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A FURTHER INVESTIGATION FOR EGOROFF'S THEOREM WITH RESPECT TO MONOTONE SET FUNCTIONS

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In this paper, we investigate Egoroff's theorem with respect to monotone set function, and show that a necessary and sufficient condition that Egoroff's theorem remain valid for monotone set function is that the monotone set function fulfill condition (E). Therefore Egoroff's theorem for non-additive measure is formulated in full generality.

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1. INTRODUCTION

Egoroff's theorem, which is one of the most important convergence theorem in classical measure theory, states that almost everywhere convergence implies almost uniform convergence on a finite measure space [1]. Wang [10] generalized the well known theorem to fuzzy measure spaces. Further researches on the theorem were made by Wang and Klir [11], Li et al [2, 3] Li and Yasuda [5] and others, and a great number of important results were obtained. In these discussions some structural characteristics of set functions, such as autocontinuity from above, null-additivity, pseudometric generating property and property (S) etc., were used.

In this paper, we further discuss the Egoroff theorem for non-additive measure. We shall use the condition (E), which is a new structural characteristic of set function, to prove Egoroff's theorem with respect to monotone set function, and we obtain an essential result: a necessary and sufficient condition that Egoroff's theorem remain valid for monotone set function is that the monotone set function fulfil condition (E). Also, we show that the above mentioned conditions are sufficient, but not necessary for Egoroff's theorem. Therefore the previous results obtained in [2, 3, 5, 10, 11] are generalized substantially. Since the continuity from below and above of set function are not required, Egoroff's theorem on fuzzy measure spaces is formulated in full generality.

2. PRELIMINARIES

Throughout this paper, we suppose that X is a non-empty set and $\mathcal{F}$ is a σ-algebra of subsets of X, and let N denote the set of all positive integers. Unless stated otherwise, all the subsets mentioned are supposed to belong to $\mathcal{F}$.

A set function $\mu : \mathcal{F} \to [0, +\infty]$ is called *monotone* [1], if $\mu(E) \le \mu(F)$ whenever $E \subset F$; *continuous from below* [1], if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $A_n \nearrow A$; *continuous from above* [1], if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ whenever $A_n \searrow A$, and $\mu(A_1) < \infty$; *strongly order-continuous* [4], if $\lim_{n\to\infty} \mu(A_n) = 0$ whenever $A_n \searrow A$, and $\mu(A) = 0$.

A *fuzzy measure* (Ralescu and Adams [7]) is a monotone set function $\mu : \mathcal{F} \to [0, +\infty]$ with $\mu(\emptyset) = 0$ and satisfies the continuity both from below and from above. If $\mu(X) < \infty$, the fuzzy measure μ is said to be *finite*.

In this paper, we always assume that μ is a monotone set function with $\mu(\emptyset) = 0$.

Let $\mathbf{F}$ be the class of all finite real-valued measurable functions on measurable space $(X, \mathcal{F})$, and let $f, f_n \in \mathbf{F}$ $(n \in N)$. We say that $\{f_n\}_n$ *converges almost everywhere to* f *on* X, and denote it by $f_n \xrightarrow{\text{a.e.}} f$, if there is subset $E \subset X$ such that $\mu(E) = 0$ and f_n converges to f on $X - E$; $\{f_n\}_n$ *converges almost uniformly to* f *on* X, and denote it by $f_n \xrightarrow{\text{a.u.}} f$, if for any $\epsilon > 0$ there is a subset $E_\epsilon \in \mathcal{F}$ such that $\mu(X - E_\epsilon) < \epsilon$ and f_n converges to f uniformly on E_ϵ.

3. CONDITION (E) OF SET FUNCTIONS

In this section, we introduce the concept of condition (E) of set function and discuss its properties. As we shall see later, this new structure plays an important role for establishing Egoroff's theorem with respect to monotone set function.

Definition 1. μ is said to fulfil condition (E), if for every double sequence $\{E_n^{(m)}\} \subset \mathcal{F}$ $(m, n \in N)$ satisfying the conditions: for any fixed $m = 1, 2, \dots,$

$$E_n^{(m)} \searrow E^{(m)} \ (n \to \infty) \ \text{ and } \ \mu\left(\bigcup_{m=1}^{+\infty} E^{(m)}\right) = 0,$$

there exist increasing sequences $\{n_i\}_{i\in N}$ and $\{m_i\}_{i\in N}$ of natural numbers, such that

$$\lim_{k\to+\infty} \mu\left(\bigcup_{i=k}^{+\infty} E_{n_i}^{(m_i)}\right) = 0.$$

Proposition 1. If μ fulfils condition (E), then it is strongly order continuous.

Proof. For any $A \in \mathcal{F}$, $\{A_n\} \subset \mathcal{F}$, with $A_n \searrow A$, and $\mu(A) = 0$, we define a double sequence $\{E_n^{(m)} \mid n \ge 1, m \ge 1\}$ of sets satisfying the following conditions: $E_n^{(m)} = A_n$, $E^{(m)} = A$, $\forall\ m, n \ge 1$. Then, there exists a subsequence $\{E_{n_i}^{(m_i)}\}$ of $\{E_n^{(m)} \mid n \ge 1, m \ge 1\}$ such that $\lim_{k\to+\infty} \mu\left(\bigcup_{i=k}^{+\infty} E_{n_i}^{(m_i)}\right) = 0$, that is

$\lim_{k\to+\infty}\mu(A_{n_k})=0$, and hence $\lim_{n\to+\infty}\mu(A_n)=0$. This shows that μ is strongly order continuous. □

Remark 1. A monotone set function fulfilling condition (E) may not be a fuzzy measure.

Example 1. Let $X=[0,1]$, $\mathcal{F}$ denote σ-algebra on X, and let m be a σ-additive measure on $\mathcal{F}$. Let $\mu:\mathcal{F}\to[0,1]$ be defined by

$$\mu(E)=\begin{cases} m(E) & \text{if } m(E)\le\frac{1}{2} \\ \frac{2}{3} & \text{if } m(E)>\frac{1}{2} \text{ and } E\neq X \\ 1 & \text{if } E=X. \end{cases}$$

It is not too difficult to verify that the set function μ is monotone and fulfills condition (E). But μ is neither continuous from below nor continuous from above.

4. EGOROFF TYPE THEOREM

In this section, we shall establish Egoroff's theorem in more general case, where μ is a monotone set function.

Theorem 1. (Egoroff's theorem) Let μ be a monotone set function with $\mu(\emptyset)=0$. Then, for any $f\in\mathbf{F}$ and $\{f_n\}_n\subset\mathbf{F}$,

$$f_n \xrightarrow{\text{a.e.}} f \quad\Longrightarrow\quad f_n \xrightarrow{\text{a.u.}} f$$

if and only if μ fulfills condition (E).

Proof. *Sufficiency*: Suppose μ fulfills condition (E). Let D be the set of these points x in X at which $\{f_n(x)\}$ dose not converge to $f(x)$. Then,

$$D=\bigcup_{m=1}^{+\infty}\bigcap_{n=1}^{+\infty}\bigcup_{i=n}^{+\infty}\left\{x\in X:\ |f_i(x)-f(x)|\ge\frac{1}{m}\right\},$$

and it follows from $f_n\xrightarrow{\text{a.e.}} f$ on X that $\mu(D)=0$. Therefore, for any fixed $m=1,2,\ldots$, we have

$$\mu\left(\bigcap_{n=1}^{+\infty}\bigcup_{i=n}^{+\infty}\left\{x\in X:\ |f_i(x)-f(x)|\ge\frac{1}{m}\right\}\right)=0.$$

If we denote

$$E_n^{(m)}=\bigcup_{i=n}^{+\infty}\left\{x\in X:\ |f_i(x)-f(x)|\ge\frac{1}{m}\right\}$$

for any $m = 1, 2, \ldots$, then $D = \bigcup_{m=1}^{+\infty} \bigcap_{n=1}^{+\infty} E_n^{(m)}$. Write $E^{(m)} = \bigcap_{n=1}^{+\infty} E_n^{(m)}$, then the double sequence $\{E_n^{(m)}\} \subset \mathcal{F}$ $(m, n \in N)$ satisfying the conditions: for any fixed $m = 1, 2, \ldots$, $E_n^{(m)} \searrow E^{(m)}$ as $n \to \infty$ and $\mu(\bigcup_{m=1}^{+\infty} E^{(m)}) = 0$.

Applying the condition (E) of μ to the double sequence $\{E_n^{(m)}\} \subset \mathcal{F}$ $(m, n \in N)$, then there exist increasing sequences $\{n_i\}_{i\in N}$ and $\{m_i\}_{i\in N}$ of natural numbers, such that

$$\lim_{k\to+\infty} \mu\left(\bigcup_{i=k}^{+\infty} E_{n_i}^{(m_i)}\right) = 0.$$

For any $\epsilon > 0$, we take k_0 such that

$$\mu\left(\bigcup_{i=k_0}^{+\infty} E_{n_i}^{(m_i)}\right) < \epsilon.$$

Let $E_\epsilon = X - \bigcup_{i=k_0}^{+\infty} E_{n_i}^{(m_i)}$, then $E_\epsilon \in \mathcal{F}$ and $\mu(X - E_\epsilon) = \mu\left(\bigcup_{i=k_0}^{+\infty} E_{n_i}^{(m_i)}\right) < \epsilon$.

Now we just need to prove that $\{f_n\}$ converges to f on E_ϵ uniformly. Since

$$E_\epsilon = \bigcap_{i=k_0}^{+\infty} \bigcap_{j=n_i}^{+\infty} \left\{x \in X : |f_j(x) - f(x)| < \frac{1}{m_i}\right\},$$

therefore, for any fixed $i \geq k_0$, $E_\epsilon \subset \bigcap_{j=n_i}^{+\infty} \{x \in X : |f_j(x) - f(x)| < 1/m_i\}$. For any given $\sigma > 0$, we take i_0 $(\geq k_0)$ such that $\frac{1}{m_{i_o}} < \sigma$. Thus, as $j > n_{i_o}$, for any $x \in E_\epsilon$,

$$|f_j(x) - f(x)| < \frac{1}{m_{i_o}} < \sigma.$$

This shows that $\{f_n\}$ converges to f on E_ϵ uniformly.

Necessity: Suppose that for any $f \in \mathbf{F}$ and $\{f_n\}_n \subset \mathbf{F}$, $f_n \xrightarrow{\text{a.e.}} f$ implies $f_n \xrightarrow{\text{a.u.}} f$. Let $\{E_n^{(m)} : m, n \in N\} \subset \mathcal{F}$ be any given double sequence of sets and let it satisfy the conditions: for any fixed $m = 1, 2, \cdots$,

$$E_n^{(m)} \searrow E^{(m)}\ (n \to \infty) \ \text{ and } \ \mu\left(\bigcup_{m=1}^{+\infty} E^{(m)}\right) = 0.$$

We put

$$\hat{E}_n^{(m)} = \bigcup_{i=1}^{m} E_n^{(i)} = E_n^{(1)} \cup E_n^{(2)} \cdots \cup E_n^{(m)} \quad (m, n \in N)$$

and

$$\hat{E}^{(m)} = \bigcap_{n=1}^{+\infty} \hat{E}_n^{(m)} \quad (m = 1, 2, \cdots).$$

Then we obtain a double sequence $\{\hat{E}_n^{(m)}\} \subset \mathcal{F}$ $(m, n \in N)$ satisfying the properties: for any fixed $n \in N$, $\hat{E}_n^{(m)} \subset \hat{E}_n^{(m+1)}$, and for any fixed $m \in N$, $\hat{E}_n^{(m)} \searrow \hat{E}^{(m)}$ as $n \to \infty$, and from $\bigcup_{m=1}^{+\infty} \hat{E}^{(m)} = \bigcup_{m=1}^{+\infty} E^{(m)}$, it follows that $\mu(\bigcup_{m=1}^{+\infty} \hat{E}^{(m)}) = 0$.

Now we construct a sequence $\{f_n\}_n \subset \mathbf{F}$: for every $n \in N$ we define

$$f_n(x) = \begin{cases} \frac{1}{m} & x \in \hat{E}_n^{(m+1)} - \hat{E}_n^{(m)} \quad m = 1, 2, \cdots \\ 2 & x \in \hat{E}_n^{(1)} \\ 0 & x \in X - \bigcup_{m=1}^{+\infty} \hat{E}_n^{(m)}. \end{cases}$$

Then for every $m = 1, 2, \cdots$,

$$\left\{x \ : \ |f_n(x)| \leq \frac{1}{m}\right\} = X - \hat{E}_n^{(m)}.$$

Therefore, we have

$$\begin{aligned} \left\{x \ : \ |f_k(x)| \leq \frac{1}{m}, \forall\, k \geq n\right\} &= \bigcap_{k=n}^{+\infty} \left\{x \ : \ |f_k(x)| \leq \frac{1}{m}\right\} \\ &= \bigcap_{k=n}^{+\infty} (X - \hat{E}_k^{(m)}) \\ &= X - \hat{E}_n^{(m)}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \bigcap_{m=1}^{+\infty} \bigcup_{n=1}^{+\infty} \bigcap_{k=n}^{+\infty} \left\{x \ : \ |f_k(x)| \leq \frac{1}{m}\right\} \\ = & \bigcap_{m=1}^{+\infty} \bigcup_{n=1}^{+\infty} (X - \hat{E}_n^{(m)}) \\ = & \bigcap_{m=1}^{+\infty} (X - \bigcap_{n=1}^{+\infty} \hat{E}_n^{(m)}) \\ = & \bigcap_{m=1}^{+\infty} (X - \hat{E}^{(m)}) \\ = & X - \bigcup_{n=1}^{+\infty} \hat{E}^{(m)}. \end{aligned}$$

Noting that $\mu(\bigcup_{m=1}^{+\infty} \hat{E}^{(m)}) = 0$, we have $f_n \xrightarrow{\text{a.e}} 0$ on X. It follows from the hypothesis that $f_n \xrightarrow{\text{a.u}} 0$ on X. By Definition 1, there exists a sequence $\{H_j\}_{j \in N}$ such that for every j, $\mu(X - H_j) < \frac{1}{j}$ and as $n \to \infty$, f_n converges to 0 uniformly on H_j. Without loss of generality, we can assume $H_1 \subset H_2 \subset \cdots$ (otherwise, we can take $\bigcup_{i=1}^{j} H_i$ instead of H_j). Thus for every $j \in N$, there exist $n_j \in N$ such that for any $x \in H_j$, we have $|f_k(x)| \le \frac{1}{j}$ whenever $k \ge n_j$. Therefore, for every $j \in N$, we have

$$H_j \subset \bigcap_{k=n_j}^{+\infty} \left\{ x \; : \; |f_k(x)| \le \frac{1}{j} \right\} = X - \hat{E}_{n_j}^{(j)},$$

and hence $\hat{E}_{n_j}^{(j)} \subset X - H_j$ $(j = 1, 2, \cdots)$. Therefore, for any $k \ge 1$, we have

$$\bigcup_{j=k}^{+\infty} \hat{E}_{n_j}^{(j)} \subset \bigcup_{j=k}^{+\infty} (X - H_j).$$

Consequently, we have

$$\begin{aligned} \mu\left(\bigcup_{j=k}^{+\infty} \hat{E}_{n_j}^{(j)}\right) &\le \mu\left(\bigcup_{j=k}^{+\infty} (X - H_j)\right) \\ &= \mu(X - H_k) \\ &< \frac{1}{k}. \end{aligned}$$

Thus we have chosen a subsequence $\{\hat{E}_{n_j}^{(j)}\}_{j \in N}$ of the double sequence $\{\hat{E}_n^{(m)}\}$ such that

$$\lim_{k \to +\infty} \mu\left(\bigcup_{j=k}^{+\infty} \hat{E}_{n_j}^{(j)}\right) = 0.$$

Noting that for $m, n \in N, E_n^m \subset \hat{E}_n^m$, then we have

$$\lim_{k \to +\infty} \mu\left(\bigcup_{j=k}^{+\infty} E_{n_j}^{(j)}\right) = 0.$$

This shows that μ fulfils condition (E). □

From Theorem 1 and Proposition 1, we can get a necessary condition for the Egoroff's theorem with respect to monotone set function.

Corollary 1. Let μ be a monotone set function with $\mu(\emptyset) = 0$. If for any $f, f_n \in \mathbf{F}$, $f_n \xrightarrow{\text{a.e.}} f$ implies $f_n \xrightarrow{\text{a.u.}} f$, then μ is strongly order continuous.

Li [3] have proved Egoroff's theorem on finite fuzzy measure space. From Theorem 1 we can obtain the following result immediately

Theorem 2. If μ is finite fuzzy measure, then μ fulfils condition (E).

Remark 2. A fuzzy measure may not fulfil condition (E). For example, Lebesgue measure m defined on real line R^1 is a fuzzy measure not fulfilling condition (E).

Remark 3. From Remark 1 (see Example 1) we know that the continuity from below and above of fuzzy measure is a sufficient, but not necessary condition for Egoroff's theorem in [3].

Remark 4. When a fuzzy measure is not necessarily finite, Li et al [2] have proved that Egoroff's theorem remains valid on fuzzy measure space possessing the order continuity and the pseudometric generating property(i.e., $\forall\ \epsilon > 0, \exists\ \delta > 0$ such that $\mu(E \cup F) < \epsilon$ whenever $\mu(E) \vee \mu(F) < \delta$)(cf. [2]). The following example indicates that a fuzzy measure fulfilling condition (E) may not possess pseudometric generating property. Therefore the order continuity and the pseudometric generating property is a sufficient, but not necessary condition for Egoroff's theorem on fuzzy measure spaces.

Example 2. Let $X = \{a, b\}$ and $\mathcal{F} = \wp(X)$. Put

$$\mu(E) = \begin{cases} 1 & \text{if } E = X \\ 0 & \text{if } E \neq X \end{cases}$$

It is obvious that fuzzy measure μ fulfils condition (E). But μ has not pseudometric generating property. In fact, $\mu(\{a\}) = \mu(\{b\}) = 0$, but $\mu(\{a\} \cup \{b\}) = 1 \neq 0$.

5. CONCLUDING REMARKS

In this paper, we have obtained a necessary and sufficient condition of Egoroff's theorem with respect to monotone set functions. The well-known Egoroff's theorem in classical measure space is formulated in full generality by using the condition (E) of set function. Therefore Theorem 1 is a substantial generalization of the related results in [2, 3, 5, 10, 11].

Proposition 1 states that the condition (E) of monotone set function implies the strong order continuity. However, we don't know whether the inverse implication remains true. Li and Yasuda have obtained an encouraging result: a necessary and sufficient condition that Egoroff's theorem remain valid on lower semicontinuous fuzzy measure space with S-compactness is that the lower semicontinuous fuzzy measure be strongly order continuous (cf. [5]). Therefore, on a S-compact space X (especially, X is countable), the strong order continuity is equivalent to condition (E).

In our further research, we intend to address this issue and to investigate whether the strong order continuity and the condition (E) are equivalent for any monotone set function.

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