 2.2 hence P is a prelattice generalized effect algebra and condition (iv) implies that it has a unique block.

Conversely, if P is a generalized MV-effect algebra then obviously (ii) – (iv) are satisfied. Let $a, b \in P$ then $a \wedge_P b$ exists by Theorem 2.2, (i) and $a \wedge_P b = a \wedge_E b$. Further, there is $(a \ominus (a \wedge b)) \leq (a \wedge b)^*$ and since $E = P \dot{\cup} P^*$ is a lattice effect algebra we obtain $[(a \ominus (a \wedge b)) \oplus (b \ominus (a \wedge b))] \oplus (a \wedge b) = [(a \ominus (a \wedge b) \vee (b \ominus (a \wedge b))] \oplus (a \wedge b) = a \vee b \in P$. □

Theorem 3.4.

(i) Every prelattice generalized effect algebra is a union of generalized MV-effect algebras (blocks).

(ii) A generalized MV-effect algebra P is an MV-effect algebra iff there exists an element $1 \in P$ such that for every $a \in P$ there exists a unique $b \in P$ for which $a \oplus b = 1$.

The proof is straightforward.

Example 3.1. The set $P_1 = \{0, 1, 2, 3, \ldots\}$ of nonnegative integers with usual addition and the set $P_2 = \langle 0, \infty)$ of nonnegative real numbers with usual addition are examples of generalized MV-effect algebras. It is easy to see that $E_1 = P_1 \dot{\cup} P_1^*$ and $E_2 = P_2 \dot{\cup} P_2^*$ are linearly (totally) ordered MV-effect algebras.

More generally, the positive cone G^+ of any partially ordered abelian group $(G; +, 0, \leq)$ is a generalized effect algebra.

Example 3.2. Let $H \neq \emptyset$ and for every $\kappa \in H$ let either $P_\kappa = P_1$ or $P_\kappa = P_2$ defined in Example 3.1. The cartesian product $\prod_{\kappa \in H} P_\kappa$ with 0 and $\oplus$ defined "coordinatewise" is a prelattice generalized effect algebra. First, it is easy to check that $P = \prod_{\kappa \in H} P_\kappa$ is a lattice in which $\leq$ and lattice operations are defined "coordinatewise" too, and every P_κ is a lattice. Further, $\oplus$ on P is a total binary operation. This proves that conditions (i) – (iv) from Theorem 2.2 are satisfied in P. Finally, for all $a, b \in P$ we have $\bigvee\{c \in P \mid c \leq b$ and $a \oplus c$ is defined$\} = b \in P$. By Theorem 2.2, $E = P \dot{\cup} P^*$ is a lattice effect algebra preserving joins and meets existing in P. Moreover, by Theorem 3.2, E is an MV-effect algebra.

A *state* on an effect algebra $(E;\oplus,0,1)$ is a mapping $\omega : E \to [0,1] \subseteq (-\infty,\infty)$ such that $\omega(0)=0$, $\omega(1)=1$ and $\omega(a\oplus b)=\omega(a)+\omega(b)$ for all $a\le b'$, $a,b\in E$.

Theorem 3.5. Let $(P;\oplus,0)$ be a generalized effect algebra and let $m : P\to[0,\infty)$ be a bounded mapping such that $m(a\oplus b)=m(a)+m(b)$ for all $a,b\in P$ with defined $a\oplus b$. Then there is a state ω on $E=P\dot{\cup}P^*$ extending $\frac{m}{k_0}$, iff $k_0=\sup\{m(a)\,|\,a\in P\}\neq 0$.

Proof. Let $k_0=\sup\{m(a)\,|\,a\in P\}\neq 0$. For every $a\in P$ let $\omega(a)=\frac{m(a)}{k_0}$ and $\omega(a^*)=\frac{k_0-\omega(a)}{k_0}$. Then $\omega(0)=0$ and $\omega(0^*)=\frac{k_0-m(0)}{k_0}=1$, because $m(0)=m(0\oplus 0)=2m(0)$, which gives $m(0)=0$. Further, for $a,b\in P$ with $a\le b^*$ we have $\omega(a\oplus b)=\frac{m(a\oplus b)}{k_0}=\omega(a)+\omega(b)$ and for a,b with $a\le b$ we have $\omega(a\oplus b^*)=\omega((b\ominus a)^*)=\frac{k_0-m(b\ominus a)}{k_0}=1-\frac{m(b)}{k_0}+\frac{m(a)}{k_0}=\omega(b^*)+\omega(a)$. This proves that ω is a state on E.

It is well known that on every MV-effect algebra (MV-algebra) there is a state.□

Theorem 3.6. On every generalized MV-effect algebra P there is a bounded mapping $m : P\to[0,1]$ such that $m(a\oplus b)=m(a)+m(b)$ for all $a,b\in P$ with defined $a\oplus b$.

Finally, note that the notion of a *central element* of a generalized effect algebra P has been introduced in [14]. Recall that $z\in P$ is central iff P is isomorphic to a direct product of $[0,z]$ and $Q_z=\{x\in P\,|\,x\wedge z=0\}$. Moreover $z\in P$ is central element of P iff it is a central element of $E=P\dot{\cup}P^*$ iff E is isomorphic to the direct product $[0,z]\times[0,z^*]$ and then for Q_z defined above we have $Q_z=P\cap[0,z^*]$ (see [14, Section 5]). Thus if $C(E)=\{z\in E\,|\,z$ is central element of $E\}$ and $C(P)=\{z\in E\,|\,z$ is central element of $P\}$ then $C(P)=C(E)\cap P$.

For a lattice effect algebra E the subset $B(E)=\bigcap\{M\subseteq E\,|\,M$ is a block of $E\}$ is called a compatibility center of E. By Theorem 3.1, for a prelattice generalized effect algebra P and lattice effect algebra $E=P\dot{\cup}P^*$ we obtain that $B(E)\cap P=\bigcap\{M\cap P\,|\,M$ is a block of $E\}$. We will call $B(E)\cap P$ a *compatibility center of* P and denote it by $B(P)$. Since for every lattice effect algebra E the equality $C(E)=S(E)\cap B(E)$ holds (see [15, Theorem 2.5, (iv)]), we obtain:

Theorem 3.7. For every prelattice generalized effect algebra P the condition $C(P)=S(P)\cap B(P)$ is satisfied. If P is a generalized MV-effect algebra then $C(P)=S(P)$.

As a consequence of Theorem 3.7 we obtain that *an element z of a prelattice generalized effect algebra P is central iff z is a sharp element of P compatible with every element of P*. Because $D\subseteq P$ is a block of P iff there is a block M of $E=P\dot{\cup}P^*$ such that $D=M\cap P$ and $M\cap P^*=(M\cap P)^*=D^*$ we obtain that $B(P)$ is a generalized MV-effect lagebra such that the MV-algebra $B(E)=B(P)\dot{\cup}(B(P))^*$. Moreover, the center $C(P)$ is a proper order ideal in the Boolean algebra $C(E)$

closed under $\oplus$ and such that for every $z \in C(E)$ either $z \in C(P)$ or $0^* \ominus z \in C(P)$ and thus $C(E) = C(P)\dot{\cup}(C(P))^* = (C(E) \cap P)\dot{\cup}(C(E) \cap P^*)$, hence $C(P)$ is a generalized Boolean algebra.

4. GENERALIZED HOMOGENEOUS EFFECT ALGEBRAS

A certain class of effect algebras called homogeneous effect algebras has been introduced by G. Jenča in [8]. Recall that an effect algebra E satisfies the *Riesz decomposition property* if for $u, v_1, v_2 \in E$ such that $u \leq v_1 \oplus v_2$ there are $u_1, u_2 \in E$ such that $u_1 \leq v_1$, $u_2 \leq v_2$ and $u = u_1 \oplus u_2$. An effect algebra is called *homogeneous* if for all $u, v_1, v_2 \in E$ such that $u \leq v_1 \oplus v_2 \leq u'$ there are $u_1 \leq v_1$ and $u_2 \leq v_2$ such that $u = u_1 \oplus u_2$. In [8] it has been proved that *every homogeneous effect algebra is a union of maximal sub-effect algebras with the Riesz decomposition property, called blocks.* The class of homogeneous effect algebras includes orthoalgebras and lattice effect algebras (hence as a subclass MV-effect algebras, orthomodular lattices, Boolean algebras and orthomodular posets). If a homogeneous effect algebra E is a lattice effect algebra then blocks are MV-effect algebras (see [16]). If E is an orthomodular lattice then blocks are Boolean algebras (see [10]).

Definition 4.1. A generalized effect algebra P is called a *generalized homogeneous effect algebra* iff the following conditions are satisfied for all $a, b, c \in P$:

(i) If $(b \oplus c) \oplus a$ is defined and $a \leq b \oplus c$ then there are $b_1, c_1 \in P$ such that $b_1 \leq b$, $c_1 \leq c$ and $a = b_1 \oplus c_1$.

(ii) If $(c \ominus b) \oplus a$ is defined and $a \leq c \ominus b$ then there are $b_1, c_1 \in P$ such that $b_1 \leq b$, $c_1 \oplus c$ is defined and $a = b_1 \oplus c_1$.

Theorem 4.1. Let P be a generalized effect algebra and $E = P\dot{\cup}P^*$ be defined as in Theorem 1.1. The following conditions are equivalent:

(i) E is a homogeneous effect algebra.

(ii) P is a generalized homogeneous effect algebra.

Proof. Let $x, y, z \in E$ and $x \leq y \oplus z \leq x^*$. Then, because for $b, c \in P$ only $b \oplus c$ or $b \oplus c^*$ can be defined ($b^* \oplus c^*$ never can be) and for no $a \in P$ the inequality $a^* \leq a$ holds, we obtain that there are $a, b, c \in P$ such that $x = a$, $y = b$ and either $z = c$ or $z = c^*$. Further, the conditions $a \leq b \oplus c^* \leq a^*$ are equivalent to the conditions $a \leq (b \oplus c^*)^* \leq a^*$ and hence to $a \leq (c \ominus b) \leq a^*$ which is equivalent to the conditions $a \leq c \ominus b$ and $(c \ominus b) \oplus a$ is defined. Moreover, $c_1 \leq c^*$ iff $c_1 \oplus c$ is defined. This proves that (i) and (ii) are equivalent. □

By Theorems 1.1 and 4.1 every homogeneous generalized effect algebra P is a proper order ideal in a homogeneous effect algebra E such that P is closed under $\oplus$ and for every $x \in E$ exactly one of elements x, x^* is in P.

Definition 4.2. A generalized effect algebra E is called a *generalized effect algebra with hereditary Riesz decomposition property* (HRDP for short) if the following conditions are satisfied for all $a, b, c \in P$:

(i) If $a \leq b \oplus c$ then there are $b_1, c_1 \in P$ such that $b_1 \leq b$, $c_1 \leq c$ and $b_1 \oplus c_1 = a$.

(ii) If $(c \ominus b) \oplus a$ is defined then there are $b_1, c_1 \in P$ such that $b_1 \leq b$, $c_1 \oplus c$ is defined and $b_1 \oplus c_1 = a$.

(iii) If $c \ominus b \leq a$ then there are $b_1, c_1 \in P$ such that $b_1 \leq b$, $c_1 \geq c$ and $a = c_1 \ominus b_1$.

Theorem 4.2. Let P be a generalized effect algebra and $E = P \dot{\cup} P^*$ be defined as in Theorem 1.1. The following conditions are equivalent:

(i) E is an effect algebra with RDP.

(ii) P is a generalized effect algebra with HRDP.

Proof. Let $x, y, z \in E$. Then $x \leq y \oplus z$ iff there are $a, b, c \in P$ such that $x = a$, $y = b$, and $z \in \{c, c^*\}$ or $x = a^*$, $y = b$ and $z = c^*$. Further, $(c \ominus b) \oplus a$ is defined iff $a \leq (c \ominus b)^* = b \oplus c^*$. Finally, $c \ominus b \leq a$ iff $a^* \leq (c \ominus b)^* = b \oplus c^*$. Moreover, if E has RDP property then: (1) $a \leq b \oplus c$ implies the existence of $b_1, c_1 \in P$ such that $b_1 \leq b$, $c_1 \leq c$ and $a = b_1 \oplus c_1$, (2) $a \leq b \oplus c^*$ implies the existence of $b_1 \in P$, $c_1 \in P$ such that $b_1 \leq b$, $c_1 \leq c^*$ and $a = b_1 \oplus c_1$, under which $c_1 \leq c^*$ iff $c_1 \oplus c$ is defined. (3) $a^* \leq b \oplus c^*$ implies that there are $b_1 \in P$ and $c_1^* \in P^*$ such that $b_1 \leq b$, $c_1^* \leq c^*$ and $a^* = b_1 \oplus c_1^* = (c_1 \ominus b_1)^*$ or, equivalently, $a = c_1 \ominus b_1$, under which $c_1 \geq c$. This proves that E has RDP iff P is a generalized effect algebra with HRDP. □

Definition 4.3. Let $(P; \oplus, 0)$ be a generalized effect algebra. If P is homogeneous and Q is a maximal sub-generalized effect algebra of P with HRDP then Q is called a *block* of P.

Theorem 4.3. Let P be a generalized homogeneous effect algebra and let $E = P \dot{\cup} P^*$.

(i) $Q \subseteq P$ is a block of P iff there is a block D of E such that $Q = D \cap P$.

(ii) P is a union of its maximal sub-generalized effect algebras with HRDP (blocks).

Proof. (i) If $D \subseteq E$ is a block of E then $x \in D$ iff $x^* \in D$ and D is closed under $\oplus$ (see [8]). It follows that $0 \in D \cap P$ and $D \cap P$ is a sub-generalized effect algebra of P. Moreover, $D = D \cap (P \dot{\cup} P^*) = (D \cap P) \dot{\cup} (D \cap P^*) = (D \cap P) \dot{\cup} (D \cap P)^*$. By Theorem 4.2, $D \cap P$ has the HRDP iff D has the RDP. If $D \cap P \subseteq Q_1 \subseteq P$ and Q_1 is a block of P then $D_1 = Q_1 \dot{\cup} Q_1^* \subseteq E$ is a block of E by Theorem 4.2 and $D \subseteq D_1$ which implies that $D = D_1$ and hence $D \cap P = Q_1$.

(ii) Since $E = \bigcup \{D \subseteq E \mid D \text{ is a block of } E\}$, we obtain that $P = \bigcup \{P \cap D \mid D \text{ is a block of } E\} = \{Q \subseteq P \mid Q \text{ is a block of } P\}$ by (i). □

Theorem 4.4. Every prelattice generalized effect algebra and, in particular, every generalized MV-effect algebra is homogeneous. Every generalized MV-effect algebra is a generalized effect algebra with HRDP. Every generalized effect algebra with HRDP is homogeneous.

Proof. In [8] has been proved that every lattice effect algebra and hence also every MV-effect algebra are homogeneous. It follows by Theorem 4.1 that every prelattice generalized effect algebra is a generalized homogeneous effect algebra and since every MV-effect algebra has a unique block (which has RDP), every generalized MV-effect algebra has HRDP, by Theorem 4.3, (i). The rest is obvious. □

A lattice effect algebra E is an MV-effect algebra iff for every $c \in S(E)$ the interval $[0, c]$ is closed under $\oplus$ and, with inherited $\oplus$, it is an MV-effect algebra in its own right.

Remark 4.1. Let P be a prelattice generalized effect algebra.

(i) If for every $x \in S(P)$ the interval $[0, x]$ is closed under $\oplus$ and it is (with inherited $\oplus$) an MV-effect algebra then, in general, P need not be a generalized MV-effect algebra.

(ii) If P has RDP: $a \le b \oplus c$ implies the existence of $b_1 \le b$, $c_1 \le c$ with $b_1 \oplus c_1 = a$, then, in general, P need not have HRDP.

Example 4.1. Let $P = \{0, a, b, c, a \oplus c, b \oplus c\}$ be a generalized effect algebra. Then $E = P \dot{\cup} P^*$ is a lattice effect algebra preserving joins existing in P, hence P is a prelattice generalized effect algebra. Moreover, for every $x \in P$ the interval $[0, x]$ is closed under $\oplus$ and it is a Boolean algebra and hence an MV-effect algebra. In spite of that, E is not an MV-effect algebra because $a \oplus c \not\leftrightarrow b \oplus c$ since $(a \oplus c) \wedge (b \oplus c) = c$ and $((a \oplus c) \ominus c) \oplus ((b \oplus c) \ominus c)$ is not defined in P (resp. E). Further, P does not have HRDP since $a \le c^* = (b \oplus c)^* \oplus b$ while $a \wedge b = 0 = a \wedge (b \oplus c)^*$. Evidently P has RDP.

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Zdenka Riečanová and Ivica Marinová, Department of Mathematics, Faculty of Electrical Engineering and Information Technology STU, Ilkovičova 3, 812 19 Bratislava 1. Slovak Republic.
e-mails: zriecan@elf.stuba.sk, marinova@kmat.elf.stuba.sk