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A DISCUSSION ON AGGREGATION OPERATORS

DANIEL GÓMEZ AND JAVIER MONTERO

It has been lately made very clear that aggregation processes can not be based upon a unique binary operator. *Global aggregation operators* have been therefore introduced as families of aggregation operators $\{T_n\}_n$, being each one of these T_n the n-ary operator actually amalgamating information whenever the number of items to be aggregated is n. Of course, some mathematical restrictions can be introduced, in order to assure an appropriate meaning, consistency and key mathematical capabilities. In this paper we shall discuss these standard conditions, pointing out their respective relevance.

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1. INTRODUCTION

In his seminal paper, Zadeh [44] proposed a basic logical structure generalizing classical binary logic allowing the continuous $[0,1]$ range for the degrees of verification and degrees of membership, in such a way that being X the set of objects under consideration, a mapping

$$\mu_A : X \to [0,1]$$

was defined, $\mu_A(x)$ meaning the degree to which each object $x \in X$ verifies certain "fuzzy" property A.

In particular, Zadeh [44] modeled the generalized conjunction of two fuzzy sets $A \cap B$ by means of the minimum operator,

$$\mu_{A\cap B}(x) = \min[\mu_A(x), \mu_B(x)] \qquad \forall x \in X$$

being its membership function. Disjunction $A \cup B$ was generalized by means of the maximum operator

$$\mu_{A\cup B}(x) = \max[\mu_A(x), \mu_B(x)] \qquad \forall x \in X$$

And negation A^c was generalized as

$$\mu_{A^c}(x) = 1 - \mu_A(x) \qquad \forall x \in X$$

Such a simple structure indeed contains classical binary structure as a particular case, producing it whenever the valuation range is restricted to $\{0,1\}$ values.

But the fuzzy framework is much more complex that the crisp framework, and soon quite a number of operators where proposed, as different alternatives to those initially proposed by Zadeh [44].

In fact, as it can be easily checked, if $\{0,1\}$ are the only allowed degrees of verification, there are only two *relevant* binary operators

$$\circ : \{0,1\} \times \{0,1\} \to \{0,1\}$$

being *monotonous*

$$a_1 \leq a_2, b_1 \leq b_2 \Rightarrow \circ(a_1, b_1) \leq \circ(a_2, b_2)$$

and *informative*, i. e. non-constant,

$$\exists (a_1, b_1), (a_2, b_2) / \circ (a_1, b_1) \neq \circ(a_2, b_2)$$

Those two unique relevant binary operators are usually called

- *conjunction*, if
$$\circ(a, b) = 1 \Longleftrightarrow a = b = 1$$
($\circ(a, b) = 0$, otherwise);
- *disjunction*, if
$$\circ(a, b) = 0 \Longleftrightarrow a = b = 0$$
($\circ(a, b) = 1$, otherwise).

Since there is only one logical structure based upon the binary $\{0,1\}$ valuation space, one may be tempted to assume that every aggregation procedure within this context should be based either upon the above (crisp) conjunction or the above (crisp) disjunction. But this is not the case: in crisp Reliability Theory (see, e. g., [7]), apart from the series systems and parallel systems (respectively associated to the minimum operator and the maximum operator), we find that perhaps the most important family of structure functions are the so-called "k-out-n" systems, where the system functions if and only if there are at least k functioning components (see [7]). These "k-out-n" structures are usually applied in politics, within voting procedures (see, e. g., [37]). And we frequently classify objects within a certain class whenever the considered object verifies *most* of the properties defining such a class (see, e. g., [2]). We should not be accepting so easily that the aggregation of crisp information should be based upon a successive application of the above binary (crisp) conjunction or (crisp) disjunction.

In addition, once the space of degrees of verification is extended into the unit interval, of course we should be expecting that both conjunction and disjunction can be modeled in many different alternative ways.

In particular, most of the present literature on Fuzzy Sets Theory stresses the role of triangular norms and triangular conorms (t-norms and t-conorms for short, see [38]) as natural generalizations for conjunction and disjunction, respectively, in case the valuation space is the unit interval (see also Klement et al [20] and Calvo et al [9]). Hence, the classical structure of binary logic can be translated into the fuzzy context, once we choose a particular triple

$$(T, S, n)$$

being T a t-norm, S a t-conorm and n a negation function (see [40]) such that

$$S(a, b) = n(T(n(a), n(b))) \qquad \forall\, a, b \in [0, 1]$$

Again, negation was generalized into the $[0, 1]$-valued context showing the potential existence of a rich family of solutions: the application

$$n : \{0, 1\} \to \{0, 1\}$$

such that $n(0) = 1$ and $n(1) = 0$ is the only one-to-one decreasing mapping for the binary $\{0, 1\}$ valuation space. But it is clear that Zadeh's initial proposal [44] is not the only possible decreasing and one-to-one mapping in case the valuation space is the unit interval, as shown in Trillas [40].

In the next sections we shall discuss in details main assumptions of those models based upon a unique binary operator, pointing out that consistent approaches to aggregation procedures need to be developed.

2. STANDARD ASSUMPTIONS ON FUZZY CONNECTIVES

As pointed out above, the standard fuzzy connectives are modeled as t-norms for conjunction and t-conorms for disjunction. These two families of binary connectives are conceived (see [38]) as mappings

$$\odot : [0, 1] \times [0, 1] \to [0, 1]$$

Standard assumptions are the following properties (see also [20]):

1. Commutativity: $\odot(a, b) = \odot(b, a), \quad \forall\, a, b \in [0, 1]$.
2. Monotonicity: $a_1 \leq a_2, b_1 \leq b_2 \Rightarrow \odot(a_1, b_1) \leq \odot(a_2, b_2)$.
3. Associativity: $\odot(\odot(a, b), c)) = \odot(a, \odot(b, c)), \quad \forall\, a, b, c \in [0, 1]$.

The following two *extreme* boundary conditions use to be also assumed, by definition (see, e. g., [15]):

4α. $\odot(1, 1) = 1$.

4β. $\odot(0, 0) = 0$.

Moreover, many results are obtained imposing some in principle desirable complementary properties (for more possible assumptions, see [20]):

5. Continuity: given a sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$, with $a_n, b_n \in [0,1], \forall n$, such that $\lim_{n\to\infty}(a_n, b_n) = (a, b)$, then

$$\lim_{n\to\infty} \odot(a_n, b_n) = \odot(a, b).$$

6. Idempotency: $\odot(a, a) = a, \quad \forall a \in [0,1]$.

The particular conjunction or disjunction role is then given by one of the following additional exclusive boundary condition:

7t. For t-norms: $\odot(a, 0) = \odot(0, a) = 0, \forall a \in [0,1]$.

7c. For t-conorms: $\odot(a, 1) = \odot(1, a) = 1, \forall a \in [0,1]$.

Let us make some critical comments on each one of the above properties.

2.1. Commutativity

Commutativity refers to irrelevancy of data ordering in the aggregation process, in such a way that aggregation will be invariant respecto to permutation (see [22]). It is indeed a quite standard *mathematical* property, but it implies a severe restriction, in some cases against real description of the problem.

In fact, in many real applications the decision maker wants to keep track on how data were obtained, at least to be able to locate them in time, place and other circumstances. It should not have the same meaning a high temperature observed yesterday than a high temperature measured one or ten years ago. The opinion of a specialist today may deserve different weight than the opinion the same specialist gave yesterday (and of course we should take into account who each one of those consulted specialists is).

Assuming that aggregation does not depend on ordering implies that result is not being affected neither by the time data are being produced or by the time data arrive to decision maker, and it is also suggesting that result may not depend on key circumstances surrounding data that most probably should have been included as data themselves. *Data* are not just *numbers*.

Commutativity can be properly assumed only when our experiment has been designed in order to fulfill commutativity (randomness, for example, can be more or less guaranteed by means of an appropriate simulation procedure based upon pseudo-random numbers). But most often data are not being produced in a controlled laboratory, and ordering can not be considered as irrelevant (as many other circumstances surrounding data).

2.2. Monotonicity

Monotonicity looks in principle like an obvious property (if every degree of truth increases, the global degree of truth should increase, or at least never decrease). But this property deserves much more attention.

In fact, it is well known in Reliability Theory (see [7]) that monotonicity does not necessarily holds when dealing with physical systems subject to failure. It is indeed a standard assumption even in a non binary context (see, e. g., [8] and [36]), but monotonicity should not be accepted without realizing that some interesting systems are put away.

In particular, we must point out that some good solutions to certain problems can be obtained by introducing a big number of items with (not too extreme) bad behaviour. For example, some firms get their independence assuming many small dependencies which compensate between them (instead of rejecting dependencies which in turn will isolate the firm and provoke bankruptcy). Some economic theories seem to suggest that *justice* can be reached allowing individuals to behave as small egoists. It is also well known that the success of some species is partially explained in their inefficient *random* reproduction process, which guarantees a heterogeneous population (an *optimal* population is not reached by aggregation of *optimal* individuals).

Equilibrium, in general, is more stable if it is based upon a big number of small forces. *Good* things can be obtained from *bad* things, under the right restrictions. This is an interesting paradox that may deserve more attention from aggregation researchers, but it requires to explore aggregation rules not imposing any monotonicity condition.

2.3. Associativity

As pointed out in [20], the very first definition of triangular norm, as proposed by Menger [29], did not require any associativity condition.

The problem is that, in order to be useful in practice, we need to be able to extend binary connectives to operations with an arbitrary number of arguments. Associativity takes care of this issue, allowing to extend each t-norm in a unique way, just by induction [20], no matter if the sequential calculus is made from left to right or from right to left.

Associativity can be therefore viewed as a necessary restriction, but only once we accept that our aggregation process should be based upon a unique binary operator. Such assumption is not obvious even in the crisp case, as pointed out above, but it is difficult to accept in a more general context. Of course aggregation should take into account which pieces of information have being actually aggregated (see, e. g., [33, 34], where the size was introduced in order to avoid Fung and Fu restrictive result [17]). If assuming that all our aggregation processes should be based upon binary operations is not the only option (see [39] for an interesting alternative approach in decision making avoiding comparison by pairs of alternatives), we should not be accepting that aggregation operators can not *evolve* in time or that they simply remain constant along the complete aggregation process (see also [19]).

Moreover, if we assume that aggregation rules should be able to aggregate homogeneous information, no matter how many chunks of information we are faced to, we still need to address two key issues:

1. Operationality (decision maker should be able to implement implied computations, or at least able to find out some good approximate solution).

2. Consistency (underlying arguments supporting each aggregation should remain constant, so we can properly talk about an aggregation rule).

Operationality can be assured allowing a successive reckoning of binary operators. Consistency can be assured, for example, imposing recursiveness, as introduced in [13] (see also [4]). We shall review this approach below, but of course there are alternative approaches.

2.4. Extreme boundary conditions

Again, analogous arguments to those conditions above relative to monotonicity may apply here: behavior in extreme situations, when all elements show the best behavior or the worst behavior may be considered natural except in some paradoxical context (it may even happen that the concepts of *best* and *worst* have to be revised in order to meet these two extreme boundary conditions).

2.5. Continuity

Quite often, continuity is considered a strong mathematical condition, and in fact quite a number of key results can be obtained assuming weaker forms of continuity (see, e. g., [20]).

But from a practical point of view we should stress the weakness of the above standard continuity. If the valuation space is in fact the whole unit interval, and in order to deal with a robust estimation associated procedure, we need a stronger restriction. We do desire that our binary connective is *smooth*, but it can be derivable and still show too high slopes, in such a way that a *small* input measurement error can still produce an too big change in the output. This situation is not avoided even assuming that our function is infinitely derivable. In practice we should be imposing certain smoothness restriction (the first derivative must not be to high, in absolute value), and such a smoothness restriction most probably will depend on our precision measurement level.

A possible proposal could take into account the concept introduced in [41], which focuses the attention on those curves that can be *drawn* by means of a ball, being its diameter to be fixed by the decision maker, but at this stage it seems hard to implement due to some analytical difficulties.

2.6. Idempotency

Idempotency is certainly violated in some contexts, due to some kind *attraction* behavior: a high degree, if repeated, may suggest in many contexts a higher aggregated degree, and a low degree, if repeated, may suggest a very low aggregated degree.

Anyway, if idempotency is accepted, it is being suggested that one certain value plays the same role as many equal values, which can not be accepted in many contexts (at least the *confidence* in such a value, as an estimated value, should be different).

2.7. Exclusive boundary conditions

Each one of these two conditions are introduced in order to give the aggregation a particular meaning. But again we should not be accepting there are only two possible roles for binary aggregation, as in the crisp context.

Alternative boundary conditions may lead to different aggregation operators, as binary weighted means or Yager's binary OWA operators [42] (see also [43]).

3. GLOBAL AGGREGATION OPERATORS

As already pointed out, fuzzy aggregation operators can not be restricted to *conjunction* and *disjunction.* Weighted means, for example, are neither t-norms or t-conorms, and do play a key role in aggregation processes. A key effort in this direction is the work of Yager [42] on OWA operators and Yager–Rybalov on the more general *uninorms* [43] (see also [14]). Nevertheless, as pointed out in [11] (see also [12, 13]), each OWA operator can deal only with an exact number of items, and we usually do not know such a number in advance. This problem can be solved by means of a sequential one-by-one aggregation taking into account only one binary operator, but it is clear this assumption becomes extremely restrictive in practice, so an *OWA rule* can be recursively defined.

We do need to consider *global aggregation operators* as a *consistent* family of n-ary operators, so we can always evaluate aggregation, no matter the number of items to be aggregated: a procedure for each case has to be defined (see [21, 26, 28, 31] but also [16]). A key issue, already stressed in [9] is how such a *consistency* can be assured (if understood just as a family of operators, with no additional restriction, operators in charge of the aggregation of different number of items within a given global aggregation operator can be not related at all).

The approach of Cutello and Montero [11], initially conceived only for OWA operators, was then brought into a more general framework, leading to the concept of *recursive rule* (see [12, 13]). The recursive model seems to offer nice properties: for example, taking into account classical results [1], the solution of a key generalized associativity equation (in the sense of Mak [24]) will lead to a quasi-additive recursive aggregation, as shown in Amo et al [4]. Such a recursive assumption implies a strong consistency on the family of all n-ary operators. In this way, recursiveness can be viewed as a natural way of assuring that our family of n-ary operators is a proper rule: as pointed out in [35], not every family of n-ary operators should be considered a proper rule.

Needless to say, recursiveness is not the only way of obtaining proper aggregation rules and some alternative approaches can be tried.

4. RECURSIVE RULES

The key idea of recursiveness is that an aggregation rule, in order to be operational, should be based upon an iterative application of binary operators, taking advantage of previous aggregations. Data are therefore being assumed to be aggregated one by one, and each particular arrangement of data will tell us the sequence of items to be aggregated. Hence, see [13], we first re-arrange data.

Definition 1. Let us denote

$$\pi_n(a_1, a_2, \ldots, a_n) = (a_{\pi_n(1)}, a_{\pi_n(2)}, \ldots, a_{\pi_n(n)}).$$

An *ordering rule* π is a *consistent* family of permutations $\{\pi_n\}_{n>1}$ such that for any possible finite collections of numbers, each extra item a_{n+1} is allocated keeping previous relative positions of items, i. e.,

$$\pi_{n+1}(a_1, a_2, \ldots, a_n, a_{n+1})$$

$$= (a_{\pi_n(1)}, \ldots, a_{\pi_n(j-1)}, a_{\pi_{n+1}(j)}, a_{\pi_n(j)} \cdots, a_{\pi_n(n)})$$

for some $j \in \{1, \ldots, n+1\}$.

In other words, once relative position of two elements is being fixed by means of a permutation π_n, no permutation $\pi_m, m > n$, will change it.

The following definition was then proposed in [13].

Definition 2. A left-recursive connective rule is a family of connective operators

$$\{\phi_n : [0,1]^n \to [0,1]\}_{n>1}$$

such that there exists a sequence of binary operators

$$\{L_n : [0,1]^2 \to [0,1]\}_{n>1}$$

verifying

$$\phi_2(a_1, a_2) = L_2(a_{\pi(1)}, a_{\pi(2)})$$

and

$$\phi_n(a_1, \ldots, a_n) = L_n(\phi_{n-1}(a_{\pi(1)}, \ldots, a_{\pi(n-1)}), a_{\pi(n)})$$

for all $n > 2$ and some ordering rule π.

Notice that in no way we are imposing a unique binary operator for the whole iterative process. This was in fact the main criticism argued in [33] against the restrictive result obtained by Fung–Fu [17].

Right recursiveness can be analogously defined, and then we can talk about a *recursive rule* when both left and right representations hold for the same ordering rule (a *standard* recursive rule will appear if it is based upon the identity ordering rule).

Then it follows (see [3]) that a connective rule $\{\phi_n\}_{n>1}$ is recursive if and only if a set of general associativity equations (in the sense of Mak [24]) hold for each n, once the ordering rule π has been already applied:

$$\begin{aligned}&\phi_n(a_1,\ldots,a_n)\\&= R_n(a_{\pi(1)},\phi_{n-1}(a_{\pi(2)},\ldots,a_{\pi(n)}))\\&= L_n(\phi_{n-1}(a_{\pi(1)},\ldots,a_{\pi(n-1)}),a_{\pi(n)})\end{aligned}$$

must hold for all n. Assuming certain regularity conditions in a recursive rule (mainly strict monotonicity), it was then shown by Amo et al [4] that there exist

1. a continuous and strictly monotonic function
$$p:[0,1]\to[0,\infty)$$
2. a family of continuous and strictly monotonic functions
$$\{\delta_n:[0,1]\to[0,\infty)\}_{n>1}$$
3. and a sequence of positive real numbers
$$\{c_n\}_{n\geq 1}$$

in such a way that

$$\phi_n(a_1,\ldots,a_n)=\delta_n^{-1}\left(\prod_{j=2}^{n-2}c_j\sum_{k=1}^{n}c_1^{k-1}p(a_k)\right)$$

for all $(a_1,\ldots,a_n)\in[0,1]^n$, and $n\geq 2$, being $\prod_{j=2}^{\ell}c_j=1$ whenever $\ell\leq 2$.

A key difficulty in order to apply the above result is to check those regularity restrictions assumed in [4]. But still taking advantage of Aczél's work [1] we can offer alternative similar results. The next theorem follows from Aczél [1], page 315.

Theorem 1. Let
$$\{\phi_n:[0,1]^n\to[0,1]\}_{n>1}$$
be a recursive aggregation rule. If L_n and R_n are invertible functions in both variables, for all $n>1$, then there exist:

1. $p:[0,1]\to R^+$, continuous and strictly monotonic function,
2. $\{\delta_n:[0,1]\to R^+\}_{n>1}$, family of continuous and strictly monotonic functions, and
3. $\{c_n\}_{n\geq 1}$, sequence of positive real numbers,

in such a way that

$$\phi_n(a_1, \ldots, a_n) = \delta_n^{-1}\left(\prod_{j=2}^{n-2} c_j \sum_{k=1}^{n} c_1^{k-1} p(a_k)\right)$$

for all $(a_1, \ldots, a_n) \in [0,1]^n$ and for all $n \geq 2$, where $\prod_{j=2}^{\ell} c_j$ is taken as 1 whenever $\ell \leq 2$.

Proof. From the definition of $\{\phi_n\}_{n>1}$, the following generalized associativity equation holds:

$$L_n(R_{n-1}(u,v), w) = R_n(u, L_{n-1}(v,w)).$$

Therefore, having $(x_1 \ldots, x_n) \in [0,1]^n$, and taking $u = x_1$,$v = \phi_{n-2}(x_2, \ldots, x_{n-1})$ and $w = x_n$, the above equation is assured. Hence, we know from [1], that the solution of the above general associativity equation is basically additive, in the sense that there exist $\sigma_n, \theta_n, l_n, p_n, q_n, r_n$ continuous and strictly monotonic functions over the compact interval [0,1], verifying that

$$\begin{aligned} R_{n-1}(u,v) &= \sigma_n^{-1}(p_n(u) + q_n(v)) \\ L_{n-1}(v,w) &= \theta_n^{-1}(q_n(v) + r_n(w)) \\ R_n(u,b) &= l_n(p_n(u) + \theta_n(b)) \\ L_n(a,w) &= l_n(\sigma_n(a) + r_n(w)). \end{aligned}$$

Since l is a strict monotonic function and L_n is invertible in both variables, we shall then prove that l is an invertible function: in fact, fixed $z \in [0,1]$, there exists $(x,y) \in [0,1]^2$ such that $L_n(x,y) = z$. Then from the above equation we get $l(p_n(x) + \theta_n(y)) = z$, and l is invertible. If we now denote $\delta_n = l^{-1}$, we have the following equations:

$$\begin{aligned} R_{n-1}(u,v) &= \sigma_n^{-1}(p_n(u) + q_n(v)) \\ L_{n-1}(v,w) &= \theta_n^{-1}(q_n(v) + r_n(w)) \\ R_n(u,b) &= \delta_n^{-1}(p_n(u) + \theta_n(b)) \\ L_n(a,w) &= \delta_n^{-1}(\sigma_n(a) + r_n(w).) \end{aligned}$$

And from this stage the proof is analogous to that one of Theorem 3.1 in [4]. □

Additional results can be tried taking into account other interesting results contained in Aczél [1]. Analogously to [4], we should be able to produce alternative characterizations of idempotent additive rules, homogeneous additive rules and geometric rules, now for invertible operators instead of *regular* operators (see [4]). But it is important to notice that all those results are showing that as soon as we decide the very first aggregation for the first couple of items, we are restricting our possibilities for the whole family of n-ary aggregations. Every decision we take will take away some degree of freedom, and the aggregation rule may eventually be fully characterized. This is a key consequence of the recursive approach, and we claim this is consistent with intuition.

5. SOME ALTERNATIVE APPROACHES

Some quite similar approaches can be found in the literature, and they become an alternative to recursiveness as far as some *consistency* restriction is being implied. In our opinion, an aggregation rule should never be understood just as a family of n-ary operators: all those aggregation operators must be deeply related, following some building procedure all throughout the aggregation process. There must be some unifying idea behind, and not only a mathematical expression with no particular meaning for users.

For example, in Mas el al [26] a general associativity equation plays also a key role, but their *modularity condition* appears as a particular case:

$$F(x, G(y, z)) = G(F(x, y), z)$$

where F and G are assumed uninorms and/or t-operators (see also [27]). From our point of view, neither commutativity or associativity should be assumed as granted. Commutativity is kind of contradictory with the fact that data have been previously ordered. And associativity has no support when the binary operator will not be applied into a sequential aggregation $F(x, F(y, z))$ or $F(F(x, y), z)$.

Deeply related to recursiveness seems to be the property of being *decomposable* (see [10, 16, 25]), which assumes that each item of any given subfamily of items can be substituted by the aggregated value of such a subfamily of items. Indeed, recursiveness assumes that calculus is sequentially decomposable, but recursiveness is not assuming such a particular extra behavior. Moreover, it is not clear the need of an arbitrary decomposition, if we have assumed that operationality is in some way related to a potential sequential decomposable calculus.

Finally, the compensatory condition given in Mesiar [30] (see also [21]) seems also deeply related to recursiveness (see the seminal paper of Zimmermann–Zysno [45]). In fact, a link between each n-ary operator and the next $(n + 1)$-ary operator is being introduced. The existence of an iterative calculus is therefore being assumed, but the basic definition does not properly link consecutive operators (Kolesárová–Komorníková [23] obtain important results but restricted to *triangular norm-based* iterative compensatory operators). A similar iterative approach is suggested in Mayor–Calvo [28] with their *self identity condition*, but again this condition should not be considered a proper link between consecutive operators (moreover, they assume that every operator is idempotent, by definition).

6. FINAL COMMENTS

In order to be considered a proper rule, all n-ary operators defining a global aggregation rule should be deeply related, and one should not expect too much freedom once first aggregations have been fixed. Our intuition supports the fact that each operator we include as part of our rule implies direct restrictions on concomitant operators, and most probably on the whole rule. Although analogous results can be obtained imposing some *ad hoc* cross-continuity condition, recursiveness appears to introduce such a desired link between operators within a rule, and this is being done from a particular computational argument. Of course we realize some quite standard rules are not recursive rules (the *median* rule is an important example, see [16], but notice its associated reckoning difficulties). Hence, a more general approach should be searched, still keeping some underlying *operational* principle:

- allowing a more arbitrary structure of data, as in some classification problems; for example, data in [5, 6] are organized within a surface (see [28]);
- not imposing recursiveness but the ability of taking advantage of some previous calculus that can be kept in memory (see, for example, [32]);
- introducing some computational complexity restrictions, based upon the particular calculus capacity of user (notice for example that our previous rearrangement of data has $O(n^2)$ computational complexity).

Indeed, recursiveness as introduced by Cutello and Montero [13] is not the only alternative in order to assure consistency of aggregation rules: alternative operational arguments, including those coming from computational complexity (see [18]), may produce alternative models, perhaps not based upon a successive binary reckoning. But notice that recursiveness builds up the aggregation model from below, developing the model addressing in first place an operationality restriction, instead of searching for some other global condition that may turn too artificial in practice, leading to non operational solutions, and therefore useless for the decision maker. The above results seem to show that key aggregation rules can be fully justified from a recursive approach.

A key argument supporting recursiveness is the iterative calculus. Items are aggregated one by one (implying a linear order on the sequence data to be aggregated), and therefore any finite family of items can be obtained by means of binary operators. But imposing that each one of these sequential aggregations will be based upon the previous aggregation links all n-ary aggregation operators in a very particular way. Consistency is therefore a consequence of the way operationality has been understood (the existence of a recursive calculus). And it is the above generalized associativity equation the core of this consistency. A different structure of data, a different notion of being *operational*, or just the different available calculus capabilities of each decision maker, will produce alternative models that deserve to be analyzed.

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Daniel Gómez, Department of Statistics and Operational Research III, School of Statistics, Complutense University, 28040 Madrid. Spain.
e-mail: dagomez@estad.ucm.es

Javier Montero, Department of Statistics and Operational Research I, Faculty of Mathematics, Complutense University, 28040 Madrid. Spain.
e-mail: javier_montero@mat.ucm.es