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NEARNESS RELATIONS IN LINEAR SPACES[1]

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In this paper, we consider nearness-based convergence in a linear space, where the coordinatewise given nearness relations are aggregated using weighted pseudo-arithmetic and geometric means and using continuous t-norms.

Keywords: nearness relation, pseudo-arithmetic mean, geometric mean, nearness-convergence, continuous t-norm

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1. INTRODUCTION AND BASIC NOTIONS

Relations of fuzzy nearness on the real line were introduced as a tool for a kind of fuzzy differential calculus in [6]. This concept was further developed in [4, 5, 7]. The motivation for introducing the notion of nearness relation was the fact that, roughly speaking, the expression

$$\frac{f(z) - f(x)}{z - x}$$

can be regarded as a 'fuzzy derivative' of f at x and it is as good (or better as near to the derivative itself) as near points x and z are to each other, i. e. the key notion is the 'nearness'. Hence we need some formalization of it.

The topological properties of fuzzy nearness relations on metric spaces were studied by Dobrakovová [2, 3]. In 2002, a paper by Kalina and Dobrakovová [8] was published generalizing '*nearness relations*' to Banach spaces, where the coordinatewise given nearnesses were aggregated by t-norms. Some topological properties of such nearness relations were investigated in that paper.

Now, we will continue investigating '*nearness-based*' convergences in the linear space of all sequences. The coordinatewise given nearness relations will be aggregated using various particular cases of weighted pseudo-arithmetic means as well as the geometric mean and also using continuous t-norms. I. e., the aggregation operators we will consider can be generalized also to the case of countably many inputs. That is why aggregation operators will be defined by the following:

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Definition 1. An aggregation operator $\mathcal{A}$ is a function $\mathcal{A} : \bigcup_{n\in\{1,2,\dots\infty\}}[0,1]^n \to [0,1]$, where $[0,1]^\infty$ is an infinite sequence with elements from $[0,1]$, with the following properties

A1. it is non-decreasing in all coordinates

A2. if all inputs equal to 1 then the output is 1

A3. if all inputs are 0 then the output is 0.

Definition 2. We say that a is the annihilator of an aggregation operator $\mathcal{A}$ if the output of the aggregation is a in case that at least one of the inputs is a.

For basic properties of such aggregation operators see e. g. [1, 11, 12, 13]. For basic properties of t-norms see [10].

Definition 3. Let X be a linear space. We say that $\mathcal{N} : X \times X \to [0;1]$ is a relation of weak nearness if and only if the following hold

1. $\mathcal{N}(x,x) = 1$ for any $x \in X$ (reflexivity);
2. $\mathcal{N}(x,y) = \mathcal{N}(y,x)$ for all $x, y \in X$ (symmetry);
3. for all $x, y \in X$, $y \neq \emptyset$ and all $t_1, t_2, t_3, t_4 \in \Re$ such that $t_1 \leq t_2 \leq t_3 \leq t_4$ the following holds

$$\mathcal{N}(x + t_1 y, x + t_4 y) \leq \mathcal{N}(x + t_2 y, x + t_3 y).$$

We say that $\mathcal{N}$ is a relation of nearness if and only if moreover

4. for all $x, y \in X$, $y \neq \emptyset$, the following holds

$$\lim_{t\to\infty} \mathcal{N}(x, x + ty) = 0.$$

If X is an n-dimensional vector space, we say that $\mathcal{N}$ is an n-dimensional nearness.

An important role will be played by special (weak) nearness relations, so-called strict ones, which are defined as follows

Definition 4. Let $\mathcal{N} : X \times X \to [0;1]$ be a weak nearness relation. It is called strict if and only if the following holds for all $x, y \in X$

$$\mathcal{N}(x,y) = 1 \quad \Leftrightarrow \quad x = y$$

i. e., if $\mathcal{N}$ has the separation property.

Lemma 1. Let $\{\mathcal{N}_i\}_i$ be a sequence (either finite or countable) of one-dimensional nearness relations and $\mathcal{A}$ be an aggregation operator. Then $\mathcal{M} : X \times X \to [0,1]$ defined by

$$\mathcal{M}(u,v) = \mathcal{A}\left(\mathcal{N}_1(u_1,v_1), \mathcal{N}_2(u_2,v_2), \ldots\right)$$

is a weak nearness relation. If moreover $\mathcal{A}$ is right-continuous at 0 in each coordinate and 0 is the annihilator of $\mathcal{A}$, then $\mathcal{M}$ is a nearness relation, i.e. the property 4 of Definition 2 holds for $\mathcal{M}$.

Proof. We have to prove properties 1 – 3 from Definition 2. The first property, the reflexivity of $\mathcal{M}$, is implied by the property A2 of aggregation operators and by the reflexivity of relations $\mathcal{N}_i$. The second property, the symmetry of $\mathcal{M}$, is implied by the symmetry of the nearness relations $\mathcal{N}_i$. The third property of $\mathcal{M}$ is implied by the monotonicity of aggregation operators, i.e. by the property A1.

Choose some points $u, v \in X$, $v \neq \emptyset$. Then there exists at least one coordinate i such that $v_i \neq 0$ and hence

$$\lim_{t\to\infty} \mathcal{N}_i(u_i, u_i + tv_i) = 0. \tag{1}$$

Assume that $\mathcal{A}$ is right-continuous at 0 in each coordinate and 0 is the annihilator of $\mathcal{A}$. Then the above properties of $\mathcal{A}$ and formula (1) imply

$$\lim_{t\to\infty} \mathcal{M}(u, u+tv) = 0. \qquad \square$$

Definition 5. (Dobrakovová [2, 3]) We say that a sequence $\{z_k\}_{k=1}^{\infty}$ of elements of X is $\mathcal{N}$-convergent to $x \in X$ ($\mathcal{N}$ being a (weak) nearness relation on $X \times X$) if and only if, for any $\alpha < 1$, there exists an n_0 such that for all $m > n_0$ there holds $\mathcal{N}(z_m, x) \geq \alpha$.

2. TOPOLOGY OF NEARNESS RELATIONS

2.1. General properties

Definition 6. For any $\alpha \in [0,1[$, $x_0 \in X$ and any nearness relation $\mathcal{N}$ denote

$$\mathcal{O}_{\mathcal{N},\alpha}(x_0) = \{x \in X; \mathcal{N}(x, x_0) > \alpha\}.$$

We will call $\mathcal{O}_{\mathcal{N},\alpha}(x_0)$ the $(\mathcal{N},\alpha)$-neighbourhood of x_0.

Definition 7. Let $\mathcal{O}_{\mathcal{N}}$ denote the family of all $(\mathcal{N},\alpha)$-neighbourhoods. Then there exists the least topology containing the whole system $\mathcal{O}_{\mathcal{N}}$. This topology we will denote by $\mathcal{T}_{\mathcal{N}}$. We will say that $\mathcal{T}_{\mathcal{N}}$ is generated by the family $\mathcal{O}_{\mathcal{N}}$.

For conditions under which the system $\mathcal{O}_{\mathcal{N}}$ is the basis of $\mathcal{T}_{\mathcal{N}}$ see [2, 3]. We present the next theorem for the sake of completeness. It is a slightly reformulated result by Dobrakovová [3].

Theorem 1. Let $\mathcal{N}$ be a nearness relation on $X \times X$ and $\{z_k\}_{k=1}^{\infty}$ a sequence of elements of X. Then

(a) If $\{z_k\}_{k=1}^{\infty}$ is $\mathcal{T}_{\mathcal{N}}$-convergent, then it is $\mathcal{N}$-convergent.

(b) If $\mathcal{O}_{\mathcal{N}}$ is the basis of the topological space $\mathcal{T}_{\mathcal{N}}$, then $\{z_k\}_{k=1}^{\infty}$ is $\mathcal{T}_{\mathcal{N}}$-convergent if and only if it is $\mathcal{N}$-convergent.

Example 1. If we take the one-dimensional nearness relation on $\Re \times \Re$

$$\mathcal{N}(x,y) = \max\{0; 1 - |x-y|\},$$

then the system of neighbourhoods $\mathcal{O}_{\mathcal{N}}$ is the basis of the Euclidean topology.

Example 2. (Dobrakovová [2, 3]) If we take the one-dimensional nearness relation on $\Re \times \Re$

$$\mathcal{M}(x,y) = \begin{cases} 1 - |x-y| & \text{if } |x-y| \le 0.5 \\ 0 & \text{otherwise,} \end{cases}$$

then $\mathcal{M}$-convergence coincides with that of the Euclidean space. However, the system of neighbourhoods $\mathcal{O}_{\mathcal{M}}$ generates the discrete topology. Namely, if we take $\alpha = 0.4$, then for each $x \in \Re$ there holds

$$\mathcal{O}_{\mathcal{M},0.4}(x) = [x - 0.5, x + 0.5],$$

since $\mathcal{M}(x,y) > 0.4$ if and only if $|x-y| \le 0.5$. Since the point x was chosen arbitrarily, each closed interval of the length 1 is contained in $\mathcal{O}_{\mathcal{M}}$ and hence any singleton is contained in $\mathcal{T}_{\mathcal{M}}$.

Theorem 2. Let $\mathcal{A}$ be an aggregation operator. Let $\{\mathcal{N}_i\}_i$ and $\{\mathcal{M}_i\}_i$ be two sequences of one-dimensional nearness relations such that, for all i and all $x, y \in \Re$, the following holds

$$\mathcal{N}_i(x,y) \ge \mathcal{M}_i(x,y). \tag{2}$$

Denote $\mathcal{A}_N$ and $\mathcal{A}_M$ the (weak) nearness relations defined by aggregating the systems $\{\mathcal{N}_i\}_i$ and $\{\mathcal{M}_i\}_i$, respectively, by $\mathcal{A}$. Then for any sequence $\{z_k\}_k$ of elements of X if $\{z_k\}_k$ is $\mathcal{A}_M$-convergent, then it is $\mathcal{A}_N$-convergent.

Proof. The assertion in question follows immediately from the monotonicity of the aggregation operator $\mathcal{A}$ and from formula (2). Namely, let the sequence $\{z_k\}_k$ be $\mathcal{A}_M$-convergent to $z \in X$. Then

$$\lim_{k\to\infty} \mathcal{A}_{\mathcal{M}}(z_k, z) = 1$$

and by formula (2) for any k

$$\mathcal{A}_{\mathcal{M}}(z_k, z) \le \mathcal{A}_{\mathcal{N}}(z_k, z) \le 1. \qquad \square$$

2.2. Finitely dimensional linear space

In this section we will consider just aggregation operators, $\mathcal{A}$, with the following properties

C1. $\mathcal{A}$ is left-continuous at 1 in each coordinate.

C2. 1 is the output of the aggregation if and only if all inputs equal to 1.

Theorem 3. Let X be an n-dimensional linear space. Let $\{\mathcal{N}_i\}_{i=1}^{n}$ be a system of strict one-dimensional nearness relations such that, for each $x \in \Re$ and each $1 \leq i \leq n$, the following holds

$$\lim_{y \to x} \mathcal{N}_i(x, y) = 1 \tag{3}$$

and let $\mathcal{A}$ be an aggregation operator fulfilling the properties C1 and C2. Denote $\mathcal{E} : X \times X \to [0, 1]$ the following relation

$$\mathcal{E}(u, v) = \mathcal{A}\left(\mathcal{N}_1(u_1, v_1), \mathcal{N}_2(u_2, v_2), \ldots, \mathcal{N}_n(u_n, v_n)\right).$$

Then $\mathcal{E}$ is a weak nearness relation and the $\mathcal{E}$-convergence coincides with the convergence in the n-dimensional Euclidean topological space.

Proof. The fact that $\mathcal{E}$ is a weak nearness relation is due to Lemma 1. Let $\{z_j\}_j$ be a sequence of elements of the space X, converging to some point $u \in X$ in the Euclidean topology. Then for each coordinate i the following holds

$$\lim_{j \to \infty} z_{i,j} = u_i. \tag{4}$$

Hence, formula (3) implies

$$\lim_{j \to \infty} \mathcal{N}_i(z_{i,j}, u_i) = 1. \tag{5}$$

This and the property C1 of the aggregation operator $\mathcal{A}$ imply that the sequence $\{z_j\}_j$ $\mathcal{E}$-converges to u.

On the other hand, let $\{z_j\}_j$ $\mathcal{E}$-converge to u. Then properties C1 and C2 of $\mathcal{A}$ imply that for each coordinate i formula (5) holds. This and formula (3) imply formula (4) for each coordinate and hence we get that $\{z_j\}_j$ converges to $u \in X$ in the Euclidean topology. □

Theorem 4. Let X be an n-dimensional linear space. Let $\{\mathcal{N}_i\}_{i=1}^{n}$ be a system of strict one-dimensional nearness relations such that, for each $x \in \Re$ and each $1 \leq i \leq n$, the following holds

$$\lim_{y \to x} \mathcal{N}_i(x, y) < 1 \tag{6}$$

and let $\mathcal{A}$ be an aggregation operator fulfilling the properties C1 and C2. Denote $\mathcal{D} : X \times X \to [0, 1]$ the following relation

$$\mathcal{D}(u, v) = \mathcal{A}\left(\mathcal{N}_1(u_1, v_1), \mathcal{N}_2(u_2, v_2), \ldots, \mathcal{N}_n(u_n, v_n)\right).$$

Then $\mathcal{D}$ is a weak nearness relation and the $\mathcal{D}$-convergence coincides with the convergence in the discrete n-dimensional topological space.

Proof. The fact that $\mathcal{D}$ is a weak nearness relation is due to Lemma 1. Let $\{z_j\}_j$ be a sequence of elements of the space X, converging to some point $u \in X$ in the discrete topology. Then it is constant beginning with some index j. Hence it is $\mathcal{D}$-convergent.

On the other hand, if $\{z_j\}_j$ is $\mathcal{D}$-convergent, then formula (6) and the properties C1 and C2 of the aggregation operator $\mathcal{A}$ imply that $\{z_j\}_j$ is constant beginning with some index j. Hence it is convergent in the discrete topology. □

3. THE SPACE OF ALL SEQUENCES

Now, the space X will be the space of all sequences. We will have a system of one-dimensional nearness relations, $\{\mathcal{N}_i\}_i$, and the (weak) n-dimensional nearness relation $\mathcal{M} : X \times X \to [0;1]$ will be given by

$$\mathcal{M}(\{x_i\}_i, \{y_i\}_i) = \mathcal{A}(\mathcal{N}_1(x_1;y_1), \mathcal{N}_2(x_2;y_2), \ldots),$$

where $\mathcal{A}$ is an aggregation operator (particularly, various kinds of means, and continuous t-norms). Throughout the whole rest of the paper, as the linear space X, we take the space of all sequences.

3.1. Weighted arithmetic means

In this section, we consider the weighted arithmetic means as the aggregation operator.

Definition 8. Let $\{w_i\}_{i=1}^{\infty}$ be a sequence of positive weights such that $\sum_{i=1}^{\infty} w_i = 1$ and let $\{\mathcal{N}_i\}_{i=1}^{\infty}$ be a sequence of one-dimensional nearness relations. Then the weighted arithmetic mean of the sequence $\{\mathcal{N}_i\}_{i=1}^{\infty}$ is defined by

$$\mathcal{A}\left(\{\mathcal{N}_i(u_i, v_i)\}_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} w_i \cdot \mathcal{N}_i(u_i, v_i)$$

where $\{u_i\}_i$ and $\{v_i\}_i$ are sequences of real numbers.

Obviously, Lemma 1 implies that the above defined weighted arithmetic mean of a system of one-dimensional nearness relations is a weak nearness relation.

Theorem 5. Let $\{w_i\}_i$ be a system of positive weights with their sum equal to 1. Let $\{\mathcal{N}_i\}_i$ be a system of strict one-dimensional nearness relations such that, for any $x \in \Re$,

$$\lim_{y \to x} \mathcal{N}_i(x, y) = 1. \tag{7}$$

Define $\mathcal{E}_\infty$ as the weighted arithmetic mean of the system $\{\mathcal{N}_i\}_i$. Then the $\mathcal{E}_\infty$-convergence coincides with the pointwise convergence of sequences.

P r o o f. Let $\{\{z_{ij}\}_i\}_j$ be a sequence of elements of X, which is pointwise convergent to $\{u_i\}_i \in X$. Choose some $\alpha \in]0,1[$. We are to show that there exists a j_α such that, for all $n > j_\alpha$, $\{z_{in}\}_n$ is contained in the α-neighbourhood of $\{u_i\}_i$, i.e.

$$\{z_{in}\}_n \in \mathcal{O}_{\mathcal{E}_\infty,\alpha}(\{u_i\}_i). \tag{8}$$

Denote $\varepsilon = 1-\alpha$. Then there exists a k_α such that $\sum_{i=1}^{k_\alpha} w_i \geq 1 - \frac{\varepsilon}{2}$. The pointwise convergence of $\{\{z_{ij}\}_i\}_j$ and formula (7) imply that the index j_α can be chosen in such a way that, for all $n > j_\alpha$ and all $1 \leq i \leq k_\alpha$,

$$\mathcal{N}_i(z_{in}, u_i) > 1 - \frac{\varepsilon}{2}.$$

and hence

$$\sum_{i=1}^{k_\alpha} w_i \cdot \mathcal{N}_i(z_{in}, u_i) > \left(1 - \frac{\varepsilon}{2}\right)^2$$

and this implies formula (8).

On the other hand, assume that $\{\{z_{ij}\}_i\}_j$ is not pointwise convergent to $\{u_i\}_i \in X$. Then there exists some i such that

$$\limsup_{j\to\infty} |z_{ij} - u_i| > 0.$$

Then the strictness of the nearness relation $\mathcal{N}_i$ implies

$$\liminf_{j\to\infty} \mathcal{N}_i(z_{ij}, u_i) < 1.$$

This implies that $\{\{z_{ij}\}_i\}_j$ is not $\mathcal{E}_\infty$-convergent. □

Theorem 8. Let $\{w_i\}_i$ be a system of positive weights with their sum equal to 1 and let $\{\mathcal{N}_i\}_i$ be a system of strict one-dimensional nearness relations such that, for any $x \in \Re$, the following holds

$$\lim_{y\to x} \mathcal{N}_i(x,y) < 1. \tag{9}$$

Define $\mathcal{D}_\infty$ as the weighted arithmetic mean of the system $\{\mathcal{N}_i\}_i$. Then a sequence $\{\{z_{ij}\}_i\}_j$ of points of the space of X $\mathcal{D}_\infty$-converges to $\{u_i\}_i$ if and only if, for any i, there exists a j_i such that for all $m > j_i$ there holds $z_{im} = u_i$.

P r o o f. The idea of the proof is similar to that of the proof of Theorem 5. Let $\{\{z_{ij}\}_i\}_j$ fulfil the requirements of the assertion in question. Then, for any $\alpha \in]0,1[$, we can find some k_α such that $\sum_{i=1}^{k_\alpha} w_i > \alpha$. Moreover, there exists some n such that, for all $m > n$ and for all $1 \leq i \leq k_\alpha$, there holds $z_{im} = u_i$. This implies that $\{\{z_{ij}\}_i\}_j$ is $\mathcal{D}_\infty$-convergent.

If the sequence $\{\{z_{ij}\}_i\}_j$ does not fulfil the requirements of the assertion in question, then there exists some i such that the sequence $\{z_{ij}\}_{j=m}^\infty$ is not constant for any m. Hence formula (9) implies that $\{\{z_{ij}\}_i\}_j$ is not $\mathcal{D}_\infty$-convergent. □

Example 3. Consider the following sequences

$$z_1 = (1,1,1,\ldots), \quad z_2 = (0,1,1,\ldots), \quad z_3 = (0,0,1,\ldots),\ldots \tag{10}$$

and

$$u_1 = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots\right), \quad u_2 = \left(\frac{1}{4},\frac{1}{2},\frac{1}{2},\ldots\right), \quad u_3 = \left(\frac{1}{6},\frac{1}{6},\frac{1}{2},\ldots\right),\ldots \tag{11}$$

Then we get the following

— The sequence (10) is both $\mathcal{E}_\infty$- and $\mathcal{D}_\infty$-convergent.

— The sequence (11) is $\mathcal{E}_\infty$-convergent, but it is not $\mathcal{D}_\infty$-convergent.

3.2. Weighted harmonic means

The weighted harmonic mean is defined by the following:

Definition 9. Let $\{w_i\}_{i=1}^\infty$ be a sequence of positive weights such that $\sum_{i=1}^\infty w_i = 1$ and let $\{\mathcal{N}_i\}_{i=1}^\infty$ be a sequence of one-dimensional nearness relations. Then the weighted harmonic mean of the sequence $\{\mathcal{N}_i\}_{i=1}^\infty$ is defined by

$$\mathcal{H}\left(\{\mathcal{N}_i(u_i,v_i)\}_{i=1}^\infty\right) = \frac{1}{\sum_{i=1}^\infty \frac{w_i}{\mathcal{N}_i(u_i,v_i)}},$$

with the convention $\frac{1}{0} = \infty$.

Obviously, Lemma 1 implies that the above defined weighted harmonic mean of a system of one-dimensional nearness relations is a nearness relation.

Example 4. Consider again the sequences (10) and (11) and the following one-dimensional nearness relations

$$\begin{aligned} \mathcal{N}_1(x,y) &= \max\{0, 1-|x-y|\}, \\ \mathcal{N}_2(x,y) &= \begin{cases} 1 & \text{if } x = y \\ \max\left\{0, \frac{1}{2}(1-|x-y|)\right\} & \text{if } x \neq y \end{cases} \\ \mathcal{N}_3 &= \max\left\{0, 1-\tfrac{1}{2}|x-y|\right\}. \end{aligned}$$

For any positive sequence of weights $\{w_j\}_j$ denote

$$\mathcal{H}_1 = \frac{1}{\sum_i \frac{w_i}{\mathcal{N}_1(x_i,y_i)}}, \quad \mathcal{H}_2 = \frac{1}{\sum_i \frac{w_i}{\mathcal{N}_2(x_i,y_i)}}, \quad \mathcal{H}_3 = \frac{1}{\sum_i \frac{w_i}{\mathcal{N}_3(x_i,y_i)}}.$$

Then the sequence (10) is $\mathcal{H}_3$-convergent and the corresponding limit is the sequence $(0,0,0,\ldots)$.

The sequence (11) is $\mathcal{H}_1$- and $\mathcal{H}_3$-convergent and the corresponding limit is $(0,0,0,\ldots)$.

Concerning the nearness relation $\mathcal{H}_2$ none of the sequences, (10) and (11), is convergent.

Theorem 7. Let $\{\mathcal{N}_i\}_i$ be a sequence of one-dimensional nearness relations and $\{w_i\}_i$ be a sequence of positive weights. Then, for any sequence $\{\{z_{ij}\}_i\}_j$ of elements of X, if $\{z_{.j}\}_j$ is $\mathcal{H}_N$-convergent then $\{z_{.j}\}_j$ is $\mathcal{A}_N$-convergent. $\mathcal{H}_N$ and $\mathcal{A}_N$ are the corresponding weighted harmonic and weighted arithmetic means, respectively, of the given system of one-dimensional nearness relations $\{\mathcal{N}_i\}_i$. On the other hand, assume that the nearness relations from the system $\{\mathcal{N}_i\}_i$ are strict and let $\{\{z_{ik}\}_i\}_k$ be $\mathcal{A}_N$-convergent. Then there exists a system of one-dimensional strict nearness relations $\{\mathcal{M}_i\}_i$ such that $\{\{z_{ik}\}_i\}_k$ is $\mathcal{H}_M$-convergent, where $\mathcal{H}_M$ is the corresponding weighted harmonic mean.

Proof. Assume that some sequence $\{\{z_{ij}\}_i\}_j$ of elements of X $\mathcal{H}_N$-converges to $\{u_i\}_i \in X$. The weighted harmonic mean of a sequence of nonnegative numbers is less then or equal to the weighted arithmetic mean of that sequence. This implies that if, for some j and some α, $\mathcal{H}_N(\{z_{ij}\}_i, \{u_i\}_i) > \alpha$ then $\mathcal{A}_N(\{z_{ij}\}_i, \{u_i\}_i) > \alpha$. Hence we get that the $\mathcal{H}_N$-convergence implies the $\mathcal{A}_N$-convergence.

Let $\{\{z_{ij}\}_i\}_j$ be $\mathcal{A}_N$-convergent to a sequence $\{u_i\}_i \in X$ and let the nearness relations $\mathcal{N}_i$ be strict. We can construct the system of one-dimensional strict nearness relations $\{\mathcal{M}_i\}_i$ by

$$\mathcal{M}_i(x,y) \geq 1 - \frac{1}{j} \Leftrightarrow \sup\{|z_{in} - u_i|; n > j\} \geq |x - y| \tag{12}$$

for $j = 1, 2, \ldots$ Since, for each i, the sequence $\{z_{ij}\}_j$ is $\mathcal{A}_N$-convergent (and hence pointwise convergent), the nearness relation $\mathcal{M}_i$ is strict. The construction of the system of nearnes relations $\{\mathcal{M}_i\}_i$ implies that the sequence $\{\{z_{ij}\}_i\}_j$ is $\mathcal{H}_M$-convergent. □

Remark 1. (a) In fact, Theorem 7 says that if the nearness relations from the system $\{\mathcal{N}_i\}_i$ are strict then $\mathcal{H}_N$-convergence implies the pointwise convergence. And conversely, if we have a pointwise convergent sequence $\{\{z_{ij}\}_i\}_j$ then we are able to find some system $M = \{\mathcal{M}_i\}_i$ of strict nearness relations such that $\{\{z_{ij}\}_i\}_j$ is $\mathcal{H}_M$-convergent.

(b) Unlike the weighted arithmetic means, the weighted harmonic means have 0 as their annihilator and they are right-continuous at 0. This is the main difference between those two kinds of means which results also into different corresponding convergences.

3.3. General weighted pseudo-arithmetic means

The pseudo-arithmetic mean which is proposed by the author is a generalization of the well-known quasi-arithmetic mean, where the function f is not assumed to be strictly monotone (since, from the point of view of the convergence, the strict monotonicity is not necessary). This section gives a global viewpoint at the convergence based on the general pseudo-arithmetic means. The results from the previous Sections 3.1 and 3.2 play an important role, since the convergences based on the weighted arithmetic and weighted harmonic means are some kinds of 'prototypes' of the studied one.

First, let us define those weighted pseudo-arithmetic means.

Definition 10. Let $f : [0,1] \to [0,\infty]$ be a monotone function such that $f(0) \neq f(1)$. Let $\{w_i\}_i$ be a sequence of positive weights with

$$\sum_{i=1}^{\infty} w_i = 1.$$

Then the weighted pseudo-arithmetic mean of a sequence $\{t_i\}_i$ of numbers from $[0,1]$ is defined by

$$\mathcal{P}_f(\{t_i\}_i) = f^{(-1)}\left(\sum_{i=1}^{\infty} w_i f(t_i)\right). \tag{13}$$

$f^{(-1)}$ is the pseudo-inverse of f defined as

$$f^{(-1)}(y) = \begin{cases} \inf\{x \in [0,1]; f(x) > y\} & \text{if } f \text{ is non-decreasing} \\ \inf\{x \in [0,1]; f(x) < y\} & \text{if } f \text{ is non-increasing} \end{cases}$$

with the convention $\inf \emptyset = 1$.

For the properties of pseudo-inverse functions see e. g. [9] or [14].

In fact, it is not important what the value of $f(1)$ is. It is just necessary that $f(1)$ is finite and $f(1) \neq f(0)$. In the following we will assume $f(1) = 1$.

The next lemma is a direct consequence of the properties of pseudo-inverse functions. Therefore its proof is omitted.

Lemma 2. Let $f : [0,1] \to [1,\infty]$ be a non-increasing function with $f(1) = 1$ and $f(0) = \infty$. Denote $\mathcal{P}_f$ the weighted pseudo-arithmetic mean defined by formula (13), applied to a sequence of one-dimensional nearness relations $\{\mathcal{N}_i\}_i$. Then, for any $u, v \in X$, $v \neq \emptyset$, there holds

$$\lim_{t\to\infty} \mathcal{P}_f(u; u + tv) = \sup\{s; f(s) = \infty\}.$$

An assertion similar to Lemma 2 could be formulated also for non-decreasing functions f.

The following lemma is a technical tool for the next theorem.

Lemma 3. Let $f : [0,1] \to [1,\infty]$ be a decreasing function with $f(1) = 1$ and $f(0) = \infty$. Assume that both, f and $f^{(-1)}$, are left-continuous at 1, f is right-continuous at 0 and $f^{(-1)}$ is left-continuous at ∞. Let $\mathcal{N}$ be a one-dimensional nearness relation. Then $\mathcal{M}$, defined by

$$\mathcal{M}(u,v) = f^{(-1)}\left(\frac{1}{\mathcal{N}_i(u,v)}\right),$$

is a nearness relation.

P r o o f. The properties 1. and 2. of Definition 2, the reflexivity and symmetry, are obvious. The monotonicity of f implies the monotonicity of $f^{(-1)}$. This implies the property 3. The property 4, the limit one, is due to the left-continuity of $f^{(-1)}$ at ∞. □

Theorem 8. Let $\{\mathcal{N}_i\}_i$ be a system of strict one-dimensional nearness relations, $\{w_i\}_i$ a system of positive weights and $f : [0,1] \to [0,\infty]$ be a monotone function such that $f(1) = 1$ and $f(0) \neq 1$. Let $\{\{z_{ij}\}_i\}_j$ be a sequence of elements of the linear space X. Let $\mathcal{P}_f$ be the above defined weighted pseudo-arithmetic mean applied to the system $\{\mathcal{N}_i\}_i$. Then the following holds

(a) Let f be not left-continuous at 1. Then the sequence $\{z_{.j}\}_j$ is $\mathcal{P}_f$-convergent if and only if for any n

$$\lim_{(j,k)\to(\infty,\infty)} \sum_{i=1}^{\infty} w_i \cdot f\left(\mathcal{N}_i(z_{ij}; z_{i,j+k})\right) \geq \lim_{x\to 1_-} f(x)$$

in case f is non-decreasing and

$$\lim_{(j,k)\to(\infty,\infty)} \sum_{i=1}^{\infty} w_i \cdot f\left(\mathcal{N}_i(z_{ij}; z_{i,j+k})\right) \leq \lim_{x\to 1_-} f(x)$$

in case f is non-increasing.

(b)Let $f^{(-1)}$ be not left-continuous at 1. Then the sequence $\{z_{.j}\}_j$ is $\mathcal{P}_f$-convergent if and only if there is a j_0 such that for all $j > j_0$, all n and all i

$$f\left(\mathcal{N}_k(z_{ij}, z_{i,j+k})\right) = 1.$$

(c) Let both, f and $f^{(-1)}$, be left-continuous at 1 and $f(0) \neq \infty$. Assume $f(0) = 0$. Then $\{z_{.j}\}_j$ is $\mathcal{P}_f$-convergent if and only if it is convergent with respect to the weighted arithmetic mean, applied to the system of nearness relations $\{\mathcal{N}_i\}_i$.

(d) Let both, f and $f^{(-1)}$, be left-continuous at 1, $f(0) = \infty$ and $\sup\{x; f(x) = \infty\} = 0$ and let f be right-continuous at 0. Then $\{z_{.j}\}_j$ is $\mathcal{P}_f$-convergent if and only if it is convergent with respect to the weighted harmonic mean applied to the system of one-dimensional nearness relations $\left\{f^{(-1)}\left(\frac{1}{\mathcal{N}_i}\right)\right\}_i$.

P r o o f. Parts (a) and (b) of this theorem are implied by the fact that if f is left-discontinuous at 1 then $f^{(-1)}$ is continuous at $f(1)$ and it has an interval of constantness and vice versa. For parts (c) and (d), one must realize that from the point of view of the convergence the only important points of the domain of f are 1 and 0. Hence, if both f and $f^{(-1)}$ are continuous at 1 and, in part (c) $f(0) \neq \infty$, the pseudo-arithmetic mean is just a transformations of the weighted arithmetic mean and because of the continuity of f and $f^{(-1)}$ at 1 for any sequence $\{\{z_{ij}\}_i\}_j$ of elements of the space X the following holds:

$$\lim_{(j,k)\to(\infty,\infty)} \mathcal{N}_i(z_{ij}, z_{ik}) = 1 \Leftrightarrow \lim_{(j,k)\to(\infty,\infty)} f\left(\mathcal{N}_i(z_{ij}, z_{ik})\right) = 1. \qquad (14)$$

In part (d), of course, formula (14) holds also and we have $f(0) = \infty$ and $\sup\{x; f(x) = \infty\} = 0$, i.e. if in at least one coordinate, i, the nearness $\mathcal{N}_i$ is equal to 0, then the whole weighted pseudo-inverse $\mathcal{P}_f$ equals to 0, i.e. 0 is the annihilator of the weighted pseudo-inverse mean and, since f is right-continuous at 0, the weighted pseudo-arithmetic mean is also right-continuous at 0. This completes the proof of part (d). □

Example 5. (a) Let $f : [0,1] \to [0,1]$ be the following function

$$f(x) = \begin{cases} 1 & \text{for } x = 1 \\ 0.9x & \text{otherwise.} \end{cases}$$

Consider the nearness relation $\mathcal{N}_1$ from Example 4 in each coordinate. Put the weight $w_1 = 0.9$. Then the sequence (10) from Example 3 is $\mathcal{P}_f$-convergent, since for any $j, k \geq 2$

$$\sum_{i=1}^{\infty} w_i f(\mathcal{N}_1(z_{i,j}; z_{i,k})) \geq 0.9$$

hence

$$\mathcal{P}_f(z_{.j}; z_{.k}) = f^{(-1)}\left(\sum_{i=1}^{\infty} w_i f(\mathcal{N}_1(z_{i,j}; z_{i,k}))\right) = 1.$$

It can be shown that under the above conditions the sequence (11) is not $\mathcal{P}_f$-convergent.

(b) Let $f : [0,1] \to [0,1]$ be the following

$$f(x) = \begin{cases} 1 & \text{for } x \geq 0.9 \\ \frac{1}{0.9}x & \text{otherwise} \end{cases}$$

and consider again the one-dimensional nearness relation $\mathcal{N}_1$ in each coordinate. Then none of the sequences (10) and (11) is $\mathcal{P}_f$-convergent. On the other hand, the sequence defined by

$$\begin{aligned} z_{2j-1} &= (0.1; 0; 0.1; 0; 0.1; 0; \ldots) \\ z_{2j} &= (0; 0.1; 0; 0.1; 0; 0.1; \ldots) \end{aligned}$$

is $\mathcal{P}_f$-convergent, since $f(\mathcal{N}_1(0; 0.1)) = f(0.9) = 1$.

If we consider the sequence

$$\begin{aligned} z_{2j-1} &= (0; 0; 0.1; 0; 0.1; 0; 0.1; \ldots) \\ z_{2j} &= (0; 0.1; 0; 0.1; 0; 0.1; \ldots) \end{aligned}$$

(i.e. the first coordinate is fixed) and we take the weight $w_1 = 0.9$, then this sequence is $\mathcal{P}_f$-convergent regardless of the fact which nearness relations we consider in the coordinates.

3.4. Geometric mean

Beside the weighted pseudo-arithmetic means there is yet one important mean, namely the geometric one. Let us examine the basic properties of the corresponding convergence and finally compare the results with those from the foregoing sections.

Definition 11. Let $\{\mathcal{N}_i\}_i$ be a system of one-dimensional nearness relations. Then by the geometric mean of this system of nearness relations we will understand the following

$$\begin{aligned} &\mathcal{G}\left((s_1, s_2, s_3, \ldots); (v_1, v_2, v_3, \ldots)\right) \\ = \;& \liminf_{i\to\infty} \sqrt[i]{\mathcal{N}_1(s_1, v_1)\mathcal{N}_2(s_2, v_2)\ldots\mathcal{N}_i(s_i, v_i)}. \qquad (15)\end{aligned}$$

Lemma 4. Let $\{a_n\}_n$ be a sequence of real numbers, all being from $]0, 1]$. Let b be such that, for all numbers a_n, beginning with the index n_0, there holds $b < a_n$. Then the geometric mean is bounded from below by b.

A direct consequence of Lemma 1 is the following:

Lemma 5. The geometric mean applied to one-dimensional nearness relations, defined by formula (15), is a nearness relation, i. e. for any $u, v \in X$ the following holds

$$\lim_{t\to\infty} \mathcal{G}(u; u + tv) = 0.$$

Theorem 9. Let $\{u_{.j}\}_j$ be a sequence of elements of X which is pointwise convergent. Then there exists a system of one-dimensional strict nearness relations $\{\mathcal{N}_i\}_i$ such that $\left\{\{u_{ij}\}_i\right\}_j$ is $\mathcal{G}$-convergent, where $\mathcal{G}$ is the nearness relation defined by formula (15).

Proof. The system of nearness relations $\{\mathcal{N}_i\}_i$ can be constructed in the same way as it has been done by formula (12). □

Lemma 6. Let $\{u_{.j}\}_j$ be a sequence of elements of X, and $\{\mathcal{N}_i\}_i$ any system of one-dimensional nearness relations such that the given sequence has a limit in the corresponding nearness relation $\mathcal{G}$ and moreover

$$\mathcal{G}\left(\{\liminf_j u_{ij}\}_i, \{\limsup_j u_{ij}\}_i\right) > 0.$$

Then this limit is not given uniquely.

The proof of this Lemma is contained in the next example.

Example 6. Consider the sequences (10) and (11) from Example 3 and the sequence defined as

$$v_k = \left(0, \left(\frac{1}{2}\right)^k, \left(\frac{2}{3}\right)^k, \left(\frac{3}{4}\right)^k, \dots\right). \tag{16}$$

Further, consider the one-dimensional nearness relations $\mathcal{N}_j$, for $j = 1,2,3$, from Example 4 and the following system of one-dimensional nearness relations

$$\mathcal{M}_k(x,y) = \max\left\{0, 1 - \frac{1}{k}|x-y|\right\}. \tag{17}$$

Let $\mathcal{G}_j$ denote the corresponding nearness relations defined by formula (15), using each time $\mathcal{N}_j$, respectively (in each coordinate), and let $\mathcal{G}_4$ denote the geometric mean of the system of nearness relations (17).

Then the following can be proved

- Sequence (10) is $\mathcal{G}_3$-convergent, moreover it is a $\mathcal{G}_3$- 'constant sequence', since $\mathcal{G}_3(z_{.,i_1}, z_{.,i_2}) = 1$ for any i_1, i_2. Sequence (10) is not convergent with respect to $\mathcal{G}_1$ and $\mathcal{G}_2$.
- Sequence (11) is convergent with respect to all nearness relations $\mathcal{G}_j$ $(j = 1,2,3)$ and moreover we get

$$\mathcal{G}_1(z_{.i_1}, z_{.j}) = \mathcal{G}_2(z_{.k}, z_{.j}) = \mathcal{G}_3(z_{.,i_1}, z_{.,i_2}) = 1$$

  for all i_1, i_2.
- Sequence (16) is convergent with respect to none of the nearness relations $\mathcal{G}_j$ $j = 1,2,3$.
- Concerning the nearness relation $\mathcal{G}_4$, all three sequences are convergent and their limits are all sequences, whose first element is from $[0,1[$ interval and all other elements are from $[0,1]$.

Remark 2. We see that the main difference between the weighted arithmetic and weighted harmonic means on one side and the geometric mean on the other side, is that using a geometric mean we have no uniquely given limit. On the other hand, if we had considered some kind of weighted geometric mean, i.e. the sum of all exponents equal to 1, then this would be just a special case of the weighted pseudo-arithmetic mean with $f = \log$ and hence the limit would be given uniquely.

Remark 3. Further, for a suitably given system of one-dimensional nearness relations $\{\mathcal{N}_i\}_i$ and a given sequence $\{z_{.j}\}_j$ of elements of X it may happen that the sequence $\{z_{.j}\}_j$ has limits with respect the given system $\{\mathcal{N}_i\}_i$ aggregated by weighted arithmetic, weighted harmonic and geometric means, but the limit(s) achieved using the corresponding geometric mean differs from that using the weighted arithmetic or weighted harmonic means.

3.5. Nearness relations aggregated by t-norms

We will now consider continuous t-norms to aggregate the system of nearness relations $\{\mathcal{N}_i\}_i$.

Theorem 10. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be strict one-dimensional nearness relations given by $\mathcal{N}_j(x,y) = f_j(|x-y|)$ for $j = 1,2$, where $f_j : [0,\infty[\to [0,1]$ are non-increasing functions with $f_j(0) = 1$ and right-continuous at 0. Then, for any sequence $\{\{z_{i,j}\}_i\}_j$ of elements of the space X, we get that $\{\{z_{i,j}\}_i\}_j$ is $\inf_{\{N_1\}}$-convergent if and only if it is $\inf_{\{N_2\}}$-convergent (this means, in each coordinate, we use the same nearness relation, $\mathcal{N}_1$ or $\mathcal{N}_2$, respectively, and aggregate them using inf.) The resulting topology is that of the uniform convergence.

Proof. By their definitions, nearness relations $\mathcal{N}_1$ and $\mathcal{N}_2$ are just transformations of metric, continuous at 0. This implies the assertion in question. □

Theorem 11. Let $\{\{z_{i,j}\}_i\}_j$ be a sequence of elements of X which is pointwise convergent. Then, for any continuous t-norm T, there exists a sequence of one-dimensional strict nearness relations $\{\mathcal{M}_i\}_i$ such that $\{z_{i,.}\}_i$ is $T_{i=1}^{\infty}\mathcal{M}_i$-convergent.

Proof. We will prove the theorem for the Lukasiewicz t-norm. We can use the slightly modified construction defined by formula (12). We choose some sequence $\{\alpha_i\}_i$ such that

$$0 < 1 + \sum_{i=1}^{\infty}(\alpha_i - 1)$$

which is the 'countable version' of Lukasiewicz t-norm. Denote the right-hand side of the above inequality by K. For the given sequence $\{\{z_{i,j}\}_i\}_j$ we put

$$1 - \mathcal{M}_i(x,y) \le (1-K)^j(1-\alpha_i) \;\Leftrightarrow\; \sup\{|z_{in} - z_{ik}|; k,n > j\} \ge |x-y|.$$

Then we get for all $k,n > j$ the following

$$1 + \sum_{i=1}^{\infty}(\mathcal{M}_i(z_{i,k};z_{i,n}) - 1) \ge 1 - (1-K)^j\sum_{i=1}^{\infty}(1-\alpha_i) = 1 - (1-K)^{1+j}$$

and hence we get that the sequence $\{\{z_{i,j}\}_i\}_j$ is L_M-convergent, where L_M is the nearness relation defined by applying Lukasiewicz t-norm to the system of one-dimensional nearness relations $\mathcal{M}_i$. □

Theorem 12. Let $\{\mathcal{N}_i\}_i$ be a sequence of one-dimensional strict nearness relations and T be a t-norm. Then there exists a sequence $\{\{z_{i,j}\}_i\}_j$ of elements of X which is pointwise convergent, but which is not convergent with respect to $T_{i=1}^{\infty}\mathcal{N}_i$.

Proof. We will construct the sequence $\{\{z_{i,j}\}_i\}_j$ to be not $\inf\{\mathcal{N}_i\}$-convergent and hence also not convergent with respect to any other t-norm. Put

$$z_{i1} = y_i \quad \text{such that } \mathcal{N}_i(0,y_i) = 0$$

and for $i > 1$ put

$$z_{ik} = \begin{cases} \sup\left\{y; \mathcal{N}_i(0,y) = 1 - \frac{1}{k}\right\} & \text{for } k \le i \\ z_{i1} & \text{for } k > i \end{cases}$$

This sequence pointwise converges to $(0,0,\ldots)$, but it is not $\inf\{\mathcal{N}_i\}$-convergent since for any i $\mathcal{N}_i(0, z_{i,i+1}) = 0$. □

Example 7. (a) Consider the sequence

$$z_i = \left(\frac{1}{i}; \frac{2}{i}; \frac{3}{i}; \frac{4}{i}; \ldots\right) \tag{18}$$

and the system of one-dimensional nearness relations

$$\mathcal{N}_j(x,y) = \max\left\{0; 1 - \frac{1}{j}|x-y|\right\}$$

Then

$$\inf_j \left\{\mathcal{N}_j\left(\frac{j}{i}; 0\right)\right\} = 1 - \frac{1}{i}.$$

Hence, the sequence (18) has a limit with respect to $\mathcal{M} = \inf_j \mathcal{N}_j$.

(b) If we take the product t-norm, then (18) is not $\Pi_{j=1}^{\infty}\mathcal{N}_j$-convergent, but we can define by induction another system of one-dimensional nearness relations in the following way

$$\mathcal{M}_1(1,0) = \frac{3}{4}; \quad \mathcal{M}_1\left(\frac{1}{i}, 0\right) = \frac{i(i+1)+1}{(i+1)^2}; \ldots$$

Assume that we know already the values of $\mathcal{M}_{k-1}\left(\frac{k-1}{i}, 0\right)$. Then we define the value of $\mathcal{M}_k\left(\frac{k}{i}, 0\right)$ by

$$\mathcal{M}_k\left(\frac{k}{i}, 0\right) = \frac{1}{\mathcal{M}_{k-1}\left(\frac{k-1}{i}, 0\right)} \frac{i \cdot (i+1)^k + 1}{(i+1)^{k+1}}.$$

We get

$$\Pi_{k=1}^{\infty}\mathcal{M}_k(z_{i,k}; 0) = \frac{i}{i+1},$$

hence the sequence (18) is $\Pi_{k=1}^{\infty}\mathcal{M}_k$-convergent.

Example 8. Consider the sequence

$$z_i = \left(\frac{1}{2^i}; \frac{1}{3^i}; \frac{1}{4^i}; \ldots\right) \tag{19}$$

and take the system of one-dimensional nearness relations

$$\mathcal{N}_k(x,y) = \max\left\{0; 1 - (k+1)^k|x-y|\right\}.$$

Then we get for each i

$$\inf_k \{\mathcal{N}_k(z_{i,k};0)\} = 0$$

since $z_{k,k} = \frac{1}{(k+1)^k}$ and

$$\mathcal{N}_k\left(\frac{1}{(k+1)^k};0\right) = 1 - (k+1)^k \frac{1}{(k+1)^k} = 0$$

and, as it is easy to see, the sequence (19) is not $\inf_k \mathcal{N}_k$-convergent (and hence it is not convergent also when replacing inf by any other t-norm).

4. CONCLUSION

The results of this paper show that we have a variety of convergences in the linear space of sequences. Each of the convergences represents a certain 'speed' with which the coordinates of the given sequence converge. The weakest topology (when aggregating strict one-dimensional nearness relations) is the topology of pointwise convergence achieved by the weighted arithmetic mean.

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