# ON APPROXIMATION IN MULTISTAGE STOCHASTIC PROGRAMS: MARKOV DEPENDENCE[1]

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A general multistage stochastic programming problem can be introduced as a finite system of parametric (one-stage) optimization problems with an inner type of dependence. Evidently, this type of the problems is rather complicated and, consequently, it can be mostly solved only approximately. The aim of the paper is to suggest some approximation solution schemes. To this end a restriction to the Markov type of dependence is supposed.

*Keywords:* multistage stochastic programming problem, approximation solution scheme, deterministic approximation, empirical estimate, Markov dependence

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## 1. INTRODUCTION

A general multistage stochastic programming problem can be in a rather general form introduced recursively (see e.g. [4, 12, 16]) as the problem:

Find

$$\varphi_{\mathcal{F}}(M) = \inf\{\mathsf{E}_{F^{\xi^0}} g^0_{\mathcal{F}}(x^0, \xi^0) | \, x^0 \in \mathcal{K}^0\}, \tag{1}$$

where the function $g^0_{\mathcal{F}}(x^0, z^0)$ is given recursively

$$\begin{aligned} g^k_{\mathcal{F}}(\bar{x}^k, \bar{z}^k) &= \inf\{\mathsf{E}_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}} \, g^{k+1}_{\mathcal{F}}(\bar{x}^{k+1}, \bar{\xi}^{k+1}) \, | \, x^{k+1} \in \mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k, \bar{z}^k)\}, \\ &\quad k = 0, 1, \ldots, M-1, \\ g^M_{\mathcal{F}}(\bar{x}^M, \bar{z}^M) &:= g^M_0(\bar{x}^M, \bar{z}^M). \end{aligned} \tag{2}$$

$\xi^j = \xi^j(\omega)$, $j = 0, 1, \ldots, M$ are $s$-dimensional random vectors defined on a probability space $(\Omega, S, P)$, $\bar{\xi}^k = \bar{\xi}^k(\omega) = [\xi^0, \ldots, \xi^k]$, $\bar{z}^k = [z^0, \ldots, z^k]$, $z^j \in R^s$, $x^j \in R^n$, $\bar{x}^k = [x^0, \ldots, x^k]$, $j = 0, 1, \ldots, k$, $k = 0, 1, \ldots, M$, $F^{\xi^j}(z^j)$, $F^{\bar{\xi}^j}(\bar{z}^j)$, $j = 0, \ldots, M$, denote the distribution functions of the $\xi^j$ and $\bar{\xi}^j$, $F^{\xi^k|\bar{\xi}^{k-1}}(z^k | \, \bar{z}^{k-1})$, $k = 1, \ldots, M$, denotes the conditional distribution function ($\xi^k$ conditioned by $\bar{\xi}^{k-1}$).

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$g_0^M(\bar{x}^M, \bar{z}^M)$ is a function defined on $R^{n(M+1)} \times R^{s(M+1)}$, $\mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k, \bar{z}^k) := \mathcal{K}_{F^{\xi^{k+1}|\bar{\xi}^k}}^{k+1}(\bar{x}^k, \bar{z}^k)$, $k = 0, 1, \ldots, M-1$, is a multifunction mapping $R^{n\,(k+1)} \times R^{s\,(k+1)}$ into the space of (mostly compact) subsets of $\mathcal{X}$. See that for every given $k \in \{0, \ldots, M-1\}$ the multifunction $\mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k, \bar{z}^k)$ can generally depend on the probability measure $P_{F^{\xi^{k+1}|\bar{\xi}^k}}(\cdot|\bar{z}^k)$ corresponding to the conditional distribution function $F^{\xi^{k+1}|\bar{\xi}^k}(\cdot|\bar{z}^k)$. $\mathcal{X}$, $\mathcal{K}^0 \subset R^n$ are nonempty sets, $\mathcal{K}^0 \subset \mathcal{X}$. $Z_{\mathcal{F}}^j \subset R^s$, $j = 0, 1, \ldots, M$, denote the supports corresponding to $F^{\xi^j}(\cdot)$, $\bar{Z}_{\mathcal{F}}^k = Z_{\mathcal{F}}^0 \times \ldots \times Z_{\mathcal{F}}^k$, $\bar{\mathcal{X}}^k = \mathcal{X} \times \ldots \times \mathcal{X}$, $k = 0, 1, \ldots, M$. The symbol $\mathsf{E}_F$ is reserved for the operator of mathematical expectation corresponding to the distribution function $F(\cdot)$. ($R^n$, $n \geq 1$, denotes the $n$-dimensional Euclidean space.)

Evidently, it can be very complicated numerical problem to solve the multistage stochastic programming programs exactly. The aim of the paper is to suggest approximative (deterministic and stochastic) solution schemes. To this end we restrict our consideration to the case when the system

$$\mathcal{F} = \{F^{\xi^0}(z^0), \quad F^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1}),\ k = 1, \ldots, M\} \tag{3}$$

corresponds to special types of the Markov dependence and to the case when there exist multifunctions $\bar{\mathcal{K}}^{k+1}(\bar{x}^k, \bar{z}^k)$, $k = 0, \ldots, M-1$ defined on $R^{(k+1)n} \times R^{(k+1)s}$ such that

$$\mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k, \bar{z}^k) = \bar{\mathcal{K}}^{k+1}(\bar{x}^k, \bar{z}^k) \quad \text{independently of the system } \mathcal{F}. \tag{4}$$

## 2. PROBLEM ANALYSIS

If we replace the system (3) by another system

$$\mathcal{G} = \{G^{\xi^0}(z^0), \quad G^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1}),\ k = 1, \ldots, M\} \tag{5}$$

we obtain a new multistage stochastic programming problem.

Of course, the problem (1) (for $k = 0$) is one-stage (nonparametric) stochastic programming problem. Moreover, if we fix successively $k \in \{0, 1, \ldots, M-1\}$, $\bar{x}^k \in \bar{\mathcal{X}}^k$, $\bar{z}^k \in \bar{Z}_{\mathcal{F}}^k$ in the relations (2) and set

$$\begin{aligned} &x := x^{k+1}, \quad z := z^{k+1}, \quad \xi := \xi^{k+1} \quad \text{with} \quad F^{\xi}(\cdot) := F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}(\cdot|\bar{z}^k), \\ &g(x, z) := g_{\mathcal{F}}^{k+1}(\bar{x}^{k+1}, \bar{z}^{k+1}), \quad X := \bar{\mathcal{K}}^{k+1}(\bar{x}^k, \bar{z}^k), \quad Z_{F^{\xi}} := Z_F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}, \end{aligned}$$

then we obtain (also) one-stage (parametric) stochastic programming problems:

Find

$$\varphi(F^{\xi}) = \inf\{\mathsf{E}_{F^{\xi}} g(x, \xi) | x \in X\}. \tag{6}$$

($Z_F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}$ denotes the measure support corresponding to $F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}(\cdot|\bar{z}^k)$.)

The system $\mathcal{F}$ is determined by $M+1$ of (mostly) conditional distribution functions. If $\mathcal{F}$ corresponds to a Markov sequence, then under some additional assumptions there exists an $s$-dimensional distribution function $F^I(\cdot)$ that can determine

$\mathcal{F}$. Consequently, it is possible to assume that a relationship between two systems $\mathcal{F}$ and $\mathcal{G}$ can be (under the assumption of the Markov dependence) "determined" by the relationship between two "corresponding" $s$-dimensional distribution functions $F^I$, $G^I$ (for more details see e. g. [12] or [13]). Consequently, in this case, to investigate the "deterministic" approximation, stability and "statistical" estimates of the problem (1), (2) the corresponding results achieved for one-stage problems (see e. g. [3, 7, 10, 18, 19, 20, 23, 25, 26]) can be employed.

## 3. ONE–STAGE PROBLEMS

In this section, first, we recall and generalize some auxiliary assertions on the stability and empirical estimates achieved for one-stage problems. Employing these results we introduce deterministic and empirical approximation schemes. To this end we employ the symbols of the problem (6).

### 3.1. Stability and "deterministic" approximation

**Proposition 1.** Let $X$ be a nonempty, compact set. If

1. there exist $a_i,\, b_i \in R^1,\; a_i < b_i,\; i = 1, \ldots, s$ such that $Z_{F^\xi} \subset \prod_{i=1}^{s} \langle a_i,\, b_i \rangle$,

2. $\mathcal{G}$ is a system of functions defined on $X \times Z_{F^\xi}$ such that

   a. $g \in \mathcal{G} \implies g(x,\, z)$ is a uniformly continuous function on $X \times Z_{F^\xi}$,

   b. $g \in \mathcal{G} \implies g(x,\, \cdot)$ is a Lipschitz function on $\prod_{i=1}^{s} \langle a_i,\, b_i \rangle$ with the Lipschitz constant $\bar{L}_g$ (w.r.t. $\mathcal{L}_2$ norm) not depending on $x \in X$,

then for every $\delta > 0$ there exist a natural number $N := N(\delta)$ and a discrete distribution $G_\delta$ function with at most $N$ atoms such that

$$|\bar{\varphi}(F^\xi) - \bar{\varphi}(G_\delta)| \le \bar{L}_g\, \delta \quad \text{for every} \quad g \in \mathcal{G}.$$

P r o o f. Since a very similar results to Proposition 1 has been already proven in [8] we can omit the proof. □

**Remark 1.** It follows from the proof of Proposition 1 that for every natural $m$ there exists a jump distribution function $\bar{G}^N(\cdot)$ with the number of jumps not greater then $N = m^s$ for which

$$\lim_{m \to +\infty} m^{1-c} |\bar{\varphi}(F^\xi) - \bar{\varphi}(\bar{G}^N)| = 0 \qquad \text{for every } c > 0\,. \tag{7}$$

To investigate the stability of the problem (6) w.r.t. a probability measures space different "distances" in the space of the probability measures can be employed (see e. g. [6, 17]). In this paper we focus on the Wasserstein metrics $d^s_{W_1}(\cdot,\cdot)$ in the space of $s$-dimensional probability measures, especially in the space $\mathcal{M}_1(R^s)$. To this end we define the system $\mathcal{M}_r(R^s)$, $r \geq 1$ by

$$\mathcal{M}_r(R^s) = \left\{ \nu \in \mathcal{P}(R^s) : \int_{R^s} \|z\|^r \nu(\mathrm{d}z) < \infty \right\},$$

where $\mathcal{P}(R^s)$ denotes the set of all (Borel) probability measures in $R^s$ and the symbol $\|\cdot\|$ denotes a suitable norm in $R^s$ (for more details see e.g. [15] or [24]). Employing the Euclidean norm $\|\cdot\|_2$ we obtain the "classical" Wasserstein metric $d^s_{W_1}(\cdot, \cdot)$ (for details see e.g. [17]) that can be defined by the relation

$$d^s_{W_1}(\nu, \mu) = \inf \left\{ \int_{R^s \times R^s} \|z - \bar{z}\|_2 \, \kappa\,(\mathrm{d}z \times \mathrm{d}\bar{z}) : \kappa \in \mathcal{D}(\mu, \nu) \right\}, \quad \nu, \mu \in \mathcal{M}_1(R^s). \tag{8}$$

$\mathcal{D}(\mu, \nu)$ is the set of those probability measures in $\mathcal{P}(R^s \times R^s)$ whose marginal distributions are $\mu$ and $\nu$.

To obtain the results of Proposition 1 the function $g(x, \cdot)$, $x \in X$ was assumed to be Lipschitz w.r.t. $\mathcal{L}_2$ norm. In [22], the function $g(x, \cdot)$, $x \in X$ is considered to be Lipschitz w.r.t. $\mathcal{L}_1$ norm. We employ this approach to introduce Proposition 2. To this end, let $m$ be a natural number; $z_{i,j} \in R^1 \cup \{-\infty\} \cup \{+\infty\}$, $i = 1, \ldots, s$, $j = 0, 1, \ldots, m$ fulfil the relations

$$-\infty = z_{i,0} < z_{i,1} \leq z_{i,2} \leq \ldots, \leq z_{i,m-1} < z_{i,m} = +\infty$$

and let $F^{\xi}_i(\cdot)$, $i = 1, 2, \ldots, s$ denote one-dimensional marginal distribution functions corresponding to $F^{\xi}(\cdot)$. If we define the probability $q^m_{j_1, j_2, \ldots, j_s}$, $j_i = 1, \ldots, m$, $i = 1, 2, \ldots, s$, $m = 1, \ldots$ by the relations

$$\begin{array}{rcl} q^m_{j_1, j_2, \ldots, j_s} & = & P_{F^{\xi}} \left\{ \xi \in \prod_{i=1}^{s} \langle z_{i, j_i - 1}, z_{i, j_i} ) \right\}, \\ \langle z_{i, j_i}, z_{i, j_i + 1} ) & := & (z_{i, j_i}, z_{i, j_i + 1}) \quad \text{everywhere when } z_{i, j_i} = -\infty, \end{array} \tag{9}$$

$s$-dimensional random vector $\zeta$ with a discrete probability measure $P_{F^{\zeta}}(\cdot)$

$$\begin{array}{lll} P_{F^{\zeta}}\{\zeta = [\,\bar{z}_{1, j_1}, \ldots \bar{z}_{s, j_s}\,]\} = & q^m_{j_1, j_2, \ldots, j_s}, & \\ \bar{z}_{i, j_i} = z_{i, j_i}, & j_i = 1, 2, \ldots, m-1, & i = 1, \ldots, s, \\ \bar{z}_{i, j_m} \in (z_{i, m-1}, +\infty) & \text{arbitrary given}, & i = 1, \ldots, s \end{array} \tag{10}$$

and if we denote the corresponding distribution function by the symbol $G^N(\cdot)$, $N = m^s$, then the following assertion follows from [22, Lemma 2].

**Proposition 2.** Let $X$ be a nonempty compact set, $P_{F^{\xi}_i}(\cdot) \in \mathcal{M}_1(R^1)$, $i = 1, \ldots, s$. Let, moreover $m$ be a natural number. If

1. $g(x, z)$ is a uniformly continuous function on $X \times R^s$ and, moreover, for every $x \in X$ a Lipschitz function on $R^s$ with the Lipschitz constant $L_g$ (corresponding to $\mathcal{L}_1$ norm) not depending on $x \in X$,

2. $G^N(\cdot)$, $N = m^s$ is an $s$-dimensional distribution function defined by the probability measure (10),

then

$$|\bar{\varphi}(F^{\xi}) - \bar{\varphi}(G^N)| \le L_g \sum_{i=1}^{s} d^1_{W_1}(F^{\xi}_i, G^m_i), \tag{11}$$

where $G^m_i(\cdot)$, $i = 1, \ldots, s$ denote one-dimensional marginal distribution functions corresponding to $G^N(\cdot)$.

Proof. Since it follows from [22, Lemma 2] that under the assumptions

$$|\mathsf{E}_{F^{\xi}} g(x, \xi) - \mathsf{E}_{G^N} g(x, \xi)| \le L_g \sum_{i=1}^{s} d^1_{W_1}(F^{\xi}_i, G^m_i) \quad \text{for every } x \in X$$

and since $X$ is a compact set the assertion of Proposition 2 is valid. □

Furthermore, it follows from the relation (11) and from the results of [24] that

$$|\bar{\varphi}(F^{\xi}) - \bar{\varphi}(G^N)| \le L_g \sum_{i=1}^{s} \int_{R^1} |F^{\xi}_i(z_i) - G^m_i(z_i)| \, \mathrm{d}z_i. \tag{12}$$

Employing the assertion of Proposition 2 and the relation (12) we can introduce a suitable approximative solution scheme. To this end, we follow [22]. First, according to [21] we can see that, under the assumptions of Proposition 2 for every $\varepsilon > 0$ there exist $\bar{a}_i < (F^{\xi}_i)^{-1}(\frac{1}{2}) < \bar{b}_i$, $i = 1, \ldots, s$ such that

$$\int_{-\infty}^{\bar{a}_i} F^{\xi}_i(z_i) \, \mathrm{d}z_i < \frac{\varepsilon}{3}, \quad \int_{\bar{b}_i}^{\infty} [1 - F^{\xi}_i(z_i)] \, \mathrm{d}z_i < \frac{\varepsilon}{3}, \; i = 1, \ldots, s.$$

Furthermore, let a symbol $(F^{\xi}_i)^{-1}(\cdot)$, $i = 1, \ldots, s$ denote a quantile function corresponding to $F^{\xi}_i(\cdot)$ (for the definition see e.g. [1]). We can for every natural $m$ and every $i \in \{1, \ldots, s\}$ define points $z_{i,j} \in R^1$, $j = 1, \ldots, m-1$ by

$$\begin{aligned} z_{i,j} &= (F^{\xi}_i)^{-1}(\frac{j}{m}), \quad j = 1, \ldots, m-1, \, j \le mF(\bar{b}_i), \\ \bar{z}_{i,j} &= \bar{b}_i \quad \text{for} \quad j \in (mF(\bar{b}_i), m) \quad \text{and} \quad j = m. \end{aligned} \tag{13}$$

Since it follows from (12) (for more details see [22]) that there exists $m_0 := m_0(\varepsilon)$ such that for every $m > m_0$

$$\int_{\bar{a}i}^{\bar{b}_i} |F^{\xi}_i(z_i) - G^m_i(z_i)| \mathrm{d}z_i < \frac{\varepsilon}{3}, \quad i = 1, 2, \ldots, s$$

and since

$$\begin{aligned} &\int_{R^1} |F^{\xi}_i(z_i) - G^m_i(z_i)| \, \mathrm{d}z_i \\ &\le \int_{-\infty}^{\bar{a}_i} F^{\xi}_i(z_i) \, \mathrm{d}z_i + \int_{\bar{a}i}^{\bar{b}_i} |F^{\xi}_i(z_i) - G^m_i(z_i)| \, \mathrm{d}z_i + \int_{\bar{b}_i}^{\infty} [1 - F^{\xi}_i(z_i)] \, \mathrm{d}z_i, \end{aligned} \tag{14}$$

we can see that

$$\int_{R^1} |F_i^{\xi}(z_i) - G_i^m(z_i)|\,\mathrm{d}z_i \le \varepsilon \quad \text{for every} \quad m > m_0.$$

Employing the last relation and the relation (12) we obtain

**Corollary 1.** Let $X$ be a nonempty compact set, $P_{F_i^{\xi}}(\cdot) \in \mathcal{M}_1(R^1)$, $i = 1, \ldots, s$. Let moreover, the assumption 1 of Proposition 2 be fulfilled. If $G^N(\cdot)$, $N = m^s$, $m = 1, 2, \ldots$ are defined by (10) with (13), then

$$\lim_{m \to +\infty} |\bar{\varphi}(F^{\xi}) - \bar{\varphi}(G^N)| = 0.$$

**Corollary 2.** Let $X$ be a nonempty compact set, $P_{F_i^{\xi}}(\cdot) \in \mathcal{M}_r(R^1)$, $i = 1, \ldots, s$ for some $r > 1$. Let, moreover, the assumption 1 of Proposition 2 be fulfilled, then there exists $G^N(\cdot)$, $N = m^s$, $m = 1, 2, \ldots$ such that

$$|\bar{\varphi}(F) - \bar{\varphi}(G^N)| = O(m^{-1+\frac{1}{r}}).$$

(For every $m$, $G^N(\cdot)$ is defined by (10) with (13) and $a_i = -m^{\frac{1}{r}}$, $b_i = m^{\frac{1}{r}}$.)

Proof. Let the assumptions of Corollary 2 be fulfilled. It was proven in [21] that then $F_i^{\xi}(z_i) = o(|z_i|^{-r})$ as $z_i \to -\infty$, $i = 1, \ldots, s$. Consequently, we can see that there exists $z_i^0 \in R^1$ such that for $-\infty < z_i < z_i^0$

$$F_i^{\xi}(z_i) \le |z_i|^{-r}, \quad \int_{-\infty}^{z_i} F_i^{\xi}(t)\mathrm{d}t \le \int_{-\infty}^{z_i} |t|^{-r}\mathrm{d}t = \frac{1}{|1-r|}|z_i|^{1-r}, \; i = 1, \ldots, s.$$

Consequently, there has to exist $m_0$ and a constant $c_r^1 \ge 0$ such that for $m > m_0$, $z_1(m) = -m^{\frac{1}{r}}$ we can obtain

$$\int_{-\infty}^{z_1(m)} F_i^{\xi}(t)\mathrm{d}t \le c_r^1 m^{\frac{(1-r)}{r}} = c_r^1 m^{-1+\frac{1}{r}}, \quad i = 1, \ldots, s.$$

Evidently, by a very similar way we can see that there exist $m^0$, a point $z^m(m)$, and a constant $c_r^2 \ge 0$ such that for $m > m^0$ we can obtain

$$\int_{z_1(m)}^{\infty} [1 - F_i^{\xi}(t)]\,\mathrm{d}t \le c_r^2 m^{\frac{(1-r)}{r}} = c_r^2 m^{-1+\frac{1}{r}}.$$

Since, furthermore, there can be found $m_0$ and a constant $c_r^0 \ge 0$ such that

$$\int_{z_1(m)}^{z_m(m)} |F(t) - G^m(t)|\,\mathrm{d}t \le c_r^0 m^{-1+\frac{1}{r}} \quad \text{for every} \quad m > m_0$$

we can see that the assertion of Corollary 2 is valid. □

If furthermore we can assume that

A.1 $P_{F_i^\xi}(\cdot)$, $i = 1, 2, \ldots, s$ are absolutely continuous with respect to one-dimensional Lebesgue measure. We denote by $f_i^\xi(\cdot)$ the probability density corresponding to $F_i^\xi(\cdot)$,

A.2 there exists constants $C_1 > 0$, $C_2 > 0$ and $T_i > 0$, $i = 1, 2, \ldots, s$ such that

$$f_i^\xi(z_i) \leq C_1 \exp\{-C_2|z_i|\} \quad \text{for} \quad z_i \notin \langle -T_i, T_i \rangle,$$

then the following assertion follows immediately from Corollary 2.

**Corollary 3.** Let the assumptions of Corollary 2 be fulfilled. If, moreover, the assumptions A.1 and A.2 are fulfilled, then for an arbitrary $c > 0$ it holds that

$$\lim_{m \to +\infty} m^{1-c} |\bar{\varphi}(F^\xi) - \bar{\varphi}(G^N)| = 0, \quad N = m^s.$$

Proof. The assertion of Corollary 3 follows from the assertion of Corollary 2 and the fact that under the assumptions A.1, A.2 the corresponding probability measure belongs to the class $\mathcal{M}_r^s$ for every $r > 1$. □

### 3.2. Empirical estimates and approximation

It happens rather often that the theoretical probability measure has to be replaced by empirical one. To recall well-known results on empirical estimates we assume.

i.1 $\{\zeta^k\}_{k=-\infty}^{\infty}$ is a sequence of $s$-dimensional independent random vectors with a common distribution function $F^\xi(\cdot)$; we denote by $F_N^\xi(\cdot)$ the empirical distribution function determined by $\{\zeta^k\}_{k=1}^N$,

i.2 there exist $a_i^{'}, b_i^{'} \in R^1$, $a_i^{'} \leq b_i^{'}$ such that $X \subset X^{'} = \prod_{i=1}^n \langle a_i^{'}, b_i^{'} \rangle$.

**Proposition 3.** [7] Let $X$ be a nonempty compact set, $t > 0$ be arbitrary, i.1 and i.2 be fulfilled. If

1. $g(x, z)$ is a uniformly continuous, bounded function on $X^{'} \times Z_{F^\xi}$,
2. for every $z \in Z_{F^\xi}$, $g(x, z)$ is a Lipschitz function on $X^{'}$ with the Lipschitz constant $L_g$ (corresponding to $\mathcal{L}_2$ norm) not depending on $z \in Z_{F^\xi}$,

then there exist constants $K(X^{'}, t)$, $k > 0$ such that

$$P\{|\bar{\varphi}(F^\xi) - \bar{\varphi}(F_N^\xi)| > t\} \leq K(X^{'}, t) \exp\{-kNt^2\}, \quad N = 1, 2, \ldots$$

**Corollary 4.** [9] Let the assumptions of Proposition 3 be fulfilled. If $v \in (0, \frac{1}{2})$, then

$$P\{N^v |\bar{\varphi}(F^\xi) - \bar{\varphi}(F_N^\xi)| > t\} \to_{(N \to +\infty)} 0.$$

It happens rather often that there exist a natural number $s_1$ and functions $h_i^*(z)$, $g_i^*(x)$, $i = 1, 2, \ldots, s_1$ defined on $R^n$ and $R^s$ fulfilling the relation

$$g(x, z) = \sum_{i=1}^{s_1} h_i^*(z) g_i^*(x), \; x \in R^n, \; z \in R^s. \tag{15}$$

**Proposition 4.** Let $X$ be a nonempty compact set, $t > 0$ and i.1 be fulfilled. If

1. $g(x, z)$ fulfills (23) with continuous and bounded $g_i^*(x)$, $i = 1, 2, \ldots, s_1$,
2. for every $i \in \{1, \ldots, s_1\}$ there exists $\theta_0(i) > 0$ and a finite $\mathsf{E}_{F^\xi} \exp\{\theta\, h_i^*(\xi)\}$ for all $0 \le \theta \le \theta_0(i)$,

then there exist constant $\beta := \beta(t) > 0$ such that

$$P\{|\bar{\varphi}(F^\xi) - \bar{\varphi}(F_N^\xi)| > t\} \le 2 s_1 \exp\{-\beta(t) N\}, \quad N = 1, 2, \ldots$$

Proof. Let $t > 0$ be arbitrary, $M(i) = \sup_{x \in X} |g_i^*(x)|$, $i = 1, \ldots, s_1$. First, it follows from the assumptions of Proposition 4 that

$$\begin{aligned} & P\{|\mathsf{E}_{F^\xi} g(x, \xi) - \mathsf{E}_{F_N^\xi} g(x, \xi)| > t \quad \text{for at least one } x \in X\} \\ & \le \sum_{i=1}^{s_1} P\{|\mathsf{E}_{F^\xi} h_i^*(\xi) - \mathsf{E}_{F_N^\xi} h_i^*(\xi)| > \frac{t}{s_1\, M(i)}\} \end{aligned} \tag{16}$$

and, simultaneously, there exists a finite

$$\mathsf{E}_{F^\xi} \exp\{\theta\, (h_i^*(\xi) - \mathsf{E}_{F^\xi} h_i^*(\xi))\} \qquad \text{far all} \quad 0 \le \theta \le \theta_0(i).$$

Following the first part of the proof of Theorem 3.1 in [2] we can see that (under the assumptions of Proposition 4) there exists a constant $\beta_i(\frac{t}{s_1 M(i)}) > 0$ such that

$$P\{|\mathsf{E}_{F^\xi} h_i^*(\xi) - \mathsf{E}_{F_N^\xi} h_i^*(\xi)| > t\} \le 2 \exp\{-\beta_i(\frac{t}{s_1 M(i)}) N\},\ i = 1, \ldots, s_1,\ N = 1, \ldots$$

Setting $\beta(t) := \min_{i \in \{1, \ldots, s_1\}} \beta_i(\frac{t}{s_1 M(i)})$ and employing the relation (16) we can see that the assertion of Proposition 4 is valid. □

It follows from the proof of Theorem 3.1 in [2] that $\beta_i(t)$ can be taken such that

$$\begin{aligned} & \beta_i(t) = -\ln\left\{1 - \tfrac{t^2}{2}\, \tfrac{\theta_0^2}{16 b_i^2} \exp\left\{-\theta_0^2 \tfrac{t^2}{4}\right\} + \tfrac{t}{4} \exp\{-t)\}\right\},\ b_i = \mathsf{E}_{F^\xi} \exp \theta_0 h_i^*(\xi) \\ & i = 1, \ldots, s_1. \end{aligned} \tag{17}$$

Employing the approach of [7] we can prove also the next assertion.

**Proposition 5.** Let $X$ be a nonempty compact set, $t > 0$, assumptions i.1 and i.2 be fulfilled. If

1. $g(x, z)$ is a uniformly continuous function on $X' \times Z_{F^\xi}$,
2. there exist $a > 0$, $\theta_0 > 0$ and a real-valued function $\nu(\cdot)$ such that

$$|g(x, z)| \le a \nu(z), \quad \mathsf{E}_{F^\xi} \exp\{\theta \nu(\xi)\} < \infty \quad \text{for all } x \in X' \quad \text{and all } 0 \le \theta \le \theta_0,$$

3. the assumption 2 of Proposition 3 is fulfilled,

then there exist constants $\beta := \beta(t) > 0, \ \alpha(t) > 0; \ \alpha(t) := \alpha(t, X)$ such that

$$P\{|\bar{\varphi}(F^{\xi}) - \bar{\varphi}(F^{\xi}_N)| > t\} \le \alpha(t) \exp\{-\beta(t) N\}, \quad N = 1, 2, \ldots$$

Proof. Since the main idea of the proof of Proposition 5 is very similar to the main idea of the proof of Proposition 3 we omit it. □

## 4. MULTISTAGE PROBLEMS

To apply the assertions of the former section to the multistage case we restrict our consideration to the case when

D.1 There exist $s$-dimensional random vectors $\xi^{-1} := \xi^{-1}(\omega), \ \eta^k := \eta(\omega), \ k = 0, \ldots, M$ defined on $(\Omega, S, P)$ and a continuous, $s$-dimensional vector function $f(\cdot)$ defined on $R^s \times R^s$ such that

a. $\{\eta^k\}_{k=0}^M$ is a sequence of independent $s$-dimensional, identically distributed random vectors,

b. $F^{\xi^0}(z^0) = F^{\eta}(f(z^0, z^{-1}))$ for every $z^0 \in R^s$ and a (known value) $\xi^{-1} := z^{-1} \in R^s$,

c. for every $\bar{z}^{k-1} \in \bar{Z}^{k-1}_{\mathcal{F}}, \ z^k \in R^s, \ F^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1}) = F^{\eta}(f(z^k, z^{k-1}))$, $k = 1, \ldots M$.

We denote by $F^{\eta}(\cdot)$, $P_{F^{\eta}}(\cdot)$ and $Z_{F^{\eta}}$ the distribution function, probability measure and the support corresponding to $\eta^0$ (consequently also $\eta^1, \ldots, \eta^M$). Evidently, $F^{\eta}(\cdot)$ corresponding to D.1 determines the system (3) $\mathcal{F} := \mathcal{F}^{\eta}$ by

$$\begin{aligned} F^{\xi^0}(z^0) &= F^{\eta}(f(z^0, z^{-1})) \quad \text{for every } z^0 \in R^s \quad \text{and known } z^{-1} \in R^s, \\ F^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1}) &= F^{\eta}(f(z^k, z^{k-1})) \quad \text{for every } \bar{z}^k \in R^{s(k+1)}, \ k = 1, \ldots, M. \end{aligned} \tag{18}$$

Of course, every other $s$-dimensional distribution function $G^{\eta}(\cdot)$ determines another system (5) $\mathcal{G} := \mathcal{G}^{\eta}$. Sometimes it is "suitable" to assume furthermore.

D.2 For every $\bar{z}^k \in \bar{Z}^k_{\mathcal{F}}, \ u \in Z_{F^{\eta}}$ there exists just one $z^{k+1} \in Z^{k+1}_{\mathcal{F}}$ fulfilling the relations $u = f(z^{k+1}, z^k), \ k = 0, 1, \ldots, M-1$.

D.3 There exist an $s$-dimensional random sequence $\{\eta^k\}_{k=-\infty}^{\infty}$ and a deterministic nonsingular matrix $A$ of the type $(s \times s)$ such that

a. $\{\eta^k\}_{k=-\infty}^{\infty}$ is a sequence of independent $s$-dimensional, identically distributed random vectors,

b. $\xi^k = A\xi^{k-1} + \eta^k, \quad k = \ldots, -1, 0, 1, \ldots$; the value $\xi^{-1} = z^{-1}, \ z^{-1} \in R^s$ is known,

c. $\xi^{k-1}, \eta^k, \ k = \ldots, -1, 0, 1, \ldots$ are stochastically independent.

### 4.1. Multistage analysis

Employing the triangular inequality and the technique used in [12] we can see that (under the assumptions D.1 and relation (4)) for every $x^0 \in \mathcal{K}^0$

$$
\begin{aligned}
&|\mathsf{E}_{F^{\xi^0}}\, g^0_{\mathcal{F}}(x^0,\, \xi^0) - \mathsf{E}_{G^{\xi^0}}\, g^0_{G}(x^0,\, \xi^0)| \\
&\leq\ |\mathsf{E}_{F^{\xi^0}} \inf_{x^1 \in \bar{\mathcal{K}}^1(x^0,\xi^0)} \mathsf{E}_{F^{\xi^1|\xi^0}}\, g^1_{\mathcal{F}}(\bar{x}^1,\, \bar{\xi}^1)\ -\ \mathsf{E}_{G^{\xi^0}} \inf_{x^1 \in \bar{\mathcal{K}}^1(x^0,\xi^0)} \mathsf{E}_{F^{\xi^1|\xi^0}}\, g^1_{\mathcal{F}}(\bar{x}^1,\, \bar{\xi}^1)| \\
&+\ |\mathsf{E}_{G^{\xi^0}} \inf_{x^1 \in \bar{\mathcal{K}}^1(x^0,\xi^0)} \mathsf{E}_{F^{\xi^1|\xi^0}} \inf_{x^2 \in \bar{\mathcal{K}}^2(\bar{x}^1,\bar{\xi}^1)} \mathsf{E}_{F^{\xi^2|\xi^1}}\, g^2_{\mathcal{F}}(\bar{x}^2,\, \bar{\xi}^2) \\
&-\ \mathsf{E}_{G^{\xi^0}} \inf_{x^1 \in \bar{\mathcal{K}}^1(x^0,\xi^0)} \mathsf{E}_{G^{\xi^1|\xi^0}} \inf_{x^2 \in \bar{\mathcal{K}}^2(\bar{x}^1,\bar{\xi}^1)} \mathsf{E}_{F^{\xi^2|\xi^1}}\, g^2_{\mathcal{F}}(\bar{x}^2,\, \bar{\xi}^2)| \\
&+|\mathsf{E}_{G^{\xi^0}} \inf_{x^1 \in \bar{\mathcal{K}}^1(x^0,\,\xi^0)} \mathsf{E}_{G^{\xi^1|\xi^0}} \inf_{x^2 \in \bar{\mathcal{K}}^2(\bar{x}^1,\bar{\xi}^1)} \mathsf{E}_{F^{\xi^2|\xi^1}} \inf_{x^3 \in \bar{\mathcal{K}}^3(\bar{x}^2,\bar{\xi}^2)} \mathsf{E}_{F^{\xi^3|\xi^2}}\, g^3_{\mathcal{F}}(\bar{x}^3,\, \bar{\xi}^3) \\
&-\ \mathsf{E}_{G^{\xi^0}} \inf_{x^1 \in \bar{\mathcal{K}}^1(x^0,\xi^0)} \mathsf{E}_{G^{\xi^1|\xi^0}} \inf_{x^2 \in \bar{\mathcal{K}}^2(\bar{x}^1,\bar{\xi}^1)} \mathsf{E}_{G^{\xi^2|\xi^1}} \inf_{x^3 \in \bar{\mathcal{K}}^3(\bar{x}^2,\bar{\xi}^2)} \mathsf{E}_{F^{\xi^3|\xi^2}}\, g^3_{\mathcal{F}}(\bar{x}^3,\, \bar{\xi}^3)| \\
&\ \vdots \\
&+\ |E_{G^{\xi^0}} \inf_{x^1 \in \bar{\mathcal{K}}^1(x^0,\xi^0)} \cdots\cdots \\
&\quad \inf_{x^{M-1} \in \bar{\mathcal{K}}^{M-1}(\bar{x}^{M-2},\bar{\xi}^{M-2})} E_{G^{\xi^{M-1}|\xi^{M-2}}} \inf_{x^M \in \bar{\mathcal{K}}^M(\bar{x}^{M-1},\bar{\xi}^{M-1})} E_{F^{\xi^M|\xi^{M-1}}}\, g^M_{\mathcal{F}}(\bar{x}^M,\, \bar{\xi}^M) \\
&-\ E_{G^{\xi^0}} \inf_{x^1 \in \bar{\mathcal{K}}^1(x^0,\xi^0)} \cdots\cdots \\
&\quad \inf_{x^{M-1} \in \bar{\mathcal{K}}^{M-1}(\bar{x}^{M-2},\bar{\xi}^{M-2})} E_{G^{\xi^{M-1}|\xi^{M-2}}} \inf_{x^M \in \bar{\mathcal{K}}^M(\bar{x}^{M-1},\bar{\xi}^{M-1})} E_{G^{\xi^M|\xi^{M-1}}}\, g^M_{G}(\bar{x}^M,\, \bar{\xi}^M)|.
\end{aligned}
\tag{19}
$$

Consequently the "value" $|\varphi_{\mathcal{F}^\eta}(M) - \varphi_{G^\eta}(M)|$ can be estimated by a suitable distance between $F^\eta$, $G^\eta$ (for more details see e.g. [12, 13, 14]).

### 4.2. "Deterministic" approximation

Let $m$ be an arbitrary natural number. It was proven in Section 3 that for every $s$-dimensional distribution function $F^\eta(\cdot)$ (with one-dimensional marginals $F^\eta_i(\cdot)$, $i = 1, \ldots, s$ fulfilling the assumptions A.1, A.2 for $\xi := \eta$) there exist points $u_{i,j} \in R^1$, $j = 1, \ldots, m$, $i = 1, \ldots s$ and one-dimensional jump distribution functions $G^{\eta,m}_i(\cdot)$, $i = 1, 2, \ldots, s$ defined by

$$
\begin{aligned}
&u_{i,j} = (F^\eta_i)^{-1}(\tfrac{j}{m}), \quad j = 1, \ldots, m-1,\ i = 1, \ldots, s, \\
&u_{i,m} = u^m(m) \quad \text{corresponding to} \quad z^m(m) \quad \text{in Corollary 2}
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
G^{\eta,m}_i(u_i) \ &= \ 0 && \text{for} \quad z_i < u_{i,1}, \\
&= \ F^\eta_i(u_{i,j}) && u_i \in (u_{i,j},\, u_{i,j+1}\rangle, \quad j = 1, \ldots, m-1, \\
&= \ 1 && u_i > u_{i,m}.
\end{aligned}
\tag{21}
$$

such that

$$\lim_{m \to +\infty} m^{1-c} \sum_{i=1}^{s} d^1_{W_1}(F^\eta_i,\, G^{\eta,m}_i) \ = \ 0 \quad \text{for arbitrary } c > 0.$$

(The symbol $(F_i^\eta)^{-1}$ denotes the quantile function corresponding to $F_i^\eta(\cdot)$, $i = 1, \dots, s$.) If we denote the corresponding $s$-dimensional distribution function by the symbol $G^{\eta, N}(\cdot)$, $N = m^s$; and by $\bar{\eta}^N$ an $s$-dimensional random vector with the distribution function $G^{\eta, N}(\cdot)$, then evidently under the assumptions D.1 and D.3 the system $\mathcal{F}$ can be approximated by the system $\mathcal{G}^{\eta, N}$ defined by

$$\mathcal{G}^{\eta, N} = \{G^{\eta, N}(u^0 - Az^{-1}),\ G^{\eta, N}(u^k - Az^{k-1}),\ k = 1, \dots, M\}, \tag{22}$$

where $z^{-1}$ is supposed to be known, $z^{k-1}$ corresponds to random vectors $\zeta^{k-1}$ determined recursively by $\zeta^k = A\zeta^{k-1} + \bar{\eta}^k, \quad k = 0, 1, \dots M, \quad \zeta^{-1} = z^{-1}$.

To present the corresponding assertion dealing with an approximation error we introduce the following system of the assumptions.

W.1 a. $g_{\mathcal{F}}^0(x^0, z^0)$ is uniformly continuous function on $\mathcal{K}^0 \times R^s$ and, moreover, for every $x^0 \in \mathcal{K}^0$ a Lipschitz function on $R^s$ with the Lipschitz constant (corresponding to $\mathcal{L}_1$ norm) not depending on $x^0 \in \mathcal{K}^0$,

b. for every $k \in \{1, \dots, M\}$, $g_F^k(\bar{x}^k, \bar{z}^k)$ is a uniformly continuous function on $\bar{\mathcal{X}}^k \times \bar{Z}_{\mathcal{F}}^k$ and, moreover for every $\bar{x}^k \in \bar{\mathcal{X}}^k$, $\bar{z}^{k-1} \in \bar{Z}_{\mathcal{F}}^{k-1}$, $g_F^k(\bar{x}^k, \bar{z}^k)$ is a Lipschitz function on $R^s$ with the Lipschitz constant (corresponding to $\mathcal{L}_1$ norm) not depending on $\bar{x}^k \in \bar{\mathcal{X}}^k$, $\bar{z}^{k-1} \in \bar{Z}_{\mathcal{F}}^{k-1}$,

B.1 the probability measures $P_{F_i^\eta}(\cdot)$, $i = 1, 2, \dots, s$ are absolutely continuous with respect to one-dimensional Lebesgue measure. We denote by the symbol $f_i^\eta(\cdot)$ the probability density corresponding to $F_i^\eta(\cdot)$,

B.2 there exist constants $\bar{C}_1 > 0$, $\bar{C}_2 > 0$ and $T_i > 0$, $i = 1, 2, \dots, s$ such that

$$f_i^\eta(z_i) \le \bar{C}_1 \exp\{-\bar{C}_2 |z_i|\} \quad \text{for} \quad z_i \notin \langle -T_i, T_i \rangle,$$

**Theorem 1.** Let the relations (4), (21) be fulfilled. If

1. the assumptions D.2, D.3 and W.1 are fulfilled and, moreover, $P_{F^\eta} \in \mathcal{M}_1(R^s)$,
2. the system $\mathcal{G}^{\eta, N}$ is defined by the relation (22), $N = m^s$, $m = 1, 2, \dots$,
3. $\mathcal{K}^0$, $\bar{\mathcal{K}}^{k+1}(\bar{x}^k, \bar{z}^k)$, $k = 0, \dots, M-1$, $\bar{x}^k \in \bar{\mathcal{X}}^k$, $\bar{z}^k \in \bar{Z}_{\mathcal{F}}^k$, $k = 0, \dots, M-1$ are nonempty compact sets,

then there exists a constant $C_{W_1} > 0$ such that

$$|\varphi_{\mathcal{F}}(M) - \varphi_{\mathcal{G}^{\eta, N}}(M)| \le C_{W_1} \sum_{i=1}^{s} d_{W_1}^1(F_i^\eta, G_i^{\eta, N}).$$

If, moreover, the assumptions B.1, B.2 are fulfilled, then also

$$\lim_{m \to +\infty} m^{1-c} |\varphi_{\mathcal{F}}(M) - \varphi_{\mathcal{G}^{\eta, N}}(M)| = 0, \ N = m^s \quad \text{for an arbitrary} \quad c > 0.$$

Proof. The proof of Theorem 1 follows from Proposition 2, Corollary 3 and the relations (19), (22) (for more details see [14]). □

### 4.3. Empirical approximation

If $F_N^{\eta}(\cdot)$ denotes the empirical distribution function determined by an independent random sample $\{\eta^i(\omega)\}_{i=-N}^{-1}$ corresponding to $F^{\eta}(\cdot)$, then

$$F_N^{\xi^0}(z^0) = F_N^{\eta}(f(z^0, z^{-1})), \; F_N^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1}) = F_N^{\eta}(f(z^k, z^{k-1})), \; k = 1, \dots, M$$

are (for every $\bar{z}^{k-1} \in \bar{Z}_{\mathcal{F}}^{k-1}$) statistical estimates of $F^{\xi^0}(z^0)$, $F^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1})$. Consequently, (under the assumptions D.1) a sequence $\{\eta^i(\omega)\}_{i=-N}^{0}$ determines an statistical estimate $\mathcal{F}(N)$ of the system (3) by

$$\mathcal{F}(N) = \{F_N^{\eta}(f(z^0, z^{-1})), \; F_N^{\eta}(f(z^k, z^{k-1})), \; k = 1, \dots, M\}, \quad N = 1, \dots . \tag{23}$$

Replacing the system (3) by the system $\mathcal{F}(N)$ we obtain an approximating problem to the multistage problem introduced by (1) and (2). The optimal value of this problem is a random variable denoted by $\varphi_{\mathcal{F}(N)}(M)$. The following Theorem 2 is a special case of Theorem 2 and Corollary in [13].

**Theorem 2.** [13] Let $t > 0$ be arbitrary, the relation (4) be fulfilled. Let, moreover, the system $\mathcal{F}(N)$, $N = 1, \dots$ be defined by the relation (25). If

1. there exist $a_i^{''} \le b_i^{''}$, $a_i^{''}$, $b_i^{''} \in R^1$, $i = 1, \dots, s$ such that $\mathcal{X} = \prod_{i=1}^{n} \langle a_i^{''}, b_i^{''} \rangle$,
2. $g_{\mathcal{F}}^k(\bar{x}^k, \bar{z}^k)$, $k = 0, 1, \dots, M$ are uniformly continuous, bounded on $\bar{\mathcal{X}}^k \times \bar{Z}_{\mathcal{F}}^k$,
3. for every $\bar{z}^k \in \bar{Z}_{\mathcal{F}}^k$, $g_{\mathcal{F}}^k(\bar{x}^k, \bar{z}^k)$, $k = 0, 1, \dots, M$ are Lipschitz functions on $\bar{\mathcal{X}}^k$ with the Lipschitz constants (corresponding on $\mathcal{L}_2$ norm) not depending on $\bar{z}^k \in \bar{Z}_{\mathcal{F}}^k$,
4. $\mathcal{K}^0$, $\bar{\mathcal{K}}^{k+1}(\bar{x}^k, \bar{z}^k)$, $k = 0, \dots, M-1$, $\bar{x}^k \in \bar{\mathcal{X}}^k$, $\bar{z}^k \in \bar{Z}_{\mathcal{F}}^k$, $k = 0, \dots, M-1$ are nonempty compact sets,
5. the systems of the assumptions D.1, D.2 are fulfilled,

then there exist constants $\bar{k} > 0, K^k(\mathcal{X}, t) > 0$, $k = 0, 1, \dots, M$ such that

a. $P\{|\varphi_{\mathcal{F}}(M) - \varphi_{\mathcal{F}(N)}(M)| > t\} \le \exp\{-Nt^2\bar{k}\} \sum_{k=0}^{M-1} K^k(\mathcal{X}, t)$, $N = 1, 2, \dots$

b. $P\{N^v|\varphi_{\mathcal{F}}(M) - \varphi_{\mathcal{F}(N)}(M)| > t\} \to_{(N \to +\infty)} 0 \qquad$ for every $v \in (0, \frac{1}{2})$.

Proof. The proof of Theorem 2 follows from Proposition 3, Corollary 4 and the relations (19), (23) (for more details see [14]). □

To obtain the assertions of Theorem 2 it was supposed $g_0(\bar{x}^M, \bar{z}^M)$ to be bounded. Following the approach used in [2] and Proposition 5 we can obtain.

**Theorem 3.** Let $t > 0$ be arbitrary, the relations (4) be fulfilled. Let, moreover, the system $\mathcal{F}(N)$, $N = 1, \dots$ be defined by the relation (23). If

1. the assumptions 1, 3, 4 and 5 of Theorem 2 are fulfilled,
2. $g_F^k(\bar{x}^k, \bar{z}^k)$, $k = 0, 1, \dots, M$ are uniformly continuous on $\bar{\mathcal{X}}^k \times \bar{Z}_F^k$,
3. for every $k = 0, \dots, M$ there exist constants $a^k > 0$, $\theta_0^k > 0$ and a real-valued function $\nu^k(\cdot)$ such that

   $$|g_F^k(\bar{x}^k, \bar{z}^k)| \le a^k \nu^k(z^k), \quad \mathsf{E}_{F_{\eta^k}} \exp\{\theta \nu^k(\xi^k)\} < \infty \quad \text{for all } 0 \le \theta \le \theta_0^k,$$

   and every $\bar{x}^k \in \bar{\mathcal{X}}^k$, $\bar{z}^{k-1} \in \bar{Z}_{\mathcal{F}}^{k-1}$,
4. $\{\xi^k\}_{k=-\infty}^{+\infty}$ is a sequence of independent random vectors and, moreover, the system $\mathcal{F}(N)$ is determined by independent random sample $\{\xi^k\}_{k=-N}^{0}$.

then there exists a constants $\beta(t)$, $K^k(\mathcal{X}, t) > 0$, $k = 0, 1, \dots, M$ such that

$$P\{|\varphi_{\mathcal{F}}(M) - \varphi_{\mathcal{F}(N)}(M)| > t\} \le \exp\{-\beta(t)N\} \sum_{k=0}^{M} K^k(\mathcal{X}, t), \quad N = 1, 2, \dots$$

Proof. The proof of Theorem 3 follows from Proposition 5 and the relations (19), (23) (for more details see [14]). □

Of course, to apply the stability and empirical estimates results achieved for one-stage problems to the multistage case the corresponding assumptions must be verified. Evidently, the uniform continuity and the Lipschitz property are the crucial assumptions. For more information see [11] or [14].

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