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# BELL–TYPE INEQUALITIES FOR PARAMETRIC FAMILIES OF TRIANGULAR NORMS

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In recent work we have shown that the reformulation of the classical Bell inequalities into the context of fuzzy probability calculus leads to related inequalities on the commutative conjunctor used for modelling pointwise fuzzy set intersection. Also, an important role has been attributed to commutative quasi-copulas. In this paper, we consider these new Bell-type inequalities for continuous t-norms. Our contribution is twofold: first, we prove that ordinal sums preserve these Bell-type inequalities; second, for the most important parametric families of continuous Archimedean t-norms and each of the inequalities, we identify the parameter values such that the corresponding t-norms satisfy the inequality considered.

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## 1. INTRODUCTION

In nature one is confronted both with phenomena that fit classical probability theory and phenomena that call for a nonclassical one. Therefore, criteria have been developed in order to distinguish the two cases. In 1964, Bell [1] introduced examples of inequalities involving probabilities which are valid in classical probability theory, but are violated by some quantum mechanical experiments. Later on, Pitowsky [7] showed that these classical conditions can be derived in a purely mathematical context without any reference to physics so that their range of applicability is by no way restricted to physical phenomena.

Pykacz and D'Hooghe [8] recently studied which of the numerous Bell-type inequalities that are necessarily satisfied by Kolmogorovian probabilities may be violated in various models of fuzzy probability calculus. They proved that if we consider fuzzy set intersection defined pointwisely by a Frank t-norm $T^{\mathbf{F}}_{\lambda}$, then the borderline between models of fuzzy probability calculus that can be distinguished from Kolmogorovian ones and models that cannot be distinguished (by the same set of inequalities) is situated at $\lambda = 9 + 4\sqrt{5}$. In this paper, we want to extend this discussion, on the one hand by considering the most important parametric families of t-norms listed in [4], on the other hand by considering all Bell-type inequalities up to four events.

This paper is organized as follows. In Section 2, we recall the definitions of quasi-copulas, copulas and t-norms. We then list the Bell inequalities up to four events and indicate which of them are generally valid for commutative (quasi-)copulas. The first major contribution of this paper can be found in Section 4: ordinal sums preserve the Bell-inequalities! This result permits to consider continuous Archimedean t-norms only. In Section 5, we therefore focus on the seven most important parametric t-norm families. For each of them, tedious, often computer-assisted calculations are needed for determining the particular parameter value that separates t-norms that do fulfil a given inequality from those that do not. In Section 6, we further refine the existing knowledge on a very particular countably infinite family of inequalities, and show that the algebraic product is not the smallest t-norm satisfying all of them, but that for instance also the Hamacher t-norm with parameter value 2 is a good candidate.

## 2. COPULAS AND TRIANGULAR NORMS

The Bell inequalities in fuzzy probability calculus are strongly related to the operation used for modelling fuzzy set intersection. Since usually not more than two fuzzy sets (events) are intersected at the same time, one could opt to consider a commutative conjunctor $f$, i. e. a commutative and increasing $[0,1]^2 \to [0,1]$ mapping that coincides with the Boolean conjunction on $\{0,1\}^2$. Not surprisingly, stronger results are obtained when considering more specific classes of commutative conjunctors, such as commutative quasi-copulas and copulas (featuring the 1-Lipschitz property, a strong kind of continuity) or continuous t-norms (featuring both associativity and continuity). Next, we recall their origin and mathematical definition.

Copulas were introduced by Sklar in 1959 and are used for combining marginal probability distributions into joint probability distributions. Triangular norms were introduced by Menger in 1942 and permit to define a kind of triangle inequality in the setting of probabilistic metric spaces. We adopt here the definitions and notations from [4, 6].

**Definition 1.** (See [2, 6].) A binary operation $C : [0,1]^2 \to [0,1]$ is called a quasi-copula if it satisfies:

(i) Neutral element 1. (i') Absorbing element 0.

(ii) Monotonicity: $C$ is increasing in each variable.

(iii) 1-Lipschitz property: for any $(x_1, x_2, y_1, y_2) \in [0,1]^4$ it holds that:

$$|C(x_1, y_1) - C(x_2, y_2)| \leq |x_1 - x_2| + |y_1 - y_2| .$$

If instead of (iii) $C$ satisfies

(iv) Moderate growth: for any $(x_1, x_2, y_1, y_2) \in [0,1]^4$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$ it holds that:

$$C(x_1, y_2) + C(x_2, y_1) \leq C(x_1, y_1) + C(x_2, y_2) ,$$

then $C$ is called a copula.

Note that in case of a quasi-copula, condition (i') is superfluous, while for a copula condition (ii) can be omitted (as it follows from (iv) and (i')). As implied by the terminology used, any copula is a quasi-copula, and therefore has the 1-Lipschitz property; the opposite is, of course, not true.

**Definition 2.** A binary operation $T : [0,1]^2 \to [0,1]$ is called a triangular norm (t-norm for short) if it satisfies for any $(x, y, z) \in [0,1]^3$:

(i) Neutral element 1.

(ii) Monotonicity: $T(x,y) \leq T(x,z)$ whenever $y \leq z$.

(iii) Associativity: $T(x, T(y,z)) = T(T(x,y), z)$.

(iv) Commutativity: $T(x,y) = T(y,x)$.

The four basic t-norms are: the minimum operator $T_{\mathbf{M}}(x,y) = \min(x,y)$, the algebraic product $T_{\mathbf{P}}(x,y) = xy$, the Łukasiewicz t-norm $T_{\mathbf{L}}(x,y) = \max(x+y-1, 0)$ and the drastic product $T_{\mathbf{D}}$:

$$T_{\mathbf{D}}(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

They can be ordered as follows: $T_{\mathbf{D}} < T_{\mathbf{L}} < T_{\mathbf{P}} < T_{\mathbf{M}}$.

It is well known that a copula is a t-norm if and only if it is associative; conversely, a t-norm is a copula if and only if it is 1-Lipschitz. Among the t-norms mentioned above, the minimum operator $T_{\mathbf{M}}$, the Łukasiewicz t-norm $T_{\mathbf{L}}$ and the algebraic product $T_{\mathbf{P}}$ are (associative and commutative) copulas. The drastic product $T_{\mathbf{D}}$ is not a copula (it is right-continuous only). In the context of Bell-inequalities, it is important to know that for any quasi-copula $C$ it holds that $T_{\mathbf{L}} \leq C \leq T_{\mathbf{M}}$ [6].

Ling [5] has shown that for every continuous t-norm $T$, either $T = T_{\mathbf{M}}$, $T$ is Archimedean or $T$ is the ordinal sum of a family of continuous Archimedean t-norms. We clarify the notions used in this statement.

**Definition 3.** Let $(T_\alpha)_{\alpha \in A}$ be a family of t-norms and $(]a_\alpha, e_\alpha[)_{\alpha \in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. The t-norm $T$ defined by

$$T(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) T_\alpha \left( \frac{x - a_\alpha}{e_\alpha - a_\alpha}, \frac{y - a_\alpha}{e_\alpha - a_\alpha} \right), & \text{if } (x,y) \in [a_\alpha, e_\alpha]^2, \\ \min(x,y), & \text{otherwise}, \end{cases}$$

is called the ordinal sum of the summands $\langle a_\alpha, e_\alpha, T_\alpha \rangle$, and we write

$$T = (\langle a_\alpha, e_\alpha, T_\alpha \rangle)_{\alpha \in A} .$$

**Definition 4.** (See [4].) A t-norm $T$ is called Archimedean if

$$(\forall (x,y) \in ]0,1[^2)(\exists n \in \mathbb{N})(x_T^{(n)} < y) ,$$

with $x_T^{(0)} = 1$ and $x_T^{(n)} = T(x, x_T^{(n-1)})$.

As the t-norms considered in this paper are always continuous, it suffices to know that

**Proposition 1.** (See [4].) A continuous t-norm $T$ is Archimedean if and only if

$$(\forall x \in ]0,1[)(T(x,x) < x)\,.$$

Note that $T_{\mathbf{M}}$ is not Archimedean, while $T_{\mathbf{P}}$, $T_{\mathbf{L}}$ and $T_{\mathbf{D}}$ are.

## 3. BELL–TYPE INEQUALITIES IN FUZZY LOGIC

In [3], we have described in detail the Bell inequalities and how they can be rewritten in the context of fuzzy probability calculus. For instance, the classical inequality

$$\mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \cap B) \leq 1\,,$$

can be expressed for fuzzy probabilities, with $A$ and $B$ fuzzy sets in a finite universe $X$ of cardinality $n$ and $A \cap B$ pointwisely modelled by means of a commutative conjunctor $I$, in the following way:

$$\frac{1}{n}\sum_u A(u) + \frac{1}{n}\sum_u B(u) - \frac{1}{n}\sum_u I(A(u), B(u)) \leq 1\,.$$

The latter inequality is fulfilled when $A(u)+B(u)-I(A(u),B(u)) \leq 1$ for any $u \in X$, which in turn is fulfilled when

$$x + y - I(x,y) \leq 1\,,$$

for any $(x,y) \in [0,1]^2$. Inequalities of this type are called Bell-type inequalities for commutative conjunctors. All Bell-type inequalities involving up to four events are collected in Table 1. To simplify the discussion of these inequalities we introduce a unique code $I_i^j$ for each inequality where $i$ denotes the number of events involved and $j$ is a sequential number.

In [3], we haven proven the following results.

**Theorem 1.**

(i) $I_2^1$, $I_3^2$ and $I_4^4$ are fulfilled for any commutative quasi-copula $C$.

(ii) $I_4^5$ is fulfilled for any commutative copula $C$.

Moreover, some generalization of $I_3^2$ and $I_4^4$ holds for any commutative quasi-copula.

**Theorem 2.** For any commutative conjunctor $I$ that satisfies $I_3^2$ and $I_4^4$, the following inequality holds for any $n \geq 3$:

$$0 \leq \sum_{i=2}^{n-1} x_i - \sum_{i=1}^{n-1} I(x_i, x_{i+1}) + I(x_1, x_n) \leq \left\lceil \frac{n}{2} \right\rceil - 1$$

for any $(x_1, \ldots, x_n) \in [0,1]^n$.

**Table 1.** Bell-type inequalities.

| code | inequality |
|---|---|
| $I_2^1$ | $T_{\mathbf{L}} \leq I \leq T_{\mathbf{M}}$ |
| $I_3^2$ | $0 \leq x - I(x,y) - I(x,z) + I(y,z)$ |
| $I_3^3$ | $x + y + z - I(x,y) - I(x,z) - I(y,z) \leq 1$ |
| $I_4^4$ | $0 \leq x + t - I(x,z) - I(x,t) - I(y,t) + I(y,z) \leq 1$ |
| $I_4^5$ | $0 \leq x + t - I(x,y) - I(x,z) + I(x,t) + I(y,z) - I(y,t) - I(z,t)$ |
| $I_4^6$ | $x + y + z + t - I(x,y) - I(x,z) - I(x,t) - I(y,z) - I(y,t) - I(z,t) \leq 1$ |
| $I_4^7$ | $2x + 2y + 2z + 2t - I(x,y) - I(x,z) - I(x,t) - I(y,z) - I(y,t) - I(z,t) \leq 3$ |
| $I_4^8$ | $0 \leq x - I(x,y) - I(x,z) - I(x,t) + I(y,z) + I(y,t) + I(z,t)$ |
| $I_4^9$ | $x + y + z - 2t - I(x,y) - I(x,z) + I(x,t) - I(y,z) + I(y,t) + I(z,t) \leq 1$ |

For the remaining inequalities $I_3^3$ and $I_4^6$–$I_4^9$, no general results are available. Remark that all inequalities are fulfilled for $T_{\mathbf{M}}$ and $T_{\mathbf{P}}$, while for $T_{\mathbf{L}}$ only inequalities $I_2^1$, $I_2^2$, $I_4^4$ and $I_4^5$ hold.

## 4. BELL–TYPE INEQUALITIES FOR ORDINAL SUMS

This section consists of a single theorem stating that ordinal sums preserve Bell-type inequalities. We conclude the section with a more general conjecture.

**Theorem 3.** Consider any of the Bell-type inequalities. The ordinal sum of a family of t-norms fulfils this inequality if and only if each of the summands fulfils this inequality.

Proof. The fact that the summands of an ordinal sum fulfil a given Bell-type inequality when the ordinal sum does, is easily verified. The converse is more tedious. Unfortunately, at this moment, each of the inequalities requires its own proof. To illustrate the line of reasoning, we consider for instance inequalities $I_3^3$ and $I_4^4$. The proofs for the other inequalities are similar and are mainly case-based.

*Inequality* $I_3^3$. First we remark that substituting $x = 0$ in $I_3^3$ yields the left part of $I_2^1$, i.e. $T_{\mathbf{L}} \leq T$. Now let $T$ be the ordinal sum of a family of t-norms that fulfil $I_3^3$. Due to the symmetry of $I_3^3$ in $x$, $y$ and $z$, we can assume without loss of generality that $x \leq y \leq z$. If $x$ and $y$, as well as $y$ and $z$, do not belong to same summand, then $I_3^3$ is fulfilled since it holds for $T_{\mathbf{M}}$.

(i) If $x$ and $z$ belong to the same summand $\langle a, b, T^* \rangle$, with $T^*$ a t-norm that fulfils inequality $I_3^3$, then also $y$ belongs to this summand. We can rewrite $I_3^3$ as follows

$$x + y + z - (a + (b-a)T^*(x', y')) - (a + (b-a)T^*(x', z')) - (a + (b-a)T^*(y', z')) \leq 1\,,$$

with $x' = \frac{x-a}{b-a}$, $y' = \frac{y-a}{b-a}$ and $z' = \frac{z-a}{b-a}$. The latter inequality is equivalent to

$$x' + y' + z' - T^*(x', y') - T^*(x', z') - T^*(y', z') \leq \frac{1}{b-a}\,.$$

Since $I_3^3$ holds for $T^*$ and $1 \leq \frac{1}{b-a}$, the above also holds.

(ii) If $x$ and $z$ do not belong to the same summand, i.e. $T(x, z) = T_{\mathbf{M}}(x, z) = x$, then we have to prove the following inequality:

$$y + z - T(x, y) - T(y, z) \leq 1\,. \tag{1}$$

(a) If $x$ and $y$ belong to the same summand $\langle a, b, T^* \rangle$, then $T(y, z) = T_{\mathbf{M}}(y, z) = y$ and (1) is equivalent to

$$\frac{z-a}{b-a} - T^*(x', y') \leq \frac{1}{b-a}\,,$$

with $x' = \frac{x-a}{b-a}$ and $y' = \frac{y-a}{b-a}$. It easily follows that

$$\frac{z-a}{b-a} - T^*(x', y') \leq \frac{1}{b-a} - T^*(x', y') \leq \frac{1}{b-a}\,.$$

(b) If $y$ and $z$ belong to the same summand $\langle a, b, T^* \rangle$, then $T(x, y) = T_{\mathbf{M}}(x, y) = x$ and (1) is equivalent to

$$y' + z' - \frac{x-a}{b-a} - T^*(y', z') \leq \frac{1}{b-a}\,,$$

with $y' = \frac{y-a}{b-a}$ and $z' = \frac{z-a}{b-a}$. Since $T^*$ fulfils $I_3^3$, it holds that $T_{\mathbf{L}} \leq T^*$ and in particular $y' + z' - T^*(y', z') \leq 1$. It then follows that

$$y' + z' - \frac{x-a}{b-a} - T^*(y', z') \leq 1 - \frac{x-a}{b-a} = \frac{b-x}{b-a} \leq \frac{1}{b-a}\,.$$

*Inequality $I_4^4$.* First we remark that substituting $y = z = 0$ in $I_4^4$ again yields the left part of $I_2^1$, i.e. $T_{\mathbf{L}} \leq T$. Now let $T$ be the ordinal sum of a family of t-norms that fulfil $I_4^4$. Due to the symmetry of $I_4^4$ in $x$ and $t$, and in $y$ and $z$, we can assume without loss of generality that $x \leq t$ and $y \leq z$. Therefore, it is sufficient to consider the following 6 cases:

| | | |
|---|---|---|
| (1) $x \leq y \leq z \leq t$ | (2) $x \leq y \leq t \leq z$ | (3) $x \leq t \leq y \leq z$ |
| (4) $y \leq x \leq t \leq z$ | (5) $y \leq x \leq z \leq t$ | (6) $y \leq z \leq x \leq t$. |

We will restrict ourselves to two of these cases only, the other ones being similar.

*The case* $x \leq y \leq z \leq t$. If $x$ and $t$ belong to the same summand $\langle a, b, T^* \rangle$, with $T^*$ a t-norm that fulfils $I_4^4$, then $y$ and $z$ also belong to this summand. Therefore, $I_4^4$ is equivalent to

$$\begin{aligned} 0 \leq x + t - (a + (b-a)T^*(x', z')) - (a + (b-a)T^*(x', t')) \\ - (a + (b-a)T^*(y', t')) + (a + (b-a)T^*(y', z')) \leq 1 \,, \end{aligned}$$

with $x' = \frac{x-a}{b-a}$, $y' = \frac{y-a}{b-a}$, $z' = \frac{z-a}{b-a}$ and $t' = \frac{t-a}{b-a}$. The latter inequality is equivalent to

$$0 \leq x' + t' - T^*(x', t') - T^*(x', t') - T^*(y', t') + T^*(y', z') \leq \frac{1}{b-a} \,.$$

Since $I_4^4$ holds for $T^*$ and $1 \leq \frac{1}{b-a}$, the above also holds.

If $x$ and $t$ do not belong to the same summand, then $T(x,t) = x$ and $I_4^4$ reduces to

$$0 \leq t - T(x,z) - T(y,t) + T(y,z) \leq 1 \,. \tag{2}$$

It is easy to see that the left inequality is always fulfilled since $t - T(y,t) \geq 0$ and $T(y,z) - T(x,z) \geq 0$. Next, we prove that also the right inequality is fulfilled. Therefore, we split up the proof into different cases.

(i) If $y$ and $z$ do not belong to the same summand (hence $T(y,z) = y$), then also $x$ and $z$, as well as $y$ and $t$, do not belong to the same summand (hence $T(x,z) = x$ and $T(y,t) = y$). Therefore, the right part of (2) reduces to $t - x \leq 1$, which is obviously fulfilled.

(ii) Suppose $y$ and $z$ belong to the same summand $\langle a, b, T^* \rangle$, with $T^*$ a t-norm that fulfils $I_4^4$. Again, we have to consider several possibilities:

(a) Also $t$ belongs to this summand, while $x$ does not (hence $T(x,z) = x$). Then the right part of (2) is equivalent to

$$t - x - (a + (b-a)T^*(y', t')) + (a + (b-a)T^*(y', z')) \leq 1 \,,$$

or also

$$\frac{a-x}{b-a} + t' - T^*(y', t') + T^*(y', z') \leq \frac{1}{b-a} \,.$$

Setting $x = 0$ in $I_4^4$ and applying it to $T^*$, we find that

$$0 \leq t' - T^*(y', t') + T^*(y', z') \leq 1 \,.$$

It then easily follows that

$$\frac{a-x}{b-a} + t' - T^*(y', t') + T^*(y', z') \leq \frac{a-x}{b-a} + 1 = \frac{b-x}{b-a} \leq \frac{1}{b-a} \,.$$

(b) Also $x$ belongs to this summand, while $t$ does not (hence $T(y,t) = y$). The right part of (2) is then equivalent to

$$t - (a + (b-a)T^*(x', z')) - y + (a + (b-a)T^*(y', z')) \leq 1 \,,$$

or also

$$\frac{t-a}{b-a} - y' - T^*(x',z') + T^*(y',z') \leq \frac{1}{b-a}\,.$$

Setting $t = 1$ in $I_4^4$ and applying it to $T^*$, we find that

$$-1 \leq -y' - T^*(x',z') + T^*(y',z') \leq 0\,.$$

It then follows that

$$\frac{t-a}{b-a} - y' - T^*(x',z') + T^*(y',z') \leq \frac{t-a}{b-a} \leq \frac{1}{b-a}\,.$$

(c) Neither $x$, nor $t$ belong to this summand (hence $T(x,z) = x$ and $T(y,t) = y$). In this case, the right part of (2) is equivalent to

$$t - x - y + (a + (b-a)T^*(y',z')) \leq 1\,,$$

or also

$$\frac{t-x}{b-a} - y' + T^*(y',z') \leq \frac{1}{b-a}\,.$$

Since $-y' + T^*(y',z') \leq 0$, it easily follows that

$$\frac{t-x}{b-a} - y' + T^*(y',z') \leq \frac{t-x}{b-a} \leq \frac{1}{b-a}\,.$$

*The case $x \leq y \leq t \leq z$.* If $x$ and $z$ belong to the same summand $\langle a, b, T^*\rangle$, with $T^*$ a t-norm that fulfils $I_4^4$, then $y$ and $t$ also belong to this summand. Therefore, inequality $I_4^4$ is fulfilled in the same way as in the previous case. If $x$ and $z$ do not belong to the same summand (hence $T(x,z) = x$), then $I_4^4$ reduces to

$$0 \leq t - T(x,t) - T(y,t) + T(y,z) \leq 1\,. \tag{3}$$

Again, it is easy to see that the left inequality is always fulfilled since $t - T(x,t) \geq 0$ and $T(y,z) - T(y,t) \geq 0$. Next, we prove that also the right inequality is fulfilled. Therefore, we split up the proof into different cases.

(i) Suppose $y$ and $t$ do not belong to the same summand. The proof is identical to case (i) above.

(ii) Suppose $y$ and $t$ belong to the same summand $\langle a, b, T^*\rangle$, with $T^*$ a t-norm that fulfils $I_4^4$. Then, we have the following possibilities:

(a) Also $z$ belongs to this summand, while $x$ does not. The proof is identical to case (ii)(a) above.

(b) Also $x$ belongs to this summand, while $z$ does not (hence $T(y,z) = y$). Then the right part of (3) is equivalent to

$$t - (a + (b-a)T^*(x',t')) - (a + (b-a)T^*(y',t')) + y \leq 1\,,$$

or also

$$t' - T^*(x',t') - T^*(y',t') + y' \leq \frac{1}{b-a}.$$

Since $T_{\mathbf{L}} \leq T^*$, it holds that $y' + t' - T^*(y',t') \leq 1$ and we can conclude that

$$y' + t' - T^*(x',t') - T^*(y',t') \leq 1 - T^*(x',t') \leq 1 \leq \frac{1}{b-a}.$$

(c) Neither $x$, nor $z$ belong to this summand (hence $T(x,t) = x$ and $T(y,z) = y$). In this case, the right part of (3) is equivalent to

$$t - x - (a + (b-a)T^*(y',t')) + y \leq 1,$$

or also

$$y' + t' - T^*(y',t') + \frac{a-x}{b-a} \leq \frac{1}{b-a}.$$

Again, since $T_{\mathbf{L}} \leq T^*$, we can conclude that

$$y' + t' - T^*(y',t') + \frac{a-x}{b-a} \leq 1 + \frac{a-x}{b-a} \leq \frac{b-x}{b-a} \leq \frac{1}{b-a}.$$

This completes the proof. □

Moreover, from our experience in proving that ordinal sums preserve the Bell-type inequalities, we can postulate the following more general conjecture.

**Conjecture.** Consider an inequality of the following form ($n \geq 2$):

$$\sum_{i=1}^{n} a_i x_i + \sum_{\substack{i=1 \\ j<i}}^{n} b_{ij}\, T(x_i, x_j) + c \geq 0,$$

with $a_i, b_{ij} \in \mathbb{R}$ for all $i = 1, \ldots, n$ and $j < i$. This inequality is preserved under ordinal sums if and only if it is fulfilled by $T_{\mathbf{M}}$, which in turn is equivalent to demanding that $c \geq 0$,

$$a_i + c \geq 0$$

for any $i$, and

$$\sum_{i=1}^{n} a_i + \sum_{\substack{i=1 \\ j<i}}^{n} b_{ij} + c \geq 0.$$

## 5. BELL–TYPE INEQUALITIES FOR PARAMETRIC T–NORM FAMILIES

In this section, we consider the most important parametric t-norm families and investigate for which values of the parameter involved the corresponding t-norms fulfil a given Bell-type inequality. These families are taken from [4] and are listed in Table 2; the subfamilies consisting of copulas are indicated as well. In view of Theorem 3, it is sufficient to concentrate on continuous Archimedean t-norms only. As the Mayor–Torrens t-norm family consists of continuous non-Archimedean t-norms, it is excluded from our study, while it does appear in the list of Klement, Mesiar and Pap. Note that all t-norms in Table 2 are Archimedean (except for $T_{\mathbf{M}}$, which appears as a limit case in some families).

In the following subsections, we consider the Bell-type inequalities one by one and identify for each of the families in Table 2, the range of parameters for which the corresponding t-norms fulfil the given inequality. The results of this study are summarized in Table 3. The delimiting parameter values for the Frank t-norm family are taken from [3].

### 5.1. Inequalities $I_2^1$, $I_3^2$, $I_4^4$ and $I_4^5$

Thanks to Theorem 1, we already know that inequalities $I_2^1$, $I_3^2$, $I_4^4$ and $I_4^5$ are fulfilled for any commutative copula. We have verified that for the parametric families considered, none of its non-copula members satisfies any of the inequalities considered.

It can easily be shown that inside the unit square $]0,1[^2$ the first-order derivatives of the function $f(x,y) = x + y - T(x,y) - 1$, with $T$ belonging to one of the families in Table 2, can only be zero in the symmetric case $x = y$. Therefore, inequality $I_2^1$ is equivalent to

$$2x - T(x,x) \leq 1\,.$$

Let us consider for instance the Yager family. The above inequality can be written explicitly as

$$2x - \max(0, 1 - 2^{1/\lambda}(1-x)) - 1 \leq 0\,. \tag{4}$$

Obviously, we have to consider two cases:

(i) The case $\max(0, 1 - 2^{1/\lambda}(1-x)) = 0$. It then holds that $1 - 2^{1/\lambda}(1-x) \leq 0$ and the latter inequality reads $2x - 1 \leq 0$, which holds for $\lambda \geq 1$.

(ii) The case $\max(0, 1 - 2^{1/\lambda}(1-x)) > 0$. In that case, inequality (4) reads $(2 - 2^{1/\lambda})(x-1) \leq 0$. It is easy to see that this inequality is fulfilled if $2 - 2^{1/\lambda} \geq 0$, or equivalently, if $\lambda \geq 1$.

Both cases lead to the same restriction on $\lambda$, and we can conclude that inside the Yager family, inequality $I_2^1$ only holds for its copula members.

Similarly, inequality $I_4^4$ is equivalent to

$$2x - 2T(x,y) - T(x,x) + T(y,y) \leq 1\,,$$

**Table 2.** Different t-norm families used throughout this work.

| T-norm family | $T_\lambda(x,y) =$ | Copulas for |
|---|---|---|
| Frank ($\lambda \in [0,+\infty]$) | $\begin{cases} T_{\mathbf{M}}(x,y) & \text{, if } \lambda = 0 \\ T_{\mathbf{P}}(x,y) & \text{, if } \lambda = 1 \\ T_{\mathbf{L}}(x,y) & \text{, if } \lambda = \infty \\ \log_\lambda(1 + \frac{(\lambda^x-1)(\lambda^y-1)}{\lambda-1}) & \text{, otherwise} \end{cases}$ | $\lambda \in [0,+\infty]$ |
| Hamacher ($\lambda \in [0,+\infty]$) | $\begin{cases} T_{\mathbf{D}}(x,y) & \text{, if } \lambda = \infty \\ 0 & \text{, if } \lambda = x = y = 0 \\ \frac{xy}{\lambda+(1-\lambda)(x+y-xy)} & \text{, otherwise} \end{cases}$ | $\lambda \in [0,2]$ |
| Schweizer–Sklar ($\lambda \in [-\infty,+\infty]$) | $\begin{cases} T_{\mathbf{M}}(x,y) & \text{, if } \lambda = -\infty \\ T_{\mathbf{P}}(x,y) & \text{, if } \lambda = 0 \\ T_{\mathbf{D}}(x,y) & \text{, if } \lambda = +\infty \\ (\max(x^\lambda + y^\lambda - 1, 0))^{\frac{1}{\lambda}} & \text{, otherwise} \end{cases}$ | $\lambda \in [-\infty,1]$ |
| Sugeno–Weber ($\lambda \in [-1,+\infty]$) | $\begin{cases} T_{\mathbf{D}}(x,y) & \text{, if } \lambda = -1 \\ T_{\mathbf{P}}(x,y) & \text{, if } \lambda = +\infty \\ \max(\frac{x+y-1+\lambda xy}{1+\lambda}, 0) & \text{, otherwise} \end{cases}$ | $\lambda \in [0,+\infty]$ |
| Dombi ($\lambda \in [0,+\infty]$) | $\begin{cases} T_{\mathbf{D}}(x,y) & \text{, if } \lambda = 0 \\ T_{\mathbf{M}}(x,y) & \text{, if } \lambda = +\infty \\ \frac{1}{1+((\frac{1-x}{x})^\lambda+(\frac{1-y}{y})^\lambda)^{\frac{1}{\lambda}}} & \text{, otherwise} \end{cases}$ | $\lambda \in [1,+\infty]$ |
| Aczel–Alsina ($\lambda \in [0,+\infty]$) | $\begin{cases} T_{\mathbf{D}}(x,y) & \text{, if } \lambda = 0 \\ T_{\mathbf{M}}(x,y) & \text{, if } \lambda = +\infty \\ \exp^{-((-\log x)^\lambda+(-\log y)^\lambda)^{\frac{1}{\lambda}}} & \text{, otherwise} \end{cases}$ | $\lambda \in [1,+\infty]$ |
| Yager ($\lambda \in [0,+\infty]$) | $\begin{cases} T_{\mathbf{D}}(x,y) & \text{, if } \lambda = 0 \\ T_{\mathbf{M}}(x,y) & \text{, if } \lambda = +\infty \\ \max(1 - ((1-x)^\lambda + (1-y)^\lambda)^{\frac{1}{\lambda}}, 0) & \text{, otherwise} \end{cases}$ | $\lambda \in [1,+\infty]$ |

while inequality $I_4^5$ is equivalent to

$$-2x + 4T(x,y) - T(x,x) + T(y,y) \leq 0\,.$$

Such a simplified equivalent inequality does not exist for inequality $I_3^2$. The verification for inequalities $I_3^2$, $I_4^4$ and $I_4^5$ was done in a numerical way.

### 5.2. Inequality $I_3^3$

It can easily be shown that inside the unit cube $]0,1[^3$ the first-order derivatives of the function $f(x,y,z) = x+y+z-T(x,y)-T(x,z)-T(y,z)-1$, with $T$ belonging to one of the families in Table 2, can only be zero in the symmetric case $x=y=z$. Therefore, inequality $I_3^3$ is equivalent to

$$3x - 3T(x,x) \leq 1\,. \tag{5}$$

We focus our attention on the Dombi t-norm family. The delimiting parameter values for the other families can be obtained in a similar way. In case no exact solution to a given problem was found, the help of Maple was called in to find a numerical solution. For the Dombi t-norm family, inequality (5) reads explicitly

$$1 - 3x + \frac{3x}{x + 2^{1/\lambda}(1-x)} \geq 0\,.$$

Reducing the left-hand side of this inequality to the same (positive) denominator, it is sufficient to study the numerator, which defines a quadratic function $g$:

$$g(x) = 3x^2(2^{1/\lambda} - 1) - 4x(2^{1/\lambda} - 1) + 2^{1/\lambda}\,.$$

We determine the values of $\lambda$ such that $g(x) \geq 0$ for any $x \in [0,1]$. Solving $g'(x) = 0$, we find that $g$ reaches an extremal value in $x_s = 2/3$. Moreover, it is easy to see that the discriminant of $g$ (i.e. $2 \cdot 2^{1/\lambda} - 10 \cdot 2^{1/\lambda} + 8$) is negative or zero when $\lambda \geq 1/2$ and in this case $g(x) \geq 0$ for any $x \in [0,1]$. On the other hand, the discriminant of $g$ is positive when $\lambda < 1/2$. In this case $g(2/3) < 0$ and we can conclude that the sign of $g(x)$ will change in the interval $[0,1]$. Therefore, inequality $I_3^3$ holds for any $\lambda \in [1/2, +\infty[$.

Note that for the Dombi family inequality $I_3^3$ is fulfilled for all of its copula members. This is for instance not the case for the Frank family. Indeed, although all Frank t-norms are copulas, inequality $I_3^3$ is only fulfilled for $\lambda \in [0, 9+4\sqrt{5}]$.

### 5.3. Inequalities $I_4^6$ and $I_4^7$

In this subsection, we consider inequalities $I_4^6$ and $I_4^7$. Again, inside the unit hypercube $]0,1[^4$ the first-order derivatives of the function

$$\begin{aligned} f(x,y,z,t) = {} & x + y + z + t - T(x,y) - T(x,z) - T(x,t) \\ & - T(y,z) - T(y,t) - T(z,t)\,, \end{aligned}$$

with $T$ belonging to one of the families in Table 2, can only be zero in the symmetric case $x = y = z = t$. Therefore, inequality $I_4^6$ is equivalent to

$$4x - 6T(x, x) \leq 1 .$$

Let us consider for instance the Hamacher t-norm family. The above inequality then reads explicitly:

$$4x - 6\frac{x^2}{\lambda + (1-\lambda)(2x - x^2)} - 1 \leq 0 .$$

Reducing the left-hand side of this inequality to the same (positive) denominator, it is sufficient to study the numerator, which defines a cubic function $g$:

$$g(x) = 4(\lambda - 1)x^3 - 3(3\lambda - 1)x^2 + 2(3\lambda - 1)x - \lambda .$$

We determine the values of $\lambda$ such that $g(x) \leq 0$ for any $x \in [0, 1]$. For $\lambda = 1$, the function $g$ reduces to the quadratic function $g(x) = -6x^2 + 4x - 1$. Since its discriminant is negative, the function $g$ is negative for any $x \in [0, 1]$. Now consider $\lambda \neq 1$. The first-order derivative of $g$ is given by

$$g'(x) = 12(\lambda - 1)x^2 - 6(3\lambda - 1)x + 2(3\lambda - 1) .$$

The discriminant of this quadratic function, i. e. $3(3\lambda - 1)(\lambda + 5)$, is positive or equal to zero when $\lambda \geq 1/3$. In that case, the function $g'$ has two real roots. We consider two different cases:

(i) The case $1/3 \leq \lambda < 1$: the smallest root of $g'$ is always smaller than 0, while the other one, say $x_s$, belongs to the interval $[0, 1]$. Therefore, it is necessary and sufficient that $g(x_s) \leq 0$ to guarantee that $g(x) \leq 0$ for any $x \in [0, 1]$. Invoking Maple, we can conclude that $g(x_s) \leq 0$.

(ii) The case $1 < \lambda$: the smallest root of $g'$, say $x_s$, belongs to the interval $[0, 1]$, while the other one is always greater than 1. Again, it is necessary and sufficient that $g(x_s) \leq 0$ to guarantee that $g(x) \leq 0$ for any $x \in [0, 1]$. Invoking Maple, we can conclude that $g(x_s) \leq 0$ when $\lambda \leq 2.6529$.

Therefore, we can conclude that inequality $I_4^6$ is fulfilled for $\lambda \leq 2.6529$.

Similarly, inequality $I_4^7$ is equivalent to

$$8x - 6T(x, x) \leq 3 ,$$

which is in the Hamacher family fulfilled when $\lambda \leq 2.222$.

Note that for the Hamacher family inequalities $I_4^6$ and $I_4^7$ are fulfilled for all of its copula members, while this is clearly not the case for the Frank family.

### 5.4. Inequalities $I_4^8$ and $I_4^9$

Finally, we consider inequalities $I_4^8$ and $I_4^9$. For inequality $I_4^8$, for instance, the first-order derivatives of the function

$$f(x, y, z, t) = -x + T(x, y) + T(x, z) + T(x, t) - T(y, z) - T(y, t) - T(z, t) ,$$

with $T$ belonging to one of the families in Table 2, can only be zero in the symmetric case $y = z = t$. This renders inequality $I_4^8$ equivalent to

$$-x + 3T(x,y) - 3T(y,y) \leq 0\,.$$

This time, we consider the Sugeno–Weber family. The above inequality then reads:

$$-x + 3\,\max\left(0, \frac{x+y-1+\lambda xy}{1+\lambda}\right) - 3\,\max\left(0, \frac{2y-1+\lambda y^2}{1+\lambda}\right) \leq 0\,. \tag{6}$$

We distinguish four different cases. When both maxima are equal to zero, inequality (6) reduces to the trivial inequality $-x \leq 0$. Also, when $\max(0, \frac{x+y-1+\lambda xy}{1+\lambda}) = 0$, inequality (6) reduces to $-x - 3\left(\frac{2y-1+\lambda y^2}{1+\lambda}\right) \leq 0$ which is easily verified for any $\lambda > -1$. The third case being similar to the previous one, it only remains to consider the case that both maxima are different from zero. In that case, inequality (6) reads:

$$-x + 3\frac{x+y-1+\lambda xy}{1+\lambda} - 3\frac{2y-1+\lambda y^2}{1+\lambda} \leq 0\,. \tag{7}$$

Note that this inequality only needs to be considered in the domain enclosed by the boundaries $x = 1$, $y = 1$, $x + y - 1 + \lambda\,xy = 0$ and $y = \frac{-1+\sqrt{1+\lambda}}{\lambda}$. If we reduce the left-hand side of this inequality to the same (positive) denominator, the numerator determines a two-place function $g$:

$$g(x,y) = -3\lambda y^2 + 3\lambda xy - 3y + (2-\lambda)x\,.$$

We determine the values of $\lambda$ such that $g(x,y) \leq 0$ for any $(x,y) \in [0,1]^2$. In order to find the stationary points of $g$, we set the first-order derivatives of $g$ equal to zero, and obtain:

$$\begin{aligned} g_x(x,y) &= 3\lambda y + 2 - \lambda = 0\,,\\ g_y(x,y) &= -6\lambda y + 3\lambda x - 3 = 0\,. \end{aligned}$$

Solving this set of equations we obtain a single solution

$$(x_s, y_s) = \left(\frac{2\lambda-1}{3\lambda}, \frac{\lambda-2}{3\lambda}\right)\,,$$

which is a stationary point of $g$. Furthermore, it holds that

$$g(x_s,y_s) = \frac{2-\lambda}{3\lambda}\,.$$

We need to verify whether this stationary point is a minimum, maximum or saddle point. To that end, we compute the second-order derivatives of $g$:

$$\begin{aligned} g_{xx}(x_s,y_s) &= 0\,,\\ g_{yy}(x_s,y_s) &= -6\lambda\,,\\ g_{xy}(x_s,y_s) &= 3\lambda\,, \end{aligned}$$

from which it follows that the determinant of these derivatives of $g$ in the stationary point $(x_s, y_s)$ is given by

$$A_2 = [g_{xx}g_{yy} - g_{xy}^2](x_s, y_s) = -9\lambda^2 .$$

Since $A_2 < 0$ for any $\lambda \in ]-1, +\infty[$, the stationary point $(x_s, y_s)$ is neither a minimum, nor a maximum (it is a saddle point). Therefore, the maximum of $g$ will be reached on the boundaries of the domain of $g$.

(i) On the boundary $y = \frac{-1+\sqrt{1+\lambda}}{\lambda}$, we obtain a linear function $h$ in $x$:

$$h(x) = x\left(\frac{3\sqrt{1+\lambda}-1-\lambda}{1+\lambda}\right) + 3\frac{\lambda\sqrt{1+\lambda}-1-\lambda}{\lambda(1+\lambda)} .$$

We determine the values of $\lambda$ such that $h(x) \leq 0$ for any $x \in [\frac{1+\lambda-\sqrt{1+\lambda}}{\lambda\sqrt{1+\lambda}}, 1]$. It is easily verified that $h$ is an increasing function when $-1 < \lambda \leq 8$, while $h$ is decreasing when $\lambda > 8$. If $h$ is increasing, $h(1)$ should be negative in order that $h(x) \leq 0$, for any $x \in [\frac{1+\lambda-\sqrt{1+\lambda}}{\lambda\sqrt{1+\lambda}}, 1]$. It is easy to see that $h(1) \leq 0$ when $3 \leq \lambda \leq 8$. In the same way, if $h$ is decreasing, $h(\frac{1+\lambda-\sqrt{1+\lambda}}{\lambda\sqrt{1+\lambda}})$ should be negative. This is the case when $\lambda > 8$. Therefore, we can conclude that $h(x) \leq 0$ for any $x \in [\frac{1+\lambda-\sqrt{1+\lambda}}{\lambda\sqrt{1+\lambda}}, 1]$ when $\lambda \geq 3$.

(ii) Similarly, on the boundary $x + y - 1 + \lambda xy = 0$ (or equivalently, $y = \frac{x-1}{-\lambda x-1}$), we obtain another function $h$ in $x$:

$$h(x) = \frac{-\lambda^2(1+\lambda)x^3 + \lambda(1+\lambda)x^2 + 5(1+\lambda)x - 3(\lambda+1)}{(\lambda x+1)^2} .$$

In the same way, it holds that $h(x) \leq 0$ for any $x \in [0, \frac{1+\lambda-\sqrt{1+\lambda}}{\lambda\sqrt{1+\lambda}}]$ when $\lambda \geq 175/81$.

(iii) It is easy to see that on the boundary $y = 1$, inequality (7) reduces to $(1 + \lambda)(2x - 3) \leq 0$ and is always fulfilled, while the boundary $x = 1$ leads to inequality (5), which is certainly fulfilled when $\lambda \geq 3$.

Summarizing all cases above, we can conclude that inequality $I_4^8$ is satisfied when $\lambda \geq 3$.

Similarly, inequality $I_4^9$ is equivalent to

$$3x - 2y - 3T(x,x) + 3T(x,y) \leq 1 ,$$

which is in the Sugeno–Weber family fulfilled when $\lambda \geq 3$.

Note that it is not that easy to find the delimiting parameter values for all families. In most cases, the resulting functions were too complicated to find analytical solutions or even numerical ones. As a way out, we used contour plots to conclude that no extrema occurred inside the unit square $]0,1[^2$, and in some cases, only saddle points. Hence, for all families an extremum will be reached on the boundaries of the unit square $[0,1]^2$. Therefore, the parameter values such that $I_4^8$ and $I_4^9$ are satisfied, are the same as the ones obtained for $I_3^3$. These results are summarized in Table 3.

**Table 3.** Conditions on the parameter $\lambda$.

| Family | $I_2^1$, $I_3^2$, $I_4^4$, $I_4^5$ | $I_3^3$, $I_4^8$, $I_4^9$ | $I_4^6$ | $I_4^7$ |
|---|---|---|---|---|
| Frank | $[0,+\infty]$ | $[0,9+4\sqrt{5}]$ | $[0,9.2946]$ | $[0,9.2946]$ |
| Hamacher | $[0,2]$ | $[0,2.9386]$ | $[0,2.6529]$ | $[0,2.2220]$ |
| Schweizer–Sklar | $[-\infty,1]$ | $[-\infty,\frac{1}{2}]$ | $[-\infty,0.3435]$ | $[-\infty,\frac{1}{2}]$ |
| Sugeno–Weber | $[0,+\infty]$ | $[3,+\infty]$ | $[8,+\infty]$ | $[2,+\infty]$ |
| Dombi | $[1,+\infty]$ | $[\frac{1}{2},+\infty]$ | $[\frac{\ln 2}{\ln(3+\frac{4}{3}\sqrt{2})},+\infty]$ | $[\frac{\ln 2}{\ln 3},+\infty]$ |
| Aczel–Alsina | $[1,+\infty]$ | $[0.7379,+\infty]$ | $[0.7533,+\infty]$ | $[0.8201,+\infty]$ |
| Yager | $[1,+\infty]$ | $[\frac{\ln 2}{2\ln 2-\ln 3},+\infty]$ | $[\frac{\ln 2}{\ln 7-\ln 6},+\infty]$ | $[\frac{\ln 2}{\ln 3-\ln 2},+\infty]$ |

## 6. A FAMILY OF BELL–TYPE INEQUALITIES

Taking a closer look at the inequalities of type $c_1\, x - c_2\, T(x,x) \leq c_3$, with constants $c_1, c_2, c_3 \geq 0$, such as the inequality $3x - 3T(x,x) \leq 1$, suggests the following general form, $n \geq 2$:

$$nx - \binom{n}{2}\, T(x,x) \leq 1\,. \tag{8}$$

For $n = 2$, we obtain the inequality $2x - T(x,x) \leq 1$ and for $n = 3$, we retrieve the inequality $3x - 3T(x,x) \leq 1$. Similarly, for $n = 4$, we find $4x - 6T(x,x) \leq 1$, i.e. the inequality equivalent to $I_4^6$.

In [3], we already proved that

**Proposition 2.** The only Frank t-norms for which inequality (8) is fulfilled for all $n \geq 2$ are the t-norms between the algebraic product $T_{\mathbf{P}}$ and the minimum operator $T_{\mathbf{M}}$ (i.e. with $\lambda \in [0,1]$).

In general, the algebraic product $T_{\mathbf{P}}$ is not the smallest t-norm that satisfies inequalities (8). This is confirmed by the following example.

**Example 1.** The Hamacher t-norm with $\lambda = 2$, i.e. $T_2^{\mathbf{H}}(x,y) = \frac{xy}{2-x-y+xy}$, which is smaller than the algebraic product ($T_2^{\mathbf{H}} < T_1^{\mathbf{H}} = T_{\mathbf{P}}$), fulfills inequality (8) for any $n \geq 2$.

Proof. Writing inequality (8) explicitly, we obtain

$$nx - \frac{n(n-1)}{2}\frac{x^2}{x^2-2x+2} - 1 \leq 0\,.$$

If we reduce the left-hand side of this inequality to the same (positive) denominator, then the numerator determines a function $f$:

$$f(x) = 2nx^3 - (n^2+3n+2)x^2 + 4(n+1)x - 4\,.$$

For $n = 2$, this function reduces to $f(x) = 4(x-1)^3$ and obviously $f(x) \leq 0$ for any $x \in [0,1]$. Now suppose $n \geq 3$. The first-order derivative of $f$ is given by

$$f'(x) = 6nx^2 - (2n^2+6n+4)x + 4(n+1)\,.$$

Next we solve the equation $f'(x) = 0$. Since $n \geq 3$, the discriminant of this quadratic function ($D = (n-2)(n+1)(n^2+7n-2)$) is always positive, and therefore $f'$ has two real roots. It is easy to see that one root of this equation, say $x_s$, lies between 0 and 1, while the second one is always greater than 1. Therefore, a necessary and sufficient condition in order that $f(x) \leq 0$ for any $x \in [0,1]$ is that $f(x_s)$ should be negative for any $n \geq 3$. Straightforward computation yields

$$\begin{aligned} f(x_s) = -\frac{(n-2)}{54n^2}\Big( & (n+1)(n^2+7n-2)\sqrt{D} \\ & + (n-1)(n^4+12n^3+31n^2-12n+4)\Big)\,, \end{aligned}$$

with $D = n^4 + 6n^3 - 11n^2 - 12n + 4$, which is negative for all $n \geq 3$. This completes our proof. □

## 7. CONCLUSIONS

In this paper, we have studied in detail the Bell-type inequalities for continuous t-norms. We have shown that ordinal sums preserves the Bell-type inequalities, which was the motivation for studying continuous Archimedean t-norms only. As general results based on additive generators are unlikely to be obtained, we have discussed in an exhaustive way the major parametric t-norm families. Finally, for a particular form of these inequalities, we have shown that the algebraic product $T_{\mathbf{P}}$ is not the smallest t-norm fulfilling them.

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