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STABILITY ESTIMATES OF GENERALIZED GEOMETRIC SUMS AND THEIR APPLICATIONS

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The upper bounds of the uniform distance $\rho\left(\sum_{k=1}^{\nu} X_k, \sum_{k=1}^{\nu} \tilde{X}_k\right)$ between two sums of a random number ν of independent random variables are given. The application of these bounds is illustrated by stability (continuity) estimating in models in queueing and risk theory.

Keywords: geometric sum, upper bound for the uniform distance, stability, risk process, ruin probability

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1. INTRODUCTION

Geometric sums are proved to be a useful and efficient tool of modelling stochastic phenomena in theory of queues, risk theory, storage, dams emulation, reliability, etc. (see, for instance [1–4, 6, 7–11]. The geometric sum is defined as the random variable $\sum_{k=1}^{\nu} X_k$, where $P(\nu = k) = q(1-q)^{k-1}$, $k = 1, 2, \ldots$ and ν is independent of a sequence of independent, identically distributed random variables $X_1, X_2, \ldots$.

The problem of the stability (continuity) arises because of an inevitable uncertainty about input data, or about so-called governing "parameters" of a model. As to geometric sums, the governing "parameter" is the distribution function F of a "real" random variable X_1, which, at least to a certain extent, is unknown and, for this, is not at one's disposal to carry out a desired analysis of the output data, that is of the distribution of the sum $\sum_{k=1}^{\nu} X_k$. Consequently, an investigator should search for an available approximating distribution function G (of a random variable $\tilde{X}_1$) obtained from theoretical considerations or (and) statistical estimation. With G in hand one replaces $\sum_{k=1}^{\nu} X_k$ by the approximating sum $\sum_{k=1}^{\nu} \tilde{X}_k$ in the study of the former. The reliability of inferences obtained in the course of such replacement depends decisively on the closeness of the distribution of $\sum_{k=1}^{\nu} X_k$ and of that of $\sum_{k=1}^{\nu} \tilde{X}_k$.

Let μ and $\tilde{\mu}$ be certain metrics in the space of random variables (or rather, in the space of their distributions). It is natural to expect that $\mu\left(\sum_{k=1}^{\nu} X_k, \sum_{k=1}^{\nu} \tilde{X}_k\right)$ is a vanishing at zero function of $\tilde{\mu}(X_1, \tilde{X}_1)$ and it is even better to be able to control the accuracy of approximation, i. e. to have stability inequalities (bounds) of the form:

$$\mu\left(\sum_{k=1}^{\nu} X_k, \sum_{k=1}^{\nu} \tilde{X}_k\right) \le \psi(\tilde{\mu}(X_1, \tilde{X}_1)), \tag{1.1}$$

where $\lim_{x\to 0^+} \psi(x) = 0$.

There are several works offering the versions of (1.1) for geometric convolutions. In [8 - 10] the bounds as in (1.1) with the uniform metric $\rho = \mu = \tilde{\mu}$ are given under the condition: $EX_1 = E\tilde{X}_1$; $EX_1^2, E\tilde{X}_1^2 < \infty$. (Here and throughout $\rho(X,Y) := \sup_{x\in\mathbb{R}} |F_X(x) - F_Y(x)|$.) These bounds provide $\psi(x) = c\sqrt{x}$ in (1.1). Some inequalities related to (1.1) can be extracted from the stability results in [1, 2, 11, 13].

Under the hypotheses $EX_1 = E\tilde{X}_1$, $\mathrm{Var}(X_1) = \mathrm{Var}(\tilde{X}_1)$; $E|X_1|^3$, $E|\tilde{X}_1|^3 < \infty$ and a certain "smoothness" assumption on the known density of $\tilde{X}_1$ we obtained in [5, 6] the variants of (1.1) with a linear function ψ. There $\mu = \mathbf{V}$ was the total variation metric, while $\tilde{\mu} = \max\{\mathbf{V}, \frac{1}{6}\mathbf{k}_3\}$, with $\mathbf{k}_3$ being a difference pseudo-moment of order 3 (see (1.3)).

In the present paper we deduce (and apply to the stability study of some models) estimate (1.1) with the uniform distance $\mu = \rho$. In applications this metric is often more useful than the total variation distance.

Instead of a usual geometric sum $\sum_{k=1}^{\nu} X_k$, we treat its generalization not assuming that the random variables $X_1, X_2, \dots$ are identically distributed and that ν has the geometric distribution.

Together with the most important case of equal means (of X_k and of $\tilde{X}_k$), we pay attention to the more tight stability bounds which hold under the condition:

$$EX_k^j = E\tilde{X}_k^j, \quad k \ge 1, \quad j = 1, 2, \dots, m-1 \quad (m \ge 2).$$

Precisely, we prove the following inequality:

$$\rho\left(\sum_{k=1}^{\nu} X_k, \ \sum_{k=1}^{\nu} \tilde{X}_k\right) \le c_m E\left(\nu^{-\frac{m-2}{2}}\right) \sup_{k\ge 1} \max\left\{\rho(X_k, \tilde{X}_k), \frac{1}{m!}\mathbf{k}_m(X_k, \tilde{X}_k)\right\}, \tag{1.2}$$

where

$$\mathbf{k}_m(X,Y) := m\int_{-\infty}^{\infty} |x|^{m-1}|F_X(x) - F_Y(x)|\mathrm{d}x \tag{1.3}$$

and c_m is a constant calculated in the explicit form, which depends on certain properties of characteristic functions of $(X_k,\ k \ge 1)$ and of $(\tilde{X}_k,\ k \ge 1)$.

To illustrate applications of inequality (1.2) we offer a solution of the stability problem in the following models. Firstly, we give new stability bounds for the ruin probability in the classical risk process. Secondly, we estimate the stability of the S. Andersen risk process (see [8]). Finally, we evaluate the accuracy of the approximation of distributions of sums of random variables by Erlang's distributions.

2. THE RESULTS

In what follows let ν be an arbitrary random variable assuming integer positive values and independent of two given sequences of independent random variables $Q = (X_k,\ k \geq 1)$ and $\tilde{Q} = (\tilde{X}_k,\ k \geq 1)$ having finite second moments. Throughout the paper we shall denote:

(i) $S = \sum_{k=1}^{\nu} X_k, \quad \tilde{S} = \sum_{k=1}^{\nu} \tilde{X}_k;$

(ii) $S_n = X_1 + \cdots + X_n, \quad \tilde{S}_n = \tilde{X}_1 + \cdots + \tilde{X}_n, \quad n = 1, 2, \dots\ ;$

(iii) $\sigma_k^2 = \mathrm{Var}(X_k), \quad \tilde{\sigma}_k^2 = \mathrm{Var}(\tilde{X}_k), \quad k \geq 1, \quad \sigma_* = \inf_{k \geq 1} \sigma_k; \quad \tilde{\sigma}_* = \inf_{k \geq 1} \tilde{\sigma}_k;$

(iv) F_X, f_X and φ_X are, respectively, the distribution function, the density (if it exists) and the characteristic function of a random variable X;

(v) $\rho(X, Y) = \sup_{x \in \mathbb{R}} |F_X(x) - F_Y(x)|$, (the uniform metric);

(vi) $\mathbf{k}_m(X, Y) = m \int_{-\infty}^{\infty} |x|^{m-1} |F_X(x) - F_Y(x)| \mathrm{d}x$, (the difference pseudomoment of order $m > 0$);

(vii) $\mu_m(X, Y) = \max\left\{\rho(X, Y), \dfrac{1}{m!}\mathbf{k}_m(X, Y)\right\}$, ($m$ an integer).

Definition 1. Let $m \geq 2$, $s \geq 1$ be fixed integers and r, $0 \leq r < \infty$ be a given real number. We say that a sequence of independent random variables $Q = (X_k,\ k \geq 1)$ belongs to the class $\mathbf{K}_m(s; r)$ if the following holds:

(a) $\sigma_* > 0$ and $\sup_{k \geq 1} E|X_k|^m < \infty$;

(b) $\sup_{k \geq 1} |\varphi_{X_k/\sigma_*}(t)| \leq r|t|^{-(m+1)/s}$ for $|t| \geq 1$.

Since the hypotheses of the below theorems require sequences under consideration to be members of $\mathbf{K}_m(s; r)$, it is useful to get idea of how wide the classes $\mathbf{K}_m$ are. The following simple assertion shows that the majority of sequences of continuous random variables (identically distributed, with finite variance) accustomed in probability theory and its applications are in the class $\mathbf{K}_2(s; r)$ (for some s, r), and even in $\mathbf{K}_m(s; r)$, $m > 2$, provided $E|X_1|^m < \infty$.

Proposition 1. For a given integer $m \geq 2$ let $Q = (X_k,\ k \geq 1)$ be a sequence of independent, identically distributed random variables such that:

(i) $\mathrm{Var}(X_1) > 0, \quad E|X_1|^m < \infty$;

(ii) There is an integer ℓ such that a random variable $Z = X_1 + \cdots + X_\ell$ has a differentiable density with a derivative in $\mathbb{L}_1(\mathbb{R})$.

Then there exist $s \geq 1$ and $r < 1$ for which $Q \in \mathbf{K}_m(s; r)$.

Remark 1. Inequalities for characteristic functions obtained in [15] supply us with another way to test whether a given density is in a class $\mathbf{K}_m(s,r)$. Let a random variable $Z = X_1 + \cdots + X_\ell$ have a density f_Z of bounded variation $V(f_Z) := \lim_{a\to\infty} V_{-a}^{a}(f_Z)$, where $V_{-a}^{a}(f_Z)$ is the total variation of f_Z on the segment $[-a,a]$. Then, according to [15], the following upper bound for a characteristic function φ_Z holds:

$$|\varphi_Z(t)| \le V(f_Z)/|t|, \qquad t \in \mathbb{R}.$$

Moreover, if the density f_Z has $n-1$ derivatives, and $f_Z^{(n-1)}$ is a function of bounded variation, then

$$|\varphi_Z(t)| \le V(f_Z^{(n-1)})/|t|^n, \quad t \in \mathbb{R}.$$

Definition 2. Let μ be a simple probability metric (i.e. $\mu(X,Y) \equiv \mu(F_X, F_Y)$, see [11, 13, 16]) and $H = (Z_k,\ k \ge 1)$ $\tilde{H} = (\tilde{Z}_k,\ k \ge 1)$ be arbitrary sequences of random variables. We write (admitting infinite values):

$$\mu(H,\tilde{H}) := \sup_{k\ge 1} \mu(Z_k, \tilde{Z}_k).$$

Actually, the following Theorem 1 is a particular case of Theorem 2 below. We single out the former because it seems to be more important for applications and because of a simple formula for calculating the constant c in inequality (2.1).

Theorem 1. Suppose that $EX_k = E\tilde{X}_k$, $k = 1,2,\ldots$ and $Q, \tilde{Q} \in \mathbf{K}_2(s;r)$. Then

$$\rho(S,\tilde{S}) \le c\mu_2(Q,\tilde{Q}) < \infty, \tag{2.1}$$

where

$$\begin{aligned} c \;=\; & \max\left\{2s-1,\ \frac{1}{\pi}\left[\frac{1+s^{-1}}{\sigma_*^2} + \frac{1}{\tilde{\sigma}_*^2}\right]\right. \\ & \left. \times \inf_{\lambda: r\lambda^{-3/s}<1} \lambda^2 \left[s(r\lambda^{-3/s})^s + \frac{4}{1-(r\lambda^{-3/s})^2}\right]\right\}. \end{aligned} \tag{2.2}$$

Corollary 1. Let $Sq := (S_n,\ n \ge 1)$, $\tilde{S}q := (\tilde{S}_n,\ n \ge 1)$. Under the conditions of Theorem 1

$$\rho(Sq, \tilde{S}q) \le c\,\mu_2(Q,\tilde{Q}).$$

The usefulness of bound (2.1) is conditioned by a magnitude of c in (2.1), (2.2). The following table gives an inkling of possible values of s, r and, so of c in (2.2) for sequences of independent, identically distributed random variables with a common density f_X.

For the "best" density in this table, i.e. for the triangular one we find (taking $\sigma_* = 1.5, \tilde{\sigma}_* = 1.4$ and $\lambda = 1$ in (2.2)) that $c = \max\{3, 3.03021\} = 3.03021$. See also Section 3 for further numerical examples.

The density f_X	$Q = (X_k,\ k \geq 1) \in \mathbf{K}_2(s;r)$	$Q = (X_k, k \geq 1) \in \mathbf{K}_3(s;r)$
Normal	$\mathbf{K}_2(1; 1.81959)$, $\mathbf{K}_2(2; 0.90980)$	$\mathbf{K}_3(2; 1.21306)$
Uniform	$\mathbf{K}_2(3; 0.57735)$	$\mathbf{K}_3(4; 0.57735)$
Triangular	$\mathbf{K}_2(2; 0.66667)$	$\mathbf{K}_3(3; 0.66667)$
Exponential	$\mathbf{K}_2(3; 1)$	$\mathbf{K}_3(4; 1)$
Gamma $\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$ with $\alpha = 4$ (λ *is* arbitrary)	$\mathbf{K}_2(2; 1.2)$ $\mathbf{K}_2(3; 0.8)$	$\mathbf{K}_3(3; 1.06667)$
Gamma with $\alpha = 6$ λ *is* arbitrary)	$\mathbf{K}_2(3; 0.75470)$	
Gamma with $\alpha \geq 2$ and arbitrary λ	$\mathbf{K}_2(2; 1.36470)$ $\mathbf{K}_2(3; 0.9080)$	

Theorem 2. Let $m \geq 2$ be a fixed integer. Suppose that $EX_k^j = E\tilde{X}_k^j$, $k \geq 1, j = 1, 2, \ldots, m-1$ and that Q, $\tilde{Q} \in \mathbf{K}_m(s;r)$. Then

$$\rho(S, \tilde{S}) \leq c_m \mu_m(Q, \tilde{Q}) E\left(\nu^{-\frac{m-2}{2}}\right), \tag{2.3}$$

where

$$\begin{aligned} c_m &= \max\left\{(2s-1)^{m/2}, \frac{s^{1-m/2}}{\pi}\left[\frac{\gamma(m,s)}{\sigma_*^m} + \frac{\tilde{\gamma}(m,s)}{\tilde{\sigma}_*^m}\right]\right. \\ &\quad \left. \times \inf_{\lambda: r\lambda^{-(m+1)/s} < 1} \lambda^m \left[s\psi_1(\lambda) + \psi_2(\lambda)\right]\right\}; \end{aligned} \tag{2.4}$$

$$\gamma(m,s) := \sup_{n \geq 2s} \frac{n - [n/2]}{[n/2]^{m/2}} n^{\frac{m-2}{2}}, \quad \tilde{\gamma}(m,s) = \sup_{n \geq 2s} \frac{[n/2]}{(n - [n/2])^{m/2}} n^{\frac{m-2}{2}} < \infty;$$

$$\psi_1(\lambda) \equiv \psi_1(\lambda; r, m, s) := \sup_{k \geq s} \lambda^{-\frac{(m+1)k}{s}} \frac{r^k k^{m/2}}{k(m+1) - ms} < \infty; \tag{2.5}$$

$\psi_2(\lambda) \equiv \psi_2(\lambda; r, m)$

$$:= \begin{cases} \dfrac{2^{3m/2-1}}{\left(1 - r^2 \lambda^{-\frac{2(m+1)}{s}}\right)^{m/2}} \left(\dfrac{m-2}{2}\right)! & \text{for } m \text{ even}, \\[2ex] \dfrac{2^{m/2}}{\left(1 - r^2 \lambda^{-\frac{2(m+1)}{s}}\right)^{m/2}} \dfrac{(m-1)!}{\left(\frac{(m-1)}{2}\right)!} & \text{for } m \text{ odd}. \end{cases} \tag{2.6}$$

Corollary 2. Let $Q = (X_k, k \geq 1)$ and $\tilde{Q} = (\tilde{X}_k, k \geq 1)$ consist of identically distributed random variables. Then, under the conditions of Theorem 2,

$$\rho(S_n, \tilde{S}_n) \leq c_m \mu_m(X_1, \tilde{X}_1) n^{-\frac{m-2}{2}}, \quad n = 1, 2, \dots . \tag{2.7}$$

Remark 2. For $m = 3$ and $\tilde{X}_1, \tilde{X}_2, \dots$ being normally distributed, inequalities (2.7) give an estimate of the rate of convergence in the central limit theorem. (See, for instance, [12 – 14, 16] for other, more specialized results on this developed topic.)

Remark 3. For $m = 3$ the calculation of $\psi_1(\lambda)$ in (2.5) (for a given λ) can be carried out as follows:

(i) Calculate $\delta = r\lambda^{-4/s}$.

(ii) Compute the value of the function $\dfrac{\delta^x x^{3/2}}{4x - 3s}$ at the points $[x_0]$ and $[x_0 + 1]$, where $[x]$ stands for the integer part of x and

$$x_0 = \left(\frac{3}{2} s \log \delta - 1\right) + \left(\left(\frac{3}{2} s \log \delta - 1\right)^2 + \frac{9}{2} s\right)^{1/2}.$$

(iii) Take the greater value among the computed ones.

Also note that $\gamma(3, 1) = 2\sqrt{3}$, $\tilde{\gamma}(3; 1) = \sqrt{2}$.

Remark 4. Let us consider a family of sequences of independent identically distributed random variables $Q^{(\theta)} = (X_k^{(\theta)}, k \geq 1)$ depending on parameter $\theta \in R^\ell$, $(\ell \geq 1)$. Let a common density $f_X^{(\theta)}$ of $X_k, k \geq 1$ exist and $G(\theta)$ be some distribution function on R^ℓ with a support $B \subseteq R^\ell$. Assume that there are integers $s \geq 1$, $m \geq 2$ such that

(i) $Q^{(\theta)} \in \mathbf{K}_m(s; r_\theta)$, $\theta \in B$;

(ii) $\sup_{\theta \in B} E|X_1^{(\theta)}|^m < \infty$;

(iii) $\sup_{\theta \in B} r_\theta \leq r < \infty$.

By simple calculations one can verify that a sequence Q of independent, identically distributed random variable with a common "mixed" density $f_X(x) = \int_B f_X^{(\theta)}(x) \mathrm{d}G(\theta)$ belongs to the class $\mathbf{K}_m(s; r)$.

3. APPLICATIONS

Example 1. (The stability estimate of ruin probability.)

Let us consider the so-called classical risk process:

$$Z(t) = x + \kappa t - \sum_{k=1}^{N(t)} Z_k \tag{3.1}$$

and its approximation

$$\tilde{Z}(t) = x + \kappa t - \sum_{k=1}^{N(t)} \tilde{Z}_k, \tag{3.2}$$

(by convention, $\sum_{k=1}^{0} = 0$).

Here $x > 0$ is an initial capital, $\kappa > 0$ is a gross premium rate, $N(t)$ is the Poisson process modelling the number of claims occurred within $[0, t]$ and the sequences of nonnegative independent, identically distributed random variables $Q = (Z_k,\ k \geq 1)$, $\tilde{Q} = (\tilde{Z}_k,\ k \geq 1)$ (independent of $N(t)$) represent the costs of successive claims.

We are concerned with an upper bound for the following uniform distance

$$\rho(\Psi, \tilde{\Psi}) := \sup_{x \geq 0} |\Psi(x) - \tilde{\Psi}(x)|$$

between the ruin probabilities

$$\Psi(x) := P\big(\inf_{t \geq 0} Z(t) < 0\big),\ \tilde{\Psi}(x) := P\big(\inf_{t \geq 0} \tilde{Z}(t) < 0\big).$$

Let $\gamma = EN(1)$; $a := EZ_1$, $\tilde{a} := E\tilde{Z}_1 < \infty$ and $F_Z, F_{\tilde{Z}}$ denote, correspondingly, a distribution function of Z_1 and of $\tilde{Z}_1$. It is well known (see, for instance, [8]) that if the relative safety loading $\rho := \frac{\kappa}{\gamma a} - 1$ is positive, then

$$\Psi(x) = (1 - q) P\left(\sum_{k=1}^{\nu} X_k > x\right), \tag{3.3}$$

where the random variable ν has a geometric distribution with parameter $q = \frac{\rho}{1+\rho}$, it does not depend on sequence $(X_k,\ k \geq 1)$ and $X_1, X_2, \ldots$ are independent, identically distributed random variable with the common distribution function

$$F_{X_1}(x) = \frac{1}{a} \int_0^x \big(1 - F_Z(u)\big) \mathrm{d}u, \quad x \geq 0. \tag{3.4}$$

Let us assume that $a = \tilde{a}$, $EZ_1^2 = E\tilde{Z}_1^2$ and $E|Z_1|^3, E|\tilde{Z}_1|^3 < \infty$. Then (similarly to (3.3) and (3.4))

$$\tilde{\Psi}(x) = (1 - q) P\left(\sum_{k=1}^{\nu} \tilde{X}_k > x\right),$$

$$F_{\tilde{X}_1}(x) = \frac{1}{a} \int_0^x (1 - F_{\tilde{Z}}(u))\, \mathrm{d}u,\ x \geq 0, \tag{3.5}$$

and, moreover, $EX_1 = E\tilde{X}_1$; $E|X_1|^2, E|\tilde{X}_1|^2 < \infty$.

Consequently, to estimate $\rho(\Psi, \tilde{\Psi})$ we can apply Theorem 1, assuming that the sequences $Q = (X_k,\ k \geq 1)$ and $\tilde{Q} = (\tilde{X}_k,\ k \geq 1)$ belong to the class $\mathbf{K}_2(s, r)$ (for some s, r). Using (3.4) and (3.5) it is easy to show that:

$$\begin{aligned}\sigma^2 &= \operatorname{Var}(X_1) = (3a)^{-1}EZ_1^3 - (4a^2)^{-1}(EZ_1^2)^2,\\ \tilde{\sigma}^2 &= \operatorname{Var}(\tilde{X}_1) = (3a)^{-1}E\tilde{Z}_1^3 - (4a^2)^{-1}(E\tilde{Z}_1^2)^2,\end{aligned}$$

On the other hand,

$$\rho(X_1, \tilde{X}_1) = \frac{1}{a}\sup_{x>0}\left|\int_0^x [F_Z(u) - F_{\tilde{Z}}(u)]\,\mathrm{d}u\right| =: \frac{1}{a}\bar{\mu}(Z_1, \tilde{Z}_1)$$

and by virtue of inequality (18.3.19) in [13]

$$\zeta_2(X_1, \tilde{X}_1) \leq \frac{1}{2}\int_0^\infty x^2\left|\mathrm{d}F_{X_1}(x) - \mathrm{d}F_{\tilde{X}_1}(x)\right|$$

$$= \frac{1}{2a}\int_0^\infty x^2|F_Z(x) - F_{\tilde{Z}}(x)|\mathrm{d}x = \frac{1}{6a}\mathbf{k}_3(Z_1, \tilde{Z}_1),$$

where ζ_2 is Zolotarev's metric of order 2 defined in (4.1). As it is seen from the proof of Theorem 2 (and, so of Theorem 1; see (4.10)) the distance μ_2 in (2.1) can be replaced by the distance $\max\{\rho, \zeta_2\}$.

Thus, we get

$$\rho(\Psi, \tilde{\Psi}) \leq \frac{c}{a}\,(1-q)\max\left\{\bar{\mu}(Z_1, \tilde{Z}_1), \frac{1}{6}\,\mathbf{k}_3(Z_1, \tilde{Z}_1)\right\},$$

where the constant c is given by (2.2).

It is natural to ask: "When $Q, \tilde{Q} \in \mathbf{K}_2(s, r)$?". *The nice property of distribution functions given by (3.4), (3.5) is that $Q, \tilde{Q} \in \mathbf{K}_2\left(3, \frac{2}{a}\right)$ for every pair of random variables $Z_1, \tilde{Z}_1$.* Indeed, integrating by parts in the definition of characteristic function one can see that $|\varphi_{X_1}(t)| \leq 2/a|t|$, $t > 0$.

Example 2. (The stability estimate of a risk process.)

In the current example we consider risk processes $Z(t)$ and $\tilde{Z}(t)$ defined by (3.1), (3.2), but we relax assumptions made in the preceding example. Namely, we do not suppose that random variables $Z_1, Z_2, \ldots$ (correspondingly, $\tilde{Z}_1, \tilde{Z}_2, \ldots$) are identically distributed and we let $N(t)$ to be any integer-valued process (independent of $(Z_k), (\tilde{Z}_k)$). The goal is to manifest an upper bound for the quantity $\sup_{t\geq 0}\rho\big(Z(t), \tilde{Z}(t)\big)$ in terms of a deviation of $Q = (Z_k,\ k \geq 1)$ from $\tilde{Q} = (\tilde{Z}_k,\ k \geq 1)$.

Supposing Q and $\tilde{Q}$ to be in the class $\mathbf{K}_2(s; r)$ (for some s, r) and $EZ_k = E\tilde{Z}_k$, $k \geq 1$ we can apply inequality (2.1) in Theorem 1 to give an upper bound of $\rho(Z(t), \tilde{Z}(t))$. For instance, assume that a "real" density $f_k \equiv f_{Z_k}$ is represented as a mixture of Gamma densities with $\alpha \in [2, M]$, $\beta \in [\beta_0, B]$, where β_0, M, B are some positive finite numbers. That is

$$f_k(x) = \int_2^M \int_{\beta_0}^B \frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x} \mathrm{d}A_k(\alpha) \mathrm{d}B_k(\beta), \quad k \geq 1,$$

where A_k, B_k are given distribution functions. Let one be uncertain about A_k, B_k and approximate them by distribution functions $\tilde{A}_k, \tilde{B}_k$, i.e. the density of the approximating random variable $\tilde{Z}_k$ is

$$\tilde{f}_k(x) = \int_2^M \int_{\beta_0}^B \frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x} \mathrm{d}\tilde{A}_k(\alpha) \mathrm{d}\tilde{B}_k(\beta).$$

Assuming

$$\int_2^M \int_{\beta_0}^B \frac{\alpha}{\beta} \mathrm{d}A_k(\alpha) \mathrm{d}B_k(\beta) = \int_2^M \int_{\beta_0}^B \frac{\alpha}{\beta} \mathrm{d}\tilde{A}_k(\alpha) \mathrm{d}\tilde{B}_k(\beta)$$

for $k = 1, 2, \ldots$, we find the hypotheses of Theorem 1 to be fulfilled with $Q, \tilde{Q} \in \mathrm{K}_2(3; 0.90980)$ (see the table and Remark 4 in Section 2). Therefore, taking $\lambda = 1.3$ in (2.2) we obtain from (2.1)

$$\rho(Z(t), \tilde{Z}(t)) \leq \max\left\{5, \frac{15}{\pi}\left[\frac{1.34}{\sigma_*^2} + \frac{1}{\tilde{\sigma}_*^2}\right]\right\} \mu_2(Q, \tilde{Q}). \tag{3.6}$$

Under the above mentioned restrictions on distributions of Z_1 and on those of $\tilde{Z}_1$ bound (3.6) considerably improves the stability estimates of the risk process given in [13], Chapt. 16. In the particular case of the same distributions of numbers of claims in the original risk process and in its approximation these bounds appear in inequality (16.2.15) in [13]. Namely, requiring the existence of a bounded density ρ_t of the random variable

$$[EN(t)]^{-1} \sum_{k=1}^{N(t)} \tilde{Z}_k,$$

(16.2.15) in [13] provides the following estimate:

$$\rho(Z(t), \tilde{Z}(t)) \leq (12\sqrt{5})^{1/3}[1 + \sup_x \rho_t(x)][EN(t)]^{-1/3}[\mathrm{k}_2(Z_1, \tilde{Z}_1)]^{1/3}, \tag{3.7}$$

which holds if $EZ_1 = \tilde{Z}_1$ and $EZ_1^2 = E\tilde{Z}_1^2$.

If $N(t)$ is the Poisson process (with parameter $\gamma > 0$), then for all sufficiently large t

$$\sup_x \bar{\rho}_t(x) \geq c' > 0,$$

where $\bar{\rho}_t$ is the density of the random variable $\frac{1}{\sqrt{\gamma t}} \sum_{k=1}^{N(t)} \tilde{Z}_k$. This follows from the central limit theorem for densities (see, e.g. the supplement to Chapt. VII in [12]).

Thus $\sup_x \rho_t(x)$ is greater than $\sqrt{\gamma t}c'$ for the mentioned t, and therefore, the time-depending term in (3.7) is of order $c''t^{1/6}$ as $t \to \infty$. On the other hand, making minor modifications in the proof of (16.2.15) and taking advantage of inequality (14.1.4), [13] one can get the following inequality valid when two first moments of the random variables Z_1 and $\tilde{Z}_1$ are equal:

$$\rho(Z(t), \tilde{Z}(t)) \le \left[1 + \sup_{x\in\mathbb{R}} \bar{\rho}_t(x)\right] \left[2\mathbf{k}_2(Z_1, \tilde{Z}_1)\right]^{1/3}. \tag{3.8}$$

The main difference between (3.6) and (3.8) are the values of exponents (a linear function against $(\cdot)^{1/3}$) involving in the factors measuring the accuracy of approximation. (These factors are expected to be small in the setting of the stability problem.) To make these comments more clear and to give a numerical illustration we consider the following example.

Let Z_k, $k = 1, 2, \ldots$ be identically distributed and $\tilde{Z}_k$ $k = 1, 2, \ldots$ be so. Let B_k assigns masses $1 - p$ and $p = 0.1$ to the points $\beta = 2$ and $\beta = 6.2/3$. Also A_k allocates a mass $(1 - p)$ to $\alpha = 6$ and a mass p – to $\alpha = 6.2$. On the other hand, we choose $\tilde{A}_k$, $\tilde{B}_k$ to be concentrated at the points $\alpha = 6, \beta = 2$, respectively. We have $Q, \tilde{Q} \in \mathbf{K}_2(3; 0.73470)$ (see the table), $\sigma_*^2 > 1.49929$ and $\tilde{\sigma}_*^2 = 1.5$. Thus, we find that $c = \max\{5, 4.89405\} = 5$ and (3.6) turns into the following inequality:

$$\rho(Z(t), \tilde{Z}(t)) \le 5 \max\left\{\rho(Z_1, \tilde{Z}_1),\ \frac{1}{2}\,\mathbf{k}_2(Z_1, \tilde{Z}_1)\right\}. \tag{3.9}$$

Finally, we calculate by computer:

$$\rho(Z_1, \tilde{Z}_1) < 0.00051537, \quad \mathbf{k}_2(Z_1, \tilde{Z}_1) < 0.0053486$$

to write out the following estimate (valid for every t):

$$\rho(Z(t), \tilde{Z}(t)) \le 0.026743.$$

In contrast to this bound provided by (3.9) in the current example inequality (3.8) offers the estimate

$$\rho(Z(t), \tilde{Z}(t)) \le \left[1 + \sup_x \bar{\rho}_t(x)\right] 0.220339.$$

Remark 5. Concerning inequality (2.3) in Theorem 2 we note that if the number of summands $\nu = \max\{1, N(t)\}$, where $N(t)$ is the Poisson process with parameter γt then, for some constant $\tilde{c}$

$$E\left(\nu^{-\frac{m-2}{2}}\right) \le \tilde{c}(\gamma t)^{-\frac{m-2}{2}}, \qquad t > 0.$$

Meanwhile in this case the right-hand side of inequality (16.2.15) in [13] is of order $t^{-\frac{m-1}{m+1}+\frac{1}{2}}$, as $t \to \infty$.

Example 3. (Estimating the stability of the approximation by Erlang's distributions.)

The so-called Erlang's distributions (Gamma distributions with integer parameter α) proved to be useful to model a random service time (as well as, an input flow) in queueing systems (see, e.g. [10, 11]). For instance, these distributions appear in modelling of multiphase service. If n is a number of service phases and X_k is the duration of the kth phase, then a total service time is $S_n = X_1 + \cdots + X_n$. The problem of stability estimating arises every time when one tries to justify a use of a customary approach to modelling replacing a "real" X_k by an exponentially distributed random variable $\tilde{X}_k$ (say, with parameter $\beta_k > 0$). Hence, $\tilde{S}_n = \tilde{X}_1 + \cdots + \tilde{X}_n$ is adopted to imitate S_n. When $\beta_k = \beta$, $k = 1, 2, \ldots, n$, $\tilde{S}_n$ has the Gamma density with parameters $n\beta, \beta$. In practice, nonnegative X_k, $k = 1, 2, \ldots, n$ are different from exponential random variables, but often they are, in some sense, close to them.

Denote: $Q = (X_k, k \geq 1), \tilde{Q} = (\tilde{X}_k, k \geq 1)$, where $X_1, X_2, \ldots$ are independent (as well as, $\tilde{X}_1, \tilde{X}_2, \ldots$ are) and $\tilde{X}_k$ has the exponential distribution with parameter $\beta_k > 0, (k \geq 1)$. We assume that $EX_k = E\tilde{X}_k = \frac{1}{\beta_k}$, $k \geq 1$. There are good statistical tests to determine whether X_k is close to $\tilde{X}_k$ (in distribution). For instance, let $\hat{a}_k$ and $\hat{\sigma}_k^2$ be some statistical estimates of mean and variance of X_k and suppose that X_k belongs to the class of aging distributions called NBUE (new better that used in expectation, see [11] for the definition). If $\hat{\sigma}_k^2/\tilde{a}_k^2 \approx 1$, one can conclude that the distribution of X_k is in close proximity to the exponential one. Moreover, the following stability estimates are known [11, 13]:

$$\rho(X_k, \tilde{X}_k) \leq \left(1 - \frac{\sigma_k^2}{a_k^2}\right)^{1/2}, \quad \zeta_2(X_k, \tilde{X}_k) \leq \frac{1}{2}(a_k^2 - \sigma_k^2),$$

where $\sigma_k^2 := \mathrm{Var}(X_k) \leq a_k^2 := (EX_k)^2$ and ζ_2 is Zolotarev's metric of order 2. We additionally suppose that $Q = (X_k, k \geq 1) \in \mathbf{K}_2(3; 1)$, relaxing, if needed, the condition (a) in Definition 1, assuming, instead: $0 < \mathrm{Var}(X_k) < \infty$, $k = 1, 2, \ldots, n$. Note that all sequences of independent, identically distributed random variables with distributions given in the table of Section 2 are in $\mathbf{K}_2$ (3;1). As it follows from the proof of Theorem 1 the term $\frac{1}{2}$ $\mathbf{k}_2$ in the definition of the metric μ_2 (and in Theorem 1, respectively) can be replaced by Zolotarev's metric ζ_2. Therefore, from (2.1), (2.2) we obtain, for each $n \geq 1$:

$$\begin{aligned} \rho(S_n, \tilde{S}_n) &\leq \max\left\{5, \tfrac{2.33334}{\pi\sigma^2} \inf_{\lambda>1} \left[\tfrac{3}{\lambda} + \tfrac{4\lambda^4}{\lambda^2-1}\right]\right\} \\ &\leq \max\left\{\max_{1\leq k\leq n} \left(1 - \frac{\sigma_k^2}{a_k^2}\right)^{1/2}, \max_{1\leq k\leq n} \frac{1}{2}(a_k^2 - \sigma_k^2)\right\}, \end{aligned} \tag{3.10}$$

where $\sigma^2 := \min\left\{\min\limits_{1\leq k\leq n} \sigma_k^2, \min\limits_{1\leq k\leq n} \frac{1}{\beta_k^2}\right\}$. Bound (3.10) is "acceptable" in the case of relatively large σ and a small enough absolute deviation $\epsilon := \sup_k(a_k^2 - \sigma_k^2)$. For instance, let $\inf_k \sigma_k \geq 2$ and $\inf_k a_k = \inf_k \frac{1}{\beta_k} \geq 2$. (In some sense the former yields the latter since in the approximation considered a_k is somewhat like σ_k.) Then, assuming $\epsilon \leq 1$ and taking $\lambda = 1.5$ in (2.2) we get:

$$\rho(S_n, \tilde{S}_n) \leq 2.5\sqrt{\epsilon}.$$

4. THE PROOFS

Proposition 1 of Section 2 easily follows from the well-known fact (see, [3], Chapt. 15) that $f_Z' \in \mathbb{L}_1(\mathbb{R})$ yields $|\varphi_Z(t)| = o(|t|^{-1})$ as $t \to \infty$.

In what follows, we shall use Zolotarev's metric

$$\zeta_m(X, Y) = \sup_{\varphi \in D_m} |E\varphi(X) - E\varphi(Y)|, \tag{4.1}$$

where D_m is the class of all functions $\varphi : \mathbb{R} \to \mathbb{R}$ having (almost everywhere) the mth derivative bounded by 1. It is well known that (see [11, 13])

$$\zeta_m(X, Y) \leq \frac{1}{m!} \mathbf{k}_m(X, Y) < \infty,$$

provided that $EX^j = EY^j$, $j = 1, 2, \ldots, m-1$, $E|X|^m, E|Y|^m < \infty$.

Proof of Theorem 2. First of all, observe that

$$\rho(X + Z, Y + Z) \leq \sup_x |f_Z^{(m-1)}(x)| \zeta_m(X, Y), \tag{4.2}$$

if the random variable Z has a density f_Z such that a bounded derivative $f_Z^{(m-1)}$ exists almost everywhere on $\mathbb{R}$. Indeed,

$$\rho(X + Z, Y + Z) = \sup_{x \in \mathbb{R}} \left| \int_{-\infty}^{\infty} F_Z(x - t) \mathrm{d}\left[F_X(t) - F_Y(t)\right] \right|$$

and the functions

$$\varphi_x(t) := \frac{1}{\sup_x |f_Z^{(m-1)}(x)|} F_Z(x - t), \quad x \in \mathbb{R}$$

have almost everywhere the mth derivative bounded by 1.

Now we take any positive λ satisfying the condition $\delta := r\lambda^{-\frac{m+1}{s}} < 1$ and fix arbitrary integers r, j such that $j \geq 0$, $n \geq s$. Denoting

$$\begin{aligned} Y &= \frac{\lambda}{\sigma_* \sqrt{n}} (X_{j+1} + X_{j+2} + \cdots + X_{j+n}), \\ \tilde{Y} &= \frac{\lambda}{\tilde{\sigma}_* \sqrt{n}} (\tilde{X}_{j+1} + \tilde{X}_{j+2} + \cdots + \tilde{X}_{j+n}), \end{aligned}$$

we show that the derivatives $f_Y^{(m-1)}(x)$, $f_{\tilde{Y}}^{(m-1)}(x)$ of the corresponding densities exist everywhere on $\mathbb{R}$ and, moreover,

$$\sup_{x\in\mathbb{R}}|f_Y^{(m-1)}(x)| \le \frac{1}{\pi}[s\psi_1(\lambda) + \psi_2(\lambda)],$$
$$\sup_{x\in\mathbb{R}}|f_{\tilde{Y}}^{(m-1)}(x)| \le \frac{1}{\pi}[s\psi_1(\lambda) + \psi_2(\lambda)].$$

Since the proofs are same for f_Y and $f_{\tilde{Y}}$, we focus on f_Y. Let $\bar{\varphi}_k$, φ_k and φ_Y denote the characteristic functions of X_k/σ_*, $\lambda X_k/\sigma_*$ and Y, respectively.

By inequality (b) in Definition 1 $\int_{-\infty}^{\infty} |t|^{m-1}|\varphi_Y(t)|\mathrm{d}t < \infty$ and, hence

$$f_Y^{(m-1)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-it)^{m-1}e^{-itx}\varphi_Y(t)\mathrm{d}t \qquad \text{exists,}$$

and

$$\begin{aligned}\sup_x|f_Y^{(m-1)}(x)| &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}|t|^{m-1}\prod_{k=j+1}^{j+n}\left|\varphi_k\left(\frac{t}{\sqrt{n}}\right)\right|\mathrm{d}t \\ &= \frac{1}{2\pi}\left[n^{m/2}\int_{|t|\ge 1}|t|^{m-1}\prod_{k=j+1}^{j+n}|\varphi_k(t)|\mathrm{d}t\right. \\ &\left.+ n^{m/2}\int_{-1}^{1}|t|^{m-1}\prod_{k=j+1}^{j+n}|\varphi_k(t)|\mathrm{d}t\right] =: \frac{1}{2\pi}[I_1 + I_2].\end{aligned} \tag{4.4}$$

We estimate I_1 and I_2 separately. By virtue of (b) in Definition 1

$$|\varphi_k(t)| = |\bar{\varphi}_k(\lambda t)| \le \frac{r}{|\lambda t|^{\frac{m+1}{s}}} = \delta|t|^{-\frac{m+1}{s}}. \tag{4.5}$$

Consequently,

$$\begin{aligned} I_1 &\le 2n^{m/2}\delta^n\int_1^{\infty} t^{m-1-\frac{(m+1)n}{s}}\mathrm{d}t \\ &= 2s\frac{\delta^n n^{m/2}}{n(m+1)-ms} \le 2s\psi_1(\lambda; r, m, s),\end{aligned} \tag{4.6}$$

(see the definition of ψ_1 in (2.5)).

In view of (4.5) $|\varphi_k(t)| \le \delta < 1$ for $|t| \ge 1$. Thus, by Theorem 1, Chapt. 1 in

[12], $|\varphi_k(t)| \le 1 - \frac{1-\delta^2}{8} t^2$ for $|t| < 1$. Therefore,

$$\begin{aligned} I_2 &\le 2n^{m/2} \int_0^1 t^{m-1} \left[1 - \tfrac{(1-\delta^2)}{8} t^2\right]^n \mathrm{d}t \\ &= 2 \int_0^{\sqrt{n}} x^{m-1} \left[1 - \tfrac{(1-\delta^2)}{8} \tfrac{x^2}{n}\right]^n \mathrm{d}x \\ &\le 2 \lim_{n\to\infty} \int_0^{\sqrt{n}} x^{m-1} \left[1 - \tfrac{(1-\delta^2)}{8} \tfrac{x^2}{n}\right]^n \mathrm{d}x \\ &= 2 \int_0^{\infty} x^{m-1} \exp\left(-\tfrac{(1-\delta^2)}{8} x^2\right) \mathrm{d}x, \end{aligned} \tag{4.7}$$

by the Monotone Convergence Theorem. The last integral in (4.7) is calculated as the corresponding absolute moment of the Gaussian distribution. Consequently,

$$I_2 \le 2\psi_2(\lambda; r, m), \tag{4.8}$$

where the function ψ_2 is defined in (2.6). Combining inequalities (4.4), (4.6) and (4.8) we arrive at the desired bound (4.3).

For an arbitrary, but fixed $n \ge 2s$ let $k = [n/2]$, where $[x]$ is the integer part of x. Denoting

$$\begin{aligned} Z_k &= X_1 + \cdots + X_k; \quad Z_k' = X_{k+1} + \cdots + X_n; \\ \tilde{Z}_k &= \tilde{X}_1 + \cdots + \tilde{X}_k; \quad \tilde{Z}_k' = \tilde{X}_{k+1} + \cdots + \tilde{X}_n, \end{aligned}$$

we obtain by the triangle inequality:

$$\rho(S_n, \tilde{S}_n) \le \rho(Z_k + Z_k', Z_k + \tilde{Z}_k') + \rho(Z_k + \tilde{Z}_k', \tilde{Z}_k + \tilde{Z}_k'). \tag{4.9}$$

In view of (4.2), (4.3) we can write (using the homogeneity of ρ: $\rho(aX, aY) = \rho(X, Y)$, $a \ne 0$ and denoting $b = \frac{1}{\pi}[s\psi_1(\lambda) + \psi_2(\lambda)]$):

$$\begin{aligned} &\rho(Z_k + Z_k', Z_k + \tilde{Z}_k') \\ &= \rho\left(\lambda \frac{Z_k}{\sigma_* \sqrt{k}} + \lambda \frac{Z_k'}{\sigma_* \sqrt{k}}, \lambda \frac{Z_k}{\sigma_* \sqrt{k}} + \lambda \frac{\tilde{Z}_k'}{\sigma_* \sqrt{k}}\right) \le b\zeta_m \left(\lambda \frac{Z_k'}{\sigma_* \sqrt{k}}, \lambda \frac{\tilde{Z}_k'}{\sigma_* \sqrt{k}}\right) =: I_n. \end{aligned}$$

The well-known (see [11, 16]) property of Zolotarev's metric

$$\zeta_m\left(a \sum_{i=1}^n \xi_i, a \sum_{i=1}^n \eta_i\right) \le a^m \sum_{i=1}^n \zeta_m(\xi_i, \eta_i), \quad a \ge 0$$

allows us to bound I_n as follows:

$$\begin{aligned} I_n &\le b \left(\tfrac{\lambda}{\sigma_*}\right)^m \tfrac{1}{k^{m/2}} \textstyle\sum_{i=k+1}^n \zeta_m(X_i, \tilde{X}_i) \\ &\le b \left(\frac{\lambda}{\sigma_*}\right)^m \frac{n - [n/2]}{[n/2]^{m/2}} \frac{1}{m!} \max_{k+1 \le i \le n} \mathbf{k}_m(X_i, \tilde{X}_i) \\ &\le \frac{b}{m!} \frac{\lambda^m}{\sigma_*^m} \gamma(m, s) \mathbf{k}_m(Q, \tilde{Q}) n^{-\frac{m-2}{2}}. \end{aligned} \tag{4.10}$$

In (4.10) $\mathbf{k}_m(Q,\tilde{Q}) < \infty$ by virtue of (a) in Definition 1 and of the definition of $\mathbf{k}_m$.

The second summand on the right-hand side of (4.9) is estimated similarly:

$$\begin{aligned} &\rho(Z_k + \tilde{Z}_k, \tilde{Z}_k + \tilde{Z}'_k) = \\ = \; &\rho\left(\lambda\frac{Z_k}{\tilde{\sigma}_*\sqrt{n-k}} + \lambda\frac{\tilde{Z}'_k}{\tilde{\sigma}_*\sqrt{n-k}}, \lambda\frac{\tilde{Z}_k}{\tilde{\sigma}_*\sqrt{n-k}} + \lambda\frac{\tilde{Z}'_k}{\tilde{\sigma}_*\sqrt{n-k}}\right) \\ \leq \; &\frac{b}{m!}\frac{\lambda^m}{\tilde{\sigma}_*^m}\tilde{\gamma}(m,s)\mathbf{k}_m(Q,\tilde{Q})n^{-\frac{m-2}{2}}. \end{aligned} \tag{4.11}$$

Combining (4.9) – (4.11) shows that

$$\rho(S_n, \tilde{S}_n) \leq b\lambda^m \left[\frac{\gamma(m,s)}{\sigma_*^m} + \frac{\tilde{\gamma}(m,s)}{\tilde{\sigma}_*^m}\right]\mu_m(Q,\tilde{Q})n^{-\frac{m-2}{2}} \tag{4.12}$$

for $n = 2s, 2s+1, \ldots$.

By the regularity of the metric ρ we get for $n = 1, 2, \ldots, 2s-1$,

$$\begin{aligned} \rho(S_n, \tilde{S}_n) &\leq \sum_{k=1}^{n} \rho(X_k, \tilde{X}_k) \leq (2s-1)\rho(Q,\tilde{Q}) \\ &\leq (2s-1)\mu_m(Q,\tilde{Q}) \leq (2s-1)^{m/2}\mu_m(Q,\tilde{Q})n^{-\frac{m-2}{2}}. \end{aligned} \tag{4.13}$$

Thus, remembering the definition of b and taking into account the fact that the only restriction to choose λ in the above calculations was the inequality $r\lambda^{-\frac{m+1}{s}} < 1$, we deduce from (4.12), (4.13) and (2.4) the following inequalities:

$$\rho(S_n, \tilde{S}_n) \leq c_m\mu_m(Q,\tilde{Q})n^{-\frac{m-2}{2}}, \quad n = 1, 2, \ldots .$$

To complete the proof it is sufficient to apply the total probability formula:

$$\begin{aligned} \rho(S,\tilde{S}) &= \sup_{x\in\mathbb{R}}\left|P\left(\sum_{k=1}^{\nu} X_k \leq x\right) - P\left(\sum_{k=1}^{\nu} \tilde{X}_k \leq x\right)\right| \\ &\leq \sum_{n=1}^{\infty}\sup_{x\in\mathbb{R}}\left|P\left(\sum_{k=1}^{n} X_n \leq x\right) - P\left(\sum_{k=1}^{n} \tilde{X}_n \leq x\right)\right|P(\nu = n) \\ &\leq c_m\mu_m(Q,\tilde{Q})\sum_{n=1}^{\infty} n^{-\frac{m-2}{2}}P(\nu = n). \end{aligned}$$

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