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THE $\mathrm{d}X(t) = Xb(X)\mathrm{d}t + X\sigma(X)\mathrm{d}W$ EQUATION AND FINANCIAL MATHEMATICS I

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The existence of a weak solution and the uniqueness in law are assumed for the equation, the coefficients b and σ being generally $C(\mathbb{R}^+)$-progressive processes. Any weak solution X is called a (b,σ)-stock price and Girsanov Theorem jointly with the DDS Theorem on time changed martingales are applied to establish the probability distribution μ_σ of X in $C(\mathbb{R}^+)$ in the special case of a diffusion volatility $\sigma(X,t) = \tilde{\sigma}(X(t))$. A martingale option pricing method is presented.

Keywords: weak solution and uniqueness in law in the SDE-theory, (b,σ)-stock price, its Girsanov and DDS-reduction, investment process, option pricing

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1. INTRODUCTION

Our semilinear equation, the merits of which we shall have yet to defend, is a model for stochastic behaviour of a stock price X, say with $X(0) = x > 0$, that comes from a family called the stochastic volatility models that are generally described by

$$\mathrm{d}X(t) = b(t)X(t)\,\mathrm{d}t + \sigma(t)X(t)\,\mathrm{d}W(t), \tag{1.1}$$

where W is a Wiener process, b and σ processes for which the above stochastic differential exists, Karatzas, Shreve [9] or Steele [16]. Our equation respects a reasonable expectation that the rate of return and volatility coefficients b and σ should be fed at time t by the previous history of X. Another way how to model such a dynamics was introduced by Merton [10] in the form

$$\begin{aligned} \mathrm{d}X(t) &= b(t,X(t))\,\mathrm{d}t + \sigma(t)X(t)\,\mathrm{d}W(t) \\ \mathrm{d}\sigma(t) &= a(t,\sigma(t))\,\mathrm{d}t + c(t,\sigma(t))\,\mathrm{d}B(t), \end{aligned} \tag{1.2}$$

where W and B are Wiener processes on a filtered probability space with a linear covariation $\langle W,B\rangle(t) = \rho t$, hence in the form of a rather general two-dimensional SDE. Having an increasing (decreasing) function $f \in C^2(\mathbb{R})$, we may prove that the one dimensional model

$$\mathrm{d}X(t) = b(t,X(t))\,\mathrm{d}t + f(X(t))\,\mathrm{d}W(t)$$

may be rewritten to an equation (1.2) with $W = B$. Cox [5], Beckers [1] choose $f(x) = \sigma \cdot x^\delta$ ($\sigma > 0, \delta \in [0,1)$) to introduce so called the constant elasticity of variance diffusion models. Wiggins [17] followed by Scott [14], [15] proposed an interesting simplification of (1.2) in the form

$$\mathrm{d}X(t) = bX(t)\,\mathrm{d}t + \sigma(t)X(t)\,\mathrm{d}W(t), \quad \mathrm{d}\sigma(t) = a(\sigma(t))\,\mathrm{d}t + c\sigma(t)\,\mathrm{d}B(t)$$

that definitely does not exhaust the collection of the models in between those given by (1.1) and (1.2) equations.

One may ask what should be qualifications of a good model (1.1) for the stochastic dynamics of a stock price. We believe that the model should be stochastically invariant, i.e. that the probability distributions of its principal outputs (stock price process, option price, etc.) have to be invariant of a special choice of the solution X, or the driving Wiener process W. The generality of (1.1) cannot allow for such a requirement. On the other hand, having made minor restrictions on b and σ, a martingale method for the option pricing is available in the setting (1.1). Any modification of (1.1) should keep this ability. What we have been able to achieve in the framework of

$$\mathrm{d}X(t) = X(t)b(X,t)\,\mathrm{d}t + X(t)\sigma(X,t)\,\mathrm{d}W(t), \quad X(0) = x > 0 \tag{1.3}$$

as a model for a (b,σ)-price X, say until the market expiration time $T > 0$. We restrict ourselves to those (b,σ) for which at least one Wiener process $(\Omega, \mathcal{F}, P, W)$ exists such that there is an X to satisfy (1.3) and $\mathcal{L}(X) = \mu_{b,\sigma}$ holds for all possible (b,σ)-prices X. To make Girsanov Theorem simply applicable, we also assume that b and σ are bounded $C(\mathbb{R}^+)$-progressive processes such that $\sigma \geq \varepsilon > 0$ and $b = 0$ outside $[0,T]$. Corollary 3.3 offers a wide choice of pairs (b,σ) to satisfy the requirements. Note that any X is always a positive process and that we are able to remove the drift in (1.3) to restrict it to Engelbert–Schmidt equation

$$\mathrm{d}X(t) = X(t)\sigma(X,t)\,\mathrm{d}W(t), \quad X(0) = x. \tag{1.4}$$

Corollary 3.2 says: If X is a continuous (properly measurable) process, then there are $P_X \sim P$ and a Wiener process W_X such that X is a (b,σ)-price w.r.t. $(\Omega, \mathcal{F}, P, W)$ iff X is a $(0,\sigma)$-price w.r.t. $(\Omega, \mathcal{F}, P_X, W_X)$. In particular, a pair (b,σ) is a suitable one for our model iff the $(0,\sigma)$ has the property. Moreover, $\mathcal{L}(X|P_X) = \mu_\sigma$ for all (b,σ)-prices X.

We have many good reasons, as we shall see later on, to learn about μ_σ as much as possible: Assuming that $\sigma(X,t) = \tilde{\sigma}(X(t))$ is a *diffusion coefficient*, Theorems 4.2 and 4.3 explain why *any* solution X to (1.4) is constructed in the following way:

(a) Choose an arbitrary Wiener process $(\Omega, \mathcal{F}, P, B)$ and consider its exponential $Y(t) = x\exp\{B(t) - t/2\}$.

(b) Put $\varphi(t) = \inf\{s \geq 0\colon \int_0^s \tilde{\sigma}^{-2}(Y(u))\,\mathrm{d}u > t\}$ and $W(t) = \int_0^{\varphi(t)} \tilde{\sigma}^{-1}(Y(u))\,\mathrm{d}B(u)$.

Then $X = Y(\varphi)$ is a solution to (1.4) on $(\Omega, \mathcal{F}, P, W)$ and a (b,σ)-stock price under the $Q \sim P$ defined by $Q_X = P$.

The rest of Section 4 concerns the problem of finding one-dimensional distributions $\mathcal{L}(X(t))$ that in fact turns to be a problem of finding one-dimensional distributions $\mathcal{L}(Y(t), \tau(t) = \int_0^t \tilde{\sigma}^{-2}(Y(u))\,du)$ (See Corollary 4.5, Remark 4.6 and Example 4.7).

Observing the above construction of a (b,σ)-price X, finally described by (1.4), we find that the driving Wiener process W is not entirely responsible for X in the form $X = f(W)$, as it is in general models (1.1) (even though there is another Wiener process B to drive X as its function). On the other hand, $W = f(X)$ and we conclude that the only proper driver of our model is a stock price X itself (see Theorem 4.3 for a correct statement).

This is also the reason why we had to reprove the corner stone statements by Karatzas and Shreve (summarized in [9, 5.8.B Section]) on the price of a financial claim or option and its valuation by an investment process. See, Theorems 5.4 and 5.5. The proof heavily depends on the fact that any (b,σ)-price X is a P_X-martingale (see Lemma 4.1) and that there is no other $Q \sim P$ with the property (see Lemma 6.7). It made it possible to reform the results on the PRP-property by M. Yor [18] (see also § 4, Chapt. V in [17]) into the form of Lemma 6.8 and repeat the reasoning presented in [9]. The price of arbitrary P_X-integrable option and its valuation are proved to be stochastic invariants.

Some of more technical results to be applied in Sections 3, 4 and 5 are referred to Section 6 whose results and proofs are of course entirely independent of the earlier text. Section 7 summarizes the unsolved problems we have met having prepared the text.

Above, we have reviewed some of the stochastic volatility models that inspired our investigation. Its novelty can be seen, as we believe, in applying the time change procedure described by (a),(b) with the aim to recover as much information on the probability distribution of price process X as possible to facilitate, for example, the computations of expectations for option pricing. In this context see also Geman, Madan and Yor [7] and Borovkov, Novikov [3].

2. NOTATIONS AND CONVENTIONS

If not said otherwise explicitly, we shall assume that

$$\text{all probability spaces } (\Omega, \mathcal{F}, P) \text{ are complete} \tag{2.1}$$

$$\text{all filtrations } \mathcal{F}_t = (\Omega, \mathcal{F}, P, \mathcal{F}_t) \text{ are complete and right continuous,} \tag{2.2}$$
$$\text{denoting } \mathcal{F}_\infty := \sigma(\cup_t \mathcal{F}_t).$$

Having a process $X = (X(t), t \geq 0)$ on a probability space (2.1), we denote $\sigma_t(X) := \sigma(X(s), s \leq t)$ and by $(\mathcal{F}_t^X)$ the P-augmentation of the canonical filtration $(\sigma_t(X))$. See Section 6 for the definition. Agree to denote and call

$$Y = (\Omega, \mathcal{F}, P, \mathcal{F}_t, Y) \text{ martingale, local martingale and semimartingale} \tag{2.3}$$

if $Y = (Y(t), t \geq 0)$ is an $\mathcal{F}_t$-martingale, $\mathcal{F}_t$-local martingale and a *continuous* $\mathcal{F}_t$-semimartingale, respectively,

$$W = (\Omega, \mathcal{F}, P, \mathcal{F}_t, W) \text{ a Wiener process} \tag{2.4}$$

if $W = (W(t), t \geq 0)$ is an $\mathcal{F}_t$-Wiener process.

Our definitions are exactly those presented in [6, part III]. Having intention to observe continuous processes X mostly, recall that such X is measurable as

$$X : (\Omega, \sigma_\infty(X)) \to (C(\mathbb{R}^+), \sigma_\infty(\mathbb{X})), \quad X : (\Omega, \sigma_t(X)) \to (C(\mathbb{R}^+), \sigma_t(\mathbb{X})), \quad t \geq 0,$$

where $\mathbb{X}$ is the canonical process on $C(\mathbb{R}^+)$ and $\sigma_\infty(\mathbb{X})$ is exactly the Borel σ-algebra of the Polish space $C(\mathbb{R}^+)$. Thus, any k-dimensional continuous process X will be also understood as a $\Pi_{i=1}^k C(\mathbb{R}^+)$-random variable with the probability distribution denoted as $\mathcal{L}(X|P) = \mathcal{L}(X)$.

If μ is a completed Borel probability measure on $C(\mathbb{R}^+)$, we denote by $(\mathcal{F}_t^\mu)$ the μ-augmentation of the canonical filtration $(\sigma_t(\mathbb{X}))$.

Few words about stochastic processes on $(C(\mathbb{R}^+), \mathcal{B}(C(\mathbb{R}^+)))$, say $S = (S(t), t \geq 0)$, follow. Note that S is a continuous process iff $x \mapsto (S(x,t), t \geq 0)$ is a Borel transformation of $C(\mathbb{R}^+)$ and recall that if S is a continuous and $\sigma_t(\mathbb{X})$-adapted process, then it is a $\sigma_t(\mathbb{X})$-progressive process, i.e. such that $(x, s) \mapsto S(x, s)$ is a map measurable as

$$(C(\mathbb{R}^+) \times [0,t], \sigma_t(\mathbb{X}) \otimes \mathcal{B}[0,t]) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ for arbitrary } t \geq 0.$$

The concept of $\sigma_t(\mathbb{X})$-progressivity or simply $C(\mathbb{R}^+)$*-progressivity* is important:

If S is a $C(\mathbb{R})$-progressive process, then there exist Borel maps $S_t : C([0,t]) \to \mathbb{R}$ such that $S(x,t) = S_t(x(s), s \leq t)$ holds for all $x \in C(\mathbb{R}^+)$ and $t \geq 0$ (more generally, the statement is true for S $\sigma_t(\mathbb{X})$-adapted).

If S is a locally bounded $C(\mathbb{R}^+)$-progressive process, then $x \mapsto (\int_0^t S(x,u)\,\mathrm{d}u, t \geq 0)$ defines a continuous $\sigma_t(\mathbb{X})$-adapted process on $C(\mathbb{R}^+)$ hence a Borel transformation of $C(\mathbb{R}^+)$.

If S is a locally bounded and nonnegative $C(\mathbb{R}^+)$-progressive process, then $x \mapsto (x(\int_0^t S(x,u)\,\mathrm{d}u), t \geq 0)$ defines a Borel transformation of $C(\mathbb{R}^+)$.

If S is a $C(\mathbb{R}^+)$-progressive process and X a continuous process, then $Y = S(X)$ defines a $\sigma_t(X)$-progressive process.

Consider $C(\mathbb{R}^+)$-progressive processes B and S, also $x \in \mathbb{R}$. Then

$$X = (\Omega, \mathcal{F}, P, \mathcal{F}_t, W, X) \text{ is a weak solution to} \tag{2.5}$$

the (B, S)-stochastic differential equation

$$\mathrm{d}X(t) = B(X,t)\,\mathrm{d}t + S(X,t)\,\mathrm{d}W(t), \quad X(0) = x \tag{2.6}$$

if $W = (\Omega, \mathcal{F}, P, \mathcal{F}_t, W)$ is a Wiener process and $X = (\Omega, \mathcal{F}, P, \mathcal{F}_t, X)$ a continuous semimartingale with the stochastic differential $\mathrm{d}X(t)$ and the initial value $X(0)$ given by (2.6). The (B, S)-stochastic differential equation (2.6) is said to be unique in law if $\mathcal{L}(X_1|P^1) = \mathcal{L}(X_2|P^2)$ for an arbitrary pair

$$X_i = (\Omega^i, \mathcal{F}^i, P^i, \mathcal{F}_t^i, W^i, X_i),\ i = 1, 2 \tag{2.7}$$

of weak solutions to the equation (2.6).

Stress that uniqueness in law does not say generally that $X_1 = X_2$ almost surely if X_1, X_2 are weak solutions (2.7) where $(\Omega^1, \mathcal{F}^1, P^1, \mathcal{F}_t^1, W^1) = (\Omega^2, \mathcal{F}^2, P^2, \mathcal{F}_t^2, W^2)$. The equation (2.6) with $B \equiv 0$, i.e.

$$\mathrm{d}X(t) = S(X, t)\,\mathrm{d}W(t),\ \ X(0) = x \tag{2.8}$$

will be referred to as the Engelbert–Schmidt equation. Recall that if $S(x, t) = \tilde{S}(x(t))$ for $(x, t) \in C(\mathbb{R}^+) \times \mathbb{R}^+$ and $\tilde{S} : \mathbb{R} \to \mathbb{R}$ is a Borel function, then (2.8) has a weak solution and is unique in law iff

$$\{x \in \mathbb{R},\ \tilde{S}(x) = 0\} = \left\{x \in \mathbb{R},\ \int_x^{x^+} \tilde{S}^{-2}(y)\,\mathrm{d}y = +\infty\right\}$$

holds. (See [8, 20.1, p. 371]). Having a filtration $(\mathcal{F}_t)$, we agree to call a $\tau : \Omega \to [0, \infty]$ an $\mathcal{F}_t$-Markov time if $[\tau \le t] \in \mathcal{F}_t$ for all $t \ge 0$ and to define 'the history' of $(\mathcal{F}_t)$ up to τ as

$$\mathcal{F}_\tau = \{F \in \mathcal{F}_\infty : F \cap [\tau \le t] \in \mathcal{F}_t,\ \forall t \ge 0\}.$$

Having a continuous local martingale $M = (\Omega, \mathcal{F}, P, \mathcal{F}_t, M)$, we denote its quadratic variation as $\langle M \rangle$. If $M(0) = 0$ and $\langle M \rangle(\infty) = \infty$ a.s., we define the *DDS*[1]*-Wiener process* B of M by

$$B = \left(\Omega, \mathcal{F}, P, \mathcal{F}_\tau^M, B = M(\tau)\right),\ \ \tau(t) = \inf\{s \ge 0, \langle M \rangle(s) > t\},\ t \ge 0. \tag{2.9}$$

(See [11, Chapt. V.1] for details). If $\mathcal{F}_\infty^B = \mathcal{F}_\infty^M$, *the local martingale* M is called *pure.* (See [11, p. 204, 205] for equivalent definitions.)

3. (b, σ)–STOCK PRICES AND GIRSANOV REDUCTION

Fix $T > 0$ and $x > 0$, the expiration market time and the initial price of the stock, respectively. Denote

$B := \{b$ a bounded $C(\mathbb{R}^+)$-progressive process with $b(x, t) = 0,\ x \in C(\mathbb{R}^+),\ t > T\}$
$S := \{\sigma$ a bounded $C(\mathbb{R}^+)$-progressive process, $\sigma \ge \varepsilon > 0$ for some $\varepsilon\}$, $BS := B \times S$.

[1] DDS stays for Dambis, Dubins–Schwarz (Wiener process).

Call $(b, \sigma) \in BS$ a *stock market* if the stochastic differential equation

$$\mathrm{d}X(t) = X(t)b(X,t)\,\mathrm{d}t + X(t)\sigma(X,t)\,\mathrm{d}W(t), \quad X(0) = x \tag{3.1}$$

has a weak solution and it is unique in law. The set of all stock markets will be denoted as BS_M. If $(b,\sigma) \in BS_M$, then $b(x,t)$ and $\sigma(x,t)$ may be referred to as the *rate of return* and *the volatility* of the (b,σ)-market, any weak solution X to the equation (3.1) (detailed by (2.5)) will be called a (b,σ)*-stock price.* Our model entails that there is a Borel probability measure $\mu_{b,\sigma}$ on $C(\mathbb{R}^+)$ such that $\mathcal{L}(X|P) = \mu_{b,\sigma}$ holds for arbitrary (b,σ)-stock price X. Abbreviate $\mu_\sigma := \mu_{0,\sigma}$. Note that arbitrary (b,σ)-stock price X is a *positive process* almost surely, since

$$X(t) = x \exp\left\{N(t) - \frac{1}{2}\langle N\rangle(t)\right\}, \quad t \geq 0, \text{ almost surely,} \tag{3.2}$$

where

$$\mathrm{d}N(t) = b(X,t)\,\mathrm{d}t + \sigma(X,t)\,\mathrm{d}W(t), \quad N(0) = 0 \tag{3.3}$$

and $\langle N\rangle(t) = \int_0^t \sigma^2(X,u)\,\mathrm{d}u$ denotes the quadratic variation of the semimartingale N.

Our definition of a (b,σ)-stock market is designed to promote the Girsanov reduction as simple as possible:

Until further remark fix $(b,\sigma) \in BS$, put $a = \frac{b}{\sigma}$, consider a Wiener process W given by (2.4) and a continuous $\mathcal{F}_t$-adapted process X. Then

$$W_X = (\Omega, \mathcal{F}, P, \mathcal{F}_t, P_X, W_X), \tag{3.4}$$

where

$$\mathrm{d}P_X := D_X\,\mathrm{d}P, \quad D_X := \exp\left\{-\int_0^T a(X,t)\,\mathrm{d}W(t) - \frac{1}{2}\int_0^T a^2(X,t)\,\mathrm{d}t\right\} \tag{3.5}$$

$$W_X(t) := \int_0^t a(X,u)\,\mathrm{d}u + W(t), \quad t \geq 0 \tag{3.6}$$

defines another Wiener process W_X. Indeed, $E_P D_X = 1$ by Novikov Theorem [6, 2.4.7] as a is a bounded process. Hence, $P_X \sim P$ is a probability measure under which $(\mathcal{F}_t)$ is a complete and right-continuous filtration and finally W_X is a Wiener process by Girsanov Theorem [6, 2.4.8]. The Girsanov reduction $W \to W_X$ provides a mighty tool for the financial mathematics. Choose $b = 0$ in (3.1) to get the Engelbert–Schmidt stochastic differential equation

$$\mathrm{d}X(t) = X(t)\sigma(X,t)\,\mathrm{d}W(t), \quad X(0) = x. \tag{3.7}$$

Theorem 3.1. Consider $(b,\sigma) \in BS$, a Wiener process W in (2.4) and a continuous $\mathcal{F}_t$-adapted process X. Then

(i) $(\Omega, \mathcal{F}, P, \mathcal{F}_t, W, X)$ is a weak solution to (3.1) iff $(\Omega, \mathcal{F}, P_X, \mathcal{F}_t, W_X, X)$ is a weak solution to (3.7).

(ii) If X_1 and X_2 in (2.7) are weak solutions to (3.1), then

$$\mathcal{L}(X_1|P^1) = \mathcal{L}(X_2|P^2) \iff \mathcal{L}(X_1|P^1_{X_1}) = \mathcal{L}(X_2|P^2_{X_2}).$$

In particular, (3.1) has a weak solution (is unique in law) iff the equation (3.7) possesses the corresponding property.

Proof. If $(\Omega, \mathcal{F}, P, \mathcal{F}_t, W, X)$ is a solution to (3.1), then X is a continuous $(P, \mathcal{F}_t)$-semimartingale with the stochastic differential ($a := \frac{b}{\sigma}$)

$$\mathrm{d}X(t) = X(t)\sigma(X,t)[a(X,t)\,\mathrm{d}t + \mathrm{d}W(t)] = X(t)\sigma(X,t)\,\mathrm{d}W_X(t).$$

It follows that X is a continuous $(P_X, \mathcal{F}_t)$-local martingale with the stochastic differential $\mathrm{d}X(t) = X(t)\sigma(X,t)\,\mathrm{d}W_X(t)$. Hence, $(\Omega, \mathcal{F}, P, \mathcal{F}_t, W_X, X)$ is a solution to (3.7). Because the above reasoning may be easily reversed, the equivalence (i) is proved.

We shall prove (ii). Assume that $\mathcal{L}(X_1|P^1) = \mathcal{L}(X_2|P^2)$ and put

$$Y_i(t) := X_i(t) - \int_0^t X_i(u)b(X_i,u)\,\mathrm{d}u = \int_0^t X_i(u)\sigma(X_i,u)\,\mathrm{d}W^i(u), \quad i = 1,2.$$

Obviously, $\mathcal{L}(X_1, Y_1|P^1) = \mathcal{L}(X_2, Y_2|P^2)$, where $\mathcal{L}(X,Y|P)$ denotes the joint probability distribution of processes X and Y under P on $\mathcal{B}(C(\mathbb{R}^+)^2) = \mathcal{B}(C(\mathbb{R}^+)) \otimes \mathcal{B}(C(\mathbb{R}^+))$. Recall that each X_i is an almost surely positive process, define a $C(\mathbb{R}^+)$-progressive process

$$c(x,t) := \frac{1}{x(t)\sigma(x,t)}, \; x(t) > 0, \;\; c(x,t) := 0, \; x(t) \le 0, \; x \in C(\mathbb{R}^+), \; t \ge 0. \tag{3.8}$$

Note that triples $(X_i, N_i = Y_i, c)$ satisfy the requirements of Lemma 6.6 and therefore

$$\mathcal{L}(X_1, W^1|P^1) = \mathcal{L}(X_2, W^2|P^2)$$

holds, where $W^i = \int c(X_i,t)\,dY_i(t)$. It follows by Lemma 6.6 again, this time with triples $(X_i, N_i = W^i, c = a)$, that

$$\mathcal{L}\left(X_1, \int a(X_1)\,\mathrm{d}W^1, \int a(X_1)\,\mathrm{d}t \,\middle|\, P^1\right) = \mathcal{L}\left(X_2, \int a(X_2)\,\mathrm{d}W^2, \int a(X_2)\,\mathrm{d}t \,\middle|\, P^2\right)$$

is seen as a true statement. Hence, $\mathcal{L}(X_1, D_{X_1}|P^1) = \mathcal{L}(X_2, D_{X_2}|P^2)$ and therefore the right-hand side in (ii) is verified.

Assume that $\mathcal{L}(X_1|P^1_{X_1}) = \mathcal{L}(X_2|P^2_{X_2})$. According to (i) $X_i = (\Omega^i, \mathcal{F}^i, P^i_{X_i}, \mathcal{F}^i_t, W^i_X, X_i)$, $i = 1,2$ are solutions to (3.7) and therefore the triples $(X_i, N_i = X_i, c)$, where c is the process defined by (3.8), meet the hypotheses of Lemma 6.6. If follows that $\mathcal{L}(X_1, W^1_{X_1}|P^1_{X_1}) = \mathcal{L}(X_2, W^2_{X_2}|P^2_{X_2})$, hence, again by Lemma 6.6, $\mathcal{L}(X_1, D^{-1}_{X_1}|P^1_{X_1}) = \mathcal{L}(X_2, D^{-1}_{X_2}|P^2_{X_2})$ holds. Since $\mathrm{d}P_i = D^{-1}_{X_i}\,\mathrm{d}P^i_{X_i}$, we verify the left-hand side of (ii).

To see the remaining part as a correct statement, just note that if $(\Omega, \mathcal{F}, Q, \mathcal{F}_t, B, X)$ is a weak solution to (3.7), then

$$\mathrm{d}P = D\,\mathrm{d}Q, \quad D := \exp\left\{\int_0^T a(x)\,\mathrm{d}B - \frac{1}{2}\int_0^T a^2(x)\,\mathrm{d}t\right\},$$
$$W(t) = B(t) - \int_0^t a(u)\,\mathrm{d}u$$

define, by Girsanov Theorem, a Wiener process $(\Omega, \mathcal{F}, P, \mathcal{F}_t, W)$ such that

$$(\Omega, \mathcal{F}, Q, \mathcal{F}_t, B) = (\Omega, \mathcal{F}, P_X, \mathcal{F}_t, W_X)$$

holds. □

In the language of financial mathematics Theorem 3.1 reads

Corollary 3.2. If $(b,\sigma) \in BS$, then (b,σ) is a stock market iff $(0,\sigma)$ is a stock market. In symbols $BS_M := B \times S_M$, where $S_M = \{\sigma \in S, (0,\sigma) \in BS_M\}$.

If $(b,\sigma) \in BS_M$, then $(\Omega, \mathcal{F}, P, \mathcal{F}_t, W, X)$ is a (b,σ)-stock price if and only if $(\Omega, \mathcal{F}, P_X, \mathcal{F}_t, W_X, X)$ is a $(0,\sigma)$-stock price.

Our assumptions on the volatility parameter $\sigma \in S$ may be for example as follows:

$$\sigma(x,t) = \tilde{\sigma}(x(t)), \quad x \in C(\mathbb{R}^+), \ t \geq 0, \quad \tilde{\sigma} : \mathbb{R} \to \mathbb{R} \text{ Borel} \tag{3.9}$$

or

$$\sigma : C(\mathbb{R}^+) \times \mathbb{R}^+ \to \mathbb{R} \text{ locally Lipschitz,} \tag{3.10}$$

which means that for some $K_n < \infty$

$$|\sigma(x,t) - \sigma(y,t)| \leq K_n ||x - y||_t, \quad ||x||_t, ||y||_t \leq n, \ t \leq n, \ n \in \mathbb{N}$$

holds, where $||x||_t := \max_{s \leq t} |x(s)|$ for $x \in C(\mathbb{R}^+)$.

Corollary 3.3. If $\sigma \in S$, then either (3.9) or (3.10) implies that $(0,\sigma) \in BS_M$ and therefore (b,σ) is a stock market for arbitrary $b \in B$ according to 3.2.

Proof. Under (3.10) the equation (3.7) reads as (2.8) with $S(x,t) = x(t)\sigma(x,t)$. This process S is seen to be $C(\mathbb{R}^+)$-progressive and locally Lipschitz such that

$$|S(x,t)| \leq C(1 + ||x||_t), \quad x \in C(\mathbb{R}^+), \ t \geq 0, \ C < \infty$$

holds. Combine Itô and Yamada–Watanabe Theorems [13, 12.2, p. 132 and 17.1, p. 150] to see that (2.8) has a weak solution and it is unique in law under (3.10). Assuming (3.9), we get that (3.7) as (2.8) with $S(x,t) = \tilde{S}(x(t))$ where $\tilde{S} : x \in \mathbb{R} \mapsto x\tilde{\sigma}(x) \in \mathbb{R}$ is a Borel function from $\mathbb{R}$ to $\mathbb{R}$ such that $\int_x^{x+} \tilde{S}^{-2}(u)\,\mathrm{d}u = +\infty$ iff $x = 0$. Engelbert–Schmidt Theorem [8, 20.1, p. 371], we have already referred to, now proves that (2.8) has a weak solution and it is unique in law. □

4. DDS-REDUCTION OF $(0,\sigma)$-STOCK PRICES

As we shall see in the next section, we have many good reasons to be interested in the probability distribution of a (b,σ)-stock price X under the probability measure P_X.

Lemma 4.1. Consider $(b,\sigma) \in BS_M$ and a (b,σ)-price X given as (2.5). Then X is an L_p-martingale on $(\Omega, \mathcal{F}, P_X)$ for arbitrary $p \geq 1$.

Proof. According to Corollary 3.2, $(\Omega, \mathcal{F}, P_X, \mathcal{F}_t, W_X, X)$ is a weak solution to (3.7). Denote $\mathrm{d}M(t) := \sigma(X,t)\,\mathrm{d}W(t), M(0) = 0$ and compute

$$X(t)^p = x^p \exp\left\{pM(t) - \frac{p^2}{2}\langle M\rangle(t)\right\} \exp\left\{\frac{p^2-p}{2}\langle M\rangle(t)\right\}.$$

Since $\langle M\rangle(t) = \int_0^t \sigma^2(X,u)\,\mathrm{d}u \leq C^2 t$ for some $C < \infty$, it follows by Novikov Theorem that X is a P_X-martingale such that $E_{P_X}|X(t)|^p < \infty$ holds for all $t \geq 0$ and $p \geq 1$. □

Once again, according to Theorem 3.1, the distribution of a (b,σ)-stock price X under P_X is equivalently defined as the distribution μ_σ of an arbitrary weak solution X to the Engelbert–Schmidt equation (3.7).

Fix $\sigma \in S_M$ and a weak solution X to (3.7) specified as in (2.5). Denote

$$\mathrm{d}M(t) := \sigma(X,t)\,\mathrm{d}W(t), \; M(0) = 0. \tag{4.1}$$

Hence,

$$\mathrm{d}X(t) = X(t)\,\mathrm{d}M(t), \; X(t) = x\exp\left\{M(t) - \frac{1}{2}\langle M\rangle(t)\right\}. \tag{4.2}$$

Since X is a positive continuous process, we get $\mathrm{d}M(t) = X(t)^{-1}\,\mathrm{d}X(t)$ and (4.2) implies that $\mathcal{F}_t^M = \mathcal{F}_t^X$ holds for all $t \geq 0$ because a legal change of the underlying complete filtration does not change the stochastic integrals. On the other hand, $\sigma \geq \varepsilon > 0$ implies that $\mathrm{d}W(t) = \sigma^{-1}(X,t)\,\mathrm{d}M(t)$, and finally

$$\mathcal{F}_t^W \subseteq \mathcal{F}_t^M = \mathcal{F}_t^X, \; t \geq 0. \tag{4.3}$$

Stress that weak solutions X to (3.7) cannot be generally constructed to achieve the equality $\mathcal{F}_t^W = \mathcal{F}_t^X$ in (4.3), or in other words, to generate them as $X = f(W)$ for a suitably measurable $f : C(\mathbb{R}^+) \to C(\mathbb{R}^+)$. (See Lemma 6.4.)

To understand the problem properly, assume for the rest of this section that[2]

$$\sigma \in S \text{ is constructed via (3.9)} \quad (\Rightarrow \sigma \in S_M \text{ by Corollary 3.3}) \tag{4.4}$$

and recall a possible construction of a weak solution X to the equation (3.7): Consider a Wiener process B on a probability space $(\Omega, \mathcal{F}, P)$, in detail, $B = (\Omega, \mathcal{F}, P, \mathcal{F}_t^B, B)$, and its exponential Y defined by

$$Y(t) = x \exp\{B(t) - t/2\}, \text{ or equivalently by } \mathrm{d}Y(t) = Y(t)\,\mathrm{d}B(t),\ Y(0) = x. \tag{4.5}$$

Define a continuous $\mathcal{F}_t^B = \mathcal{F}_t^Y$-adapted, strictly increasing process φ with $\varphi(0) = 0$, $\varphi(\infty) = \infty$ and $\mathcal{F}_t^B = \mathcal{F}_t^Y$-Markov times $\varphi(t)$ by

$$\varphi(t) := \inf\left\{ s \geq 0, \int_0^s \tilde{\sigma}^{-2}(Y(u))\,\mathrm{d}u > t \right\},\ t \geq 0. \tag{4.6}$$

Recall A.5 (i) Proposition [11, p. 173] to prove that

$$X(t) := Y(\varphi(t)),\quad W(t) := \int_0^{\varphi(t)} \tilde{\sigma}^{-1}(Y(u))\,\mathrm{d}B(u),\ t \geq 0 \tag{4.7}$$

define continuous $\mathcal{F}_{\varphi(t)}^B$-local martingales on $(\Omega, \mathcal{F}, P)$, where

$$\mathcal{F}_{\varphi(t)}^B := \{F \in \mathcal{F}_\infty^B : F \cap [\varphi(t) \leq s] \in \mathcal{F}_s\ \forall s \geq 0\},\ t \geq 0, \tag{4.8}$$

is "the history" of B up to $\mathcal{F}_t^B$-Markov time $\varphi(t)$. It follows by 1.5.(i) in [11], again, that

$$\langle W\rangle(t) = \int_0^{\varphi(t)} \tilde{\sigma}^{-2}(Y(u))\,\mathrm{d}u = t \text{ holds for all } t \geq 0 \text{ almost surely}$$

and $W = (\Omega, \mathcal{F}, P, \mathcal{F}_{\varphi(t)}^B, W)$ is a Wiener process by Lévy characterization theorem. Denoting $\mathrm{d}N = \tilde{\sigma}^{-1}(Y)\,\mathrm{d}B, N(0) = 0$, we have $W = N(\varphi)$ and

$$X(t) = x + \int_0^{\varphi(t)} Y(u)\tilde{\sigma}(Y(u))\,\mathrm{d}N(u) = x + \int_0^t X(u)\tilde{\sigma}(X(u))\,\mathrm{d}W(u),\ t \geq 0$$

by Proposition 1.5 (ii) in [11] again. Thus, we have proved

[2] I.e., σ is of the form $\sigma(x,t) = \tilde{\sigma}(x(t))$, where $\tilde{\sigma} : \mathbb{R} \to [\varepsilon, \infty)$ is a Borel function and $\varepsilon > 0$.

Theorem 4.2. Let $B = (\Omega, \mathcal{F}, P, \mathcal{F}_t^B, B)$ be a Wiener process (perhaps the canonical one on $C(\mathbb{R}^+)$). Define $Y, \varphi, \mathcal{F}_{\varphi(t)}^B$ and W by (4.5), (4.6), (4.8) and (4.7), respectively. Then

$$X = \left(\Omega, \mathcal{F}, P, \mathcal{F}_{\varphi(t)}^B, W, X = Y(\varphi)\right) \tag{4.9}$$

is a weak solution to (3.7). In particular, $\mu_\sigma = \mathcal{L}(Y(\varphi))$.

Remark that $\mathcal{L}(Y) = \mu_1$ and that Theorem 4.2 suggests a weak solution to (3.7) that lives on $\mathcal{F}_\infty^B$, where B is a Wiener process but not the driver of the equation (3.7). The next theorem states that there are no other solutions to (3.7) than those given by (4.9).

Theorem 4.3. Consider a weak solution X to (3.7) specified by (2.5). Denote by B the DDS-Wiener process of $M = \int X^{-1}\,\mathrm{d}X = \int \sigma(X)\,\mathrm{d}W$ ($\langle M\rangle(\infty) = \infty$), define Y, φ and $\mathcal{F}_{\varphi(t)}^B$ by (4.5), (4.6) and (4.8), respectively. Then $Y(\varphi) = X$ almost surely and M is a pure martingale, that means[3]:

$$\mathcal{F}_\infty^X = \mathcal{F}_\infty^M = \mathcal{F}_\infty^B = \mathcal{F}_\infty^Y, \text{ or equivalently } \mathcal{F}_t^X = \mathcal{F}_{\varphi(t)}^B \text{ for } t \geq 0. \tag{4.10}$$

Proof. Specify B and τ as in (2.9) and apply [11, 1.5, p. 173] to prove that $Z := X(\tau)$ is a continuous $\mathcal{F}_{\tau(t)}^X$-local martingale on $(\Omega, \mathcal{F}, P)$ such that

$$Z(t) = x + \int_0^{\tau(t)} X(u)\,\mathrm{d}M(u) = x + \int_0^t X(\tau(u))\,\mathrm{d}M(\tau)(u) = x + \int_0^t Z(u)\,\mathrm{d}B(u) \tag{4.11}$$

holds almost surely for all $t \geq 0$. Hence, Z solves the equation $\mathrm{d}Z = Z\,\mathrm{d}B$ with $Z(0) = x$ and therefore $X(\tau) = Y$ almost surely. By [11, 1.4, p. 172] we compute that outside a P-null set and for $t \geq 0$

$$t = \langle M\rangle(\tau(t)) = \int_0^{\tau(t)} \tilde{\sigma}^2(X(u))\,\mathrm{d}u = \int_0^t \tilde{\sigma}^2(Y(u))\,\mathrm{d}\tau(u)$$

and therefore $\tau(t) = \int_0^t \tilde{\sigma}^{-2}(Y(u))\,\mathrm{d}u$ holds. It follows that $\varphi = \tau^{-1} = \langle M\rangle, X(\tau) = Y, X = Y(\varphi)$ are equalities valid almost surely, hence $\mathcal{F}_\infty^M = \mathcal{F}_\infty^X = \mathcal{F}_\infty^Y = \mathcal{F}_\infty^B$ by (i) in Lemma 6.4. □

Summary 4.4. We state (keeping assumption (4.4) about σ):

(a) Any $(0,\sigma)$-stock price X can be reduced to a $(0,1)$-stock price $Y = X(\tau)$, where $\tau(s) := \inf\{t \geq 0, \int_0^t \tilde{\sigma}^2(X(u))\,\mathrm{d}u > s\}$.

(b) Any $(0,1)$-stock price Y can be extended to a $(0,\sigma)$-stock price $X = Y(\varphi)$, where $\varphi(t) := \inf\{s \geq 0 : \int_0^s \tilde{\sigma}^{-2}(Y(u))\,\mathrm{d}u > t\}$.

[3]See Exercise 4.14.4[0] in [11, p. 205].

Both the reduction and extension does not change the underlying probability space $(\Omega, \mathcal{F}, P)$ and employ filtrations $(\mathcal{F}_t^X)$ and $(\mathcal{F}_t^Y)$ such that $\mathcal{F}_\infty^X = \mathcal{F}_\infty^Y$.

To establish the probability distribution $\mathcal{L}(X)$ of a $(0,\sigma)$-stock price X as $\mathcal{L}(Y(\varphi))$, where Y is a $(0,1)$-stock price and φ is defined in (b), or even one dimesional distributions $\mathcal{L}(X(t)) = \mathcal{L}(Y(\varphi(t)))$, may impose serious problems.

For the rest of present section fix Y, the exponential (4.5) of a Wiener process B, i.e. a $(0,1)$-stock price process. Denote

$$G_t := \mathcal{L}(Y(t), \tau(t)), \quad \text{where} \quad \tau(t) := \int_0^t \tilde{\sigma}^{-2}(Y(u))\,\mathrm{d}u, \; t \geq 0 \tag{4.12}$$

and consider $f \in C^2(\mathbb{R})$ such that random variables

$$\int_0^t |f'(Y(u))Y(u)|^2\,\mathrm{d}u, \quad \int_0^t |f''(Y(u))|Y^2(u)\,\mathrm{d}u, \quad f''(Y(t))Y^2(t) \tag{4.13}$$

are integrable for all $t \geq 0$. Having such an f, we define

$$m_{t,u}(f) := \int_{\mathbb{R}\times[0,t]} f''(y)y^2 G_u(\mathrm{d}y, \mathrm{d}\tau), \quad t, u \geq 0.$$

Corollary 4.5. Assume (4.4) about σ and let X be a $(0,\sigma)$-stock price. Then

$$Ef(X(t)) = f(x) + \frac{1}{2}\int_0^\infty m_{t,u}(f)\,\mathrm{d}u, \; t \geq 0 \tag{4.14}$$

for all $f \in C^2(\mathbb{R})$ such that (4.13) holds.

Proof. According to Theorem 4.2, we may choose without loss of generality $X = Y(\varphi)$ where $\varphi(t) = \inf\{s \geq 0 : \tau(s) > t\}$. Itô formula yields

$$\begin{aligned} f(Y(t)) &= f(x) + \int_0^t f'(Y(u))\,\mathrm{d}Y(u) + \frac{1}{2}\int_0^t f''(Y(u))\,\mathrm{d}\langle Y\rangle(u) \\ &= f(x) + \int_0^t f'(Y(u))Y(u)\,\mathrm{d}B(u) + \frac{1}{2}\int_0^t f''(Y(u))Y^2(u)\,\mathrm{d}u. \end{aligned}$$

It follows by (4.13) that the middle integral is an $\mathcal{F}_t^B$-martingale and, obviously, $\varphi(t)$ is a bounded $\mathcal{F}_t^B$-Markov time ($\sigma \leq C \Rightarrow \varphi(t) \leq tC^2$). Hence, the Stopping Theorem implies that

$$\begin{aligned} Ef(X(t)) = EY(\varphi(t)) &= f(x) + \frac{1}{2}E\int_0^{\varphi(t)} f''(Y(u))Y^2(u)\,\mathrm{d}u \\ &= f(x) + \frac{1}{2}\int_0^\infty E[f''(Y(u))Y^2(u)I_{[0,t]}(\tau(u))]\,\mathrm{d}u \end{aligned}$$

is true for all $t \geq 0$. This is easily seen to be the equality stated by (4.14). □

Remark 4.6. If X is a $(0,\sigma)$-stock price, assuming (4.4) about $0 \le \sigma \le C < \infty$, then

(a) to establish a one-dimensional distribution $\mathcal{L}(X(t))$, we need only to know two-dimensional distribution $G_u = \mathcal{L}(Y(u), \tau(u))$ for $0 \le u \le C^2 t$.

(b) If $\tau(u)$ has an absolutely continuous distribution with a density $g(u,s)$ for each $u > 0$, then we may rewrite (4.14) to

$$\begin{aligned} Ef(X(t)) &= f(x) + \frac{1}{2}\int_0^\infty \int_0^t E[f''(Y(u))Y^2(u)|\tau(u) = s]g(u,s)\,\mathrm{d}s\,\mathrm{d}u \\ &= f(x) + \frac{1}{2}\int_0^t \int_0^\infty E[f''(Y(u))Y^2(u)|\tau(u) = s]g(u,s)\,\mathrm{d}u\,\mathrm{d}s. \end{aligned}$$

Hence, almost everywhere on $\mathbb{R}^+$

$$\frac{\mathrm{d}}{\mathrm{d}t} Ef(X(t)) = \frac{1}{2}\int_0^\infty E\left[f''(Y(u))Y^2(u)|\tau(u) = t\right] g(u,t)\,\mathrm{d}u \tag{4.15}$$

holds for arbitrary $f \in C^2(\mathbb{R})$ that satisfies (4.13). Choosing $f(y) = y^p$ for $p \ge 1$, we may apply Lemma 4.1 with $X = Y$ to verify (4.13) and turn (4.15) into the equality

$$\frac{\mathrm{d}}{\mathrm{d}t} EX(t)^p = \frac{p(p-1)}{2}\int_0^\infty E\left[Y(u)^p|\tau(u) = t\right] g(u,t)\,\mathrm{d}u$$

that is true again outside a λ-null set in $\mathbb{R}^+$. Hence, there is a chance, at least theoretical, to recover the Laplace transform of $\mathcal{L}(X(t))$.. Choosing $f(y) = e^{i\lambda y}$, where $\lambda \in \mathbb{R}$, we write (4.15) as

$$\frac{\mathrm{d}}{\mathrm{d}t} Ee^{i\lambda X(t)} = -\frac{\lambda^2}{2}\int_0^\infty E\left[e^{i\lambda Y(u)}Y^2(u)|\tau(u) = t\right] g(u,t)\,\mathrm{d}u \tag{4.16}$$

$$= \frac{\lambda^2}{2}\int_0^\infty \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} E\left[e^{i\lambda Y(u)}|\tau(u) = t\right] g(u,t)\,\mathrm{d}u \tag{4.16}$$

to receive another link between $\mathcal{L}(X(t))$ and $\mathcal{L}(Y(u))$'s in terms of characteristic functions. Our chances to establish $\mathcal{L}(X(t))$ are perhaps limited to simple choices of the volatility σ as in the following

Example 4.7. Consider a volatility $\sigma(x,t) = \tilde{\sigma}(x(t))$, where

$$\tilde{\sigma}(y) := \sigma_2 I_{(-\infty,x]}(y) + \sigma_1 I_{(x,\infty)}(y), \quad y \in \mathbb{R} \tag{4.17}$$

and $\sigma_1, \sigma_2 > 0$. Having an intention to model a stock market that is very sensitive to a decrease of the stock price we could perhaps choose a (b,σ) market (4.17) with $\sigma_2 \gg \sigma_1$. Compute

$$\tau(t) = \int_0^t \tilde{\sigma}^{-2}(Y(u))\,\mathrm{d}u = (\sigma_1^{-2} - \sigma_2^{-2})\lambda\left[s \le t, B(s) - \frac{s}{2} > 0\right] + \sigma_2^{-2} t. \tag{4.18}$$

To simplify our formulas, choose $\sigma_2^2 = 1, \sigma_1^2 = \frac{1}{2}$ to get

$$\tau(t) = \lambda_t^+(W) + t, \text{ where} \tag{4.19}$$

$$W(t) = B(t) - \frac{t}{2}, \quad \lambda_t^+(y) = \lambda[s \le t, y(s) > 0], \ t \ge 0, \ y \in C(\mathbb{R}^+). \tag{4.20}$$

Thus, $G_t := \mathcal{L}(Y(t), \tau(t)) = \mathcal{L}(xe^{W(t)}, \lambda_t^+(W) + t)$ is the probability distribution asked for by the formulas (4.14) or (4.15). The distribution G_t can be easily recovered from $H_t = \mathcal{L}(W(t), \lambda_t^+(W))$. Assuming that W is a Wiener process, then H_t is an absolutely continuous distribution with the density $h_t(y, \lambda) = t^{-3/2} h_1(y/\sqrt{t}, \lambda/t)$, where

$$h_1(y, \lambda) = \int_\lambda^1 \frac{|y| e^{-\frac{y^2}{2(1-a)}}}{2\pi[a(1-a)]^{3/2}} \, da \quad \text{if} \quad y \le 0, \ \ 0 < \lambda < 1 \tag{4.21}$$

and

$$h_1(y, \lambda) = \int_{1-\lambda}^1 \frac{|y| e^{-\frac{y^2}{2(1-a)}}}{2\pi[a(1-a)]^{3/2}} \, da \quad \text{if} \quad y > 0, \ \ 0 < \lambda < 1. \tag{4.22}$$

(See [2, pp. 97–100]) and we may apply Girsanov Theorem to establish H_t if W is a shifted Wiener process defined by (4.20). By $\mathrm{d}Q_t = \exp\{\frac{1}{2}B(t) - t/8\} \, \mathrm{d}P$ we define a probability measure such that W is a Wiener process on $[0, t]$ under Q_t. Check that $\mathrm{d}P = e^{-\frac{1}{2}W(t) - \frac{t}{8}} \, \mathrm{d}Q_t$ and compute

$$H_t(a, b) = P[W(t) \le a, \lambda_t^+(W) \le b] = e^{-\frac{t}{8}} \int_{[W(t) \le a, \lambda_t^+(W) \le b]} e^{-\frac{1}{2}W(t)} \, \mathrm{d}Q_t \tag{4.23}$$

$$= e^{-\frac{t}{8}} \int_{-\infty}^a \int_0^b e^{-\frac{1}{2}y} h_t(y, \lambda) \, \mathrm{d}y \, \mathrm{d}\lambda \tag{4.24}$$

holds for $t > 0, a \in \mathbb{R}$ and $b \ge 0$.

5. INVESTMENTS, OPTION PRICING AND DISCOUNTED STOCK PRICE

Fix coefficients $(b, \sigma) \in BS_M$, *a* (b, σ)*-stock price* X *detailed by* (2.5) and denote by $\Pi(\sigma)$ the set of all $\mathcal{F}_t^{\mu_\sigma}$-progressive processes p such that

$$\textstyle\int_0^t p^2(x, u) \, \mathrm{d}u < \infty, \quad t \ge 0 \ \ \mu_\sigma\text{-almost surely} \tag{5.1}$$

$$p(x, t) = 0 \ \ \mu_\sigma \otimes \lambda\text{-almost everywhere on } C(\mathbb{R}^+) \times (T, \infty) \tag{5.2}$$

hold. Consider a $(P, \mathcal{F}_t^X)$-semimartingale C defined by

$$\mathrm{d}C(t) = p(X, t) \, \mathrm{d}X(t) = p(X, t) X(t) \sigma(X, t) \, \mathrm{d}W_X(t), \quad C(0) = c \ge 0 \tag{5.3}$$

with a $p \in \Pi(\sigma)$ and agree to call it *an* $\mathbf{X}$*-investment process* with the initial endowment c while p will be referred to as its *portfolio*. *Note* that for a $p \in \Pi(\sigma)$

the process $p(X)$ is $\mathcal{F}_t^X$-progressive by (i) in 6.5 and that, according to (5.1), the integral $\int p(X)\,\mathrm{d}X$ is defined correctly because $Xb(X)$ and $X\sigma(X)$ are processes with locally bounded trajectories. *Also note* that (5.2) says that outside a P-null set the trajectories of C are constant on $[T,\infty)$. Remark finally that (5.3) defines a self-financing investment strategy C, since its infinitesimal profits $dC(t)$ are born entirely by the infinitesimal changes $\mathrm{d}X(t)$ and by the "number of shares" of the stock $p(X,t)$ owned by the investor at time t.

Simple characterization of X-investments is provided by

Lemma 5.1. A process $C=(C(t),t\geq 0)$ is an X-investment process iff it is a continuous $(P_X,\mathcal{F}_t^X)$-local martingale such that

$$C(0)=c\geq 0 \ \ P\text{-almost surely},\ \ C(t\wedge T)=C(t),\ \ t\geq 0\ P\text{-almost surely.} \tag{5.4}$$

Proof. Assume that C is a continuous $(P_X,\mathcal{F}_t^X)$-local martingale such that (5.4) holds. Recall Corollary 3.2 to prove that $(\Omega,\mathcal{F},P_X,\mathcal{F}_t^X,W_X,X)$ is a weak solution to (3.7) and that this equation is unique in law. Because X is a true P_X-martingale by Lemma 4.1, we may apply Lemma 6.8 to verify that

$$C(t)=c+\int_0^t H(u)\,\mathrm{d}X(u),\ \ t\geq 0\ \ P_X\text{-almost surely,} \tag{5.5}$$

where H is an $\mathcal{F}_t^X$-progressive process such that

$$H(0)\in\mathbb{R},\int_0^t H^2(u)X^2(u)\sigma^2(X,u)\,\mathrm{d}u<\infty,\ \ t\geq 0, P_X\text{-almost surely.} \tag{5.6}$$

Because X is a continuous and positive process, it follows by (5.6) that also

$$H(0)\in\mathbb{R},\ \ \int_0^t H^2(u)\,\mathrm{d}u<\infty,\ \ t\geq 0,\ \ P_X\text{-almost surely} \tag{5.7}$$

holds. Hence, Lemma 6.5 (ii) applies to construct an $\mathcal{F}_t^{\mu_\sigma}$-progressive process p such that

$$H=p(X) \qquad P_X\otimes\lambda\text{ -almost everywhere on }\Omega\times\mathbb{R}^+. \tag{5.8}$$

Now, (5.8) and (5.7) obviously imply (5.1). It is a consequence of (5.8) that outside a P-null set and for all $t\geq 0$

$$\int_0^t (H(u)-p(X,u))^2\,d\langle X\rangle(u)=0,\ \ \text{hence}\ \ \int_0^t H(u)\,\mathrm{d}X(u)=\int_0^t p(X,u)\,\mathrm{d}X(u)$$

holds. This and (5.5) imply (5.3) and we need only to verify (5.2):

Since, according to (5.4), the trajectories of C are constant on $[T,\infty)$ with P_X-probability one, we conclude that outside a P-null set and for all $t \geq T$

$$\int_T^t H\,\mathrm{d}X = 0 \quad \text{and therefore} \quad \int_T^t H^2\,\mathrm{d}\langle X\rangle = 0$$

are true equalities. Since $\mathrm{d}\langle X\rangle \sim \lambda$ with P_X-probability one, we prove that $H = 0$ $P \otimes \lambda$-almost everywhere on $\Omega \times (T,\infty]$, which jointly with (5.8) verifies (5.2). □

The correspondence between an X-investment process C and its portfolio control p is bijective given a fixed (b,σ)-price X. The probability distribution $\mathcal{L}(C)$ does not depend on the choice of (b,σ)-price X given a fixed portfolio p. More precisely:

Theorem 5.2. (i) Let X be a (b,σ)-price given as in (2.5), C_1 and C_2 X-investment processes controlled by portfolios p_1 and p_2, respectively, such that $C_1(0) = C_2(0)$. Then

$$C_1 = C_2 \ P\text{-almost surely iff } p_1 = p_2 \ \ P \otimes \lambda\text{-almost everywhere on } \Omega \times \mathbb{R}^+. \tag{5.9}$$

(ii) Let p be a portfolio in $\Pi(\sigma)$, X_1 and X_2 (b,σ)-prices given as in (2.7), C_1 and C_2 X_1 and X_2-investments processes controlled by p, respectively, such that $C_1(0) = C_2(0)$ holds. Then

$$\mathcal{L}(C_1, X_1|P^1) = \mathcal{L}(C_2, X_2|P^2), \quad \mathcal{L}(C_1, X_1|P^1_{X_1}) = \mathcal{L}(C_2, X_2|P^1_{X_2}) \tag{5.10}$$

hold.

Proof. As for (i), note that $C_1 = C_2$ P-almost surely iff $C_1 = C_2$ P_X-almost surely and iff

$$\int_0^t (p_1(X) - p_2(X))^2\,\mathrm{d}\langle X\rangle = 0,\ t \geq 0,\ \ P_X\text{-almost surely.} \tag{5.11}$$

As in the proof of Lemma 5.1, (5.11) is exactly as to say that $p_1 = p_2$ holds $P_X \otimes \lambda$-almost everywhere on $\Omega \times \mathbb{R}^+$. The equivalence (5.9) is verified since $P \otimes \lambda$ and $P_X \otimes \lambda$ are equivalent measures. To prove the first statement in (5.10) apply 6.6 with $X_i, N_i = X_i, c = p$ and $\mu = \mu_{b,\sigma} = \mathcal{L}(X_i)$ and note that $c = p$ satisfies the requirements of 6.6 since $\mathcal{F}_t^{\mu_{b,\sigma}} = \mathcal{F}_t^{\mu_\sigma}$. The latter equality in (5.10) follows by the former one as $(\Omega^i, \mathcal{F}^i, P^i_{X_i}, \mathcal{F}^i_t, W_{X_i}, X_i), i = 1,2$ are $(0,\sigma)$ stock prices by Corollary 3.2. □

Example 5.3. Consider a (0,1)-price $X = (\Omega, \mathcal{F}, P, \mathcal{F}_t^X, W, X)$, note that $X(t) = \exp\{W(t) - t/2\}$ and that $\mathcal{F}_t^X = \mathcal{F}_t^W$. Let p_1 and p_2 be $\mathcal{F}_t^{\mu_1}$-progressive processes defined for positive $x \in C(\mathbb{R}^+)$ as

$$p_1(x,t) = \frac{1}{x(t)} I_{[0,T]}(t), \quad p_2(x,t) = \operatorname{sign}\left[\ln(x(t)) + \frac{t}{2}\right] p_1(x,t)$$

and observe that both p_1 and p_2 are in $\Pi(1)$. If C_1 and C_2 are X-investments controlled by p_1 and p_2, respectively and $C_1(0) = C_2(0) = 0$, then $dC_1(t) = \mathrm{d}W(t)$ and $\mathrm{d}C_2(t) = \operatorname{sign}W(t)\,\mathrm{d}W(t)$ and therefore $\mathcal{L}(C_1) = \mathcal{L}(C_2) = \mathcal{L}(W)$ while $p_1 = p_2$ $\mu_1 \otimes \lambda$ almost everywhere is obviously a very false statement. Thus, (ii) in Theorem 5.2 cannot be simply reversed. Remark that we admit negative values of a controlling portfolio to include such interesting investment operations as short sales of the stock. On the other hand, to admit investment processes taking possibly negative values necessarily means to include arbitrage investments C with $C(0) = 0$ and $C(T) > 0$ (see 3.3.1 Example in [6]), hence operations that should be prohibited in a reasonable and safe market.

Denote by $A(X)$ the set of all X-investment process C that are nonnegative almost surely (admissible investments) and consider a financial claim w to be earned at time $t = T$ such that

$$w \geq 0 \text{ almost surely, and } w \text{ is a } \sigma_t(X)\text{-measurable random variable.} \tag{5.12}$$

Naturally, we are interested in X-investments C for which $C(T) = w$ holds almost surely and their initial endowements are as small as possible. Denote

$$A(X, w) := \{C \in A(X) : C(T) = w \text{ almost surely}\}$$

and modify the celebrated Karatzas–Merton–Shreve Theorem (see [9, 5.8.A section] or [6, 3.2.10]) to our case:

Theorem 5.4. If w in (5.12) is a P_X-integrable random variable, then

$$q_X := E_{P_X} w = \min[C(0) : C \in A(X, w)]. \tag{5.13}$$

There exists an almost surely unique $C_X \in A(X, w)$ with $C_X(0) = q_X$ and it is characterized inside of $A(X, w)$ by either of the following requirements:

(i) C_X is an $(\mathcal{F}_t^X, P_X)$-martingale.

(ii) $C_X \leq C$ almost surely for arbitrary $C \in A(X, w)$.

Remark that C_X is the only true martingale in $A(X, w)$, the other members of the set are nonegative true local martingales, in particular supermartingales.

Proof. Put $G(t) = E_{P_X}[w|\sigma_t(X)]$ for $t \geq 0$ to define $(\mathcal{F}_t^X, P_X)$ martingale[4] with the properties

$$G(0) = q_X, \quad G(T) = w, \quad G(t) \geq 0, \quad G(t) = G(t \wedge T) \text{ almost surely, } t \geq 0. \tag{5.14}$$

Since $X = (\Omega, \mathcal{F}, P_X, \mathcal{F}_t^X, W_X, X)$ is a weak solution to (3.7) which is an equation that is unique in law, since X is a P-martingale, Lemma 6.8 applies to prove that C can be modified to a continuous $(\mathcal{F}_t^X, P_X)$-martingale C_X. It follows from (5.14) by Lemma 5.1 that $C_X \in A(X, w)$ is an X-investment with $C_X(0) = q_X$ to be

[4] $(\mathcal{F}_t^X)$ is the augmentation of the $(\sigma_t(X))$.

true almost surely. If C is another process in $A(X,w)$, then it is an $(\mathcal{F}_t^X, P_X)$-supermartingale and therefore

$$C(t) \geq E_{P_X}[C(T)|\mathcal{F}_t^X] = E_{P_X}[w|\mathcal{F}_t^X] = C_X(t) \text{ almost surely}, \ t \leq T. \tag{5.15}$$

It follows that

$$C \geq C_X \text{ almost surely}, \ C \in A(X,w). \tag{5.16}$$

Hence (5.13) is proved.

If $C \in A(X,w)$ is a process such that $C(0) = q_X$ holds almost surely, then, according to (5.16), $C - C_X$ is an almost surely nonnegative P_X-supermatingale with $(C - C_X)(0) = 0$ almost surely and therefore $C = C_X$ almost surely.

If $C \in A(X,w)$ is a P_X-martingale, then $C - C_X \geq 0$ almost surely is also a P_X-martingale with $(C - C_X)(T) = 0$ almost surely, hence $C = C_X$ almost surely again and the proof is completed. □

The latter result encourages the following definition:

A $C(\mathbb{R}^+)$-progressive process g will be called a (b,σ)-*option* if

$$g \geq 0 \text{ and } q_{g,T} := \int g(x,T)\mu_\sigma(\mathrm{d}x) < \infty. \tag{5.17}$$

We have on mind functionals as

$$g(x,t) = (x(t) - K)^+, \ g(x,t) = \left(\frac{1}{t}\int_0^t X(u)\,\mathrm{d}u - K\right)^+, \ g(x,0) = 0$$

that generate, for example, the European call option and the Asian call option, respectively. Note that considering the financial claims

$$w_X := g(X,T), \ X \text{ a } (b,\sigma)\text{-stock price},$$

it follows by Corollary 3.2 that $\mathcal{L}(X|P_X) = \mu_\sigma$ for arbitrary (b,σ)-stock price X and therefore

$$q_{g,T} := \int g(x,T)\mu_\sigma(\mathrm{d}x) = E_{P_X} g(X,T) = \min[C(0) : C \in A(X, g(X,T))] \tag{5.18}$$

holds for all (b,σ)-prices X.

Thus, having a (b,σ)-option g, we define *the price of* g (at time T) as $q_{g,T}$ and note that the price does not depend on the rate of return b. Another invariants in a given (b,σ)-stock market enter our theory as follows: If X is a (b,σ)-stock price, Theorem 5.4 says that there is an almost surely unique X-investment process C_X such that

$$C_X(0) = q_{g,T}, \ C \geq 0, \ C(T) = g(X,T) \text{ hold almost surely.} \tag{5.19}$$

We call a $p \in \Pi(\sigma)$ a *hedging portfolio* against a (b,σ)-option g if

$$C_X(t) = q_{g,T} + \int_0^t p(X,u)\,\mathrm{d}X(u), \ t \geq 0 \text{ almost surely} \tag{5.20}$$

is true for all (b,σ)-prices X.

Theorem 5.5. Let g be a (b,σ)-option. Then there is a hedging portfolio p against the g. If X_1 and X_2 are (b,σ)-prices given as in (2.7), then

$$\mathcal{L}(C_{X_1}|P^1) = \mathcal{L}(C_{X_2}|P^2) \text{ and } \mathcal{L}(C_{X_1}|P^1_{X_1}) = \mathcal{L}(C_{X_2}|P^2_{X_2}). \tag{5.21}$$

Note that Example 5.3, to contrast Theorem 5.2 (i), shows that there exist more than one hedging portfolio p against g.

Proof. Assume that C_{X_1} is a process controlled as

$$C_{X_1}(t) = q_{g,T} + \int_0^t p(X_1,u)\,\mathrm{d}X_1(u),\ t \geq 0,$$

where $p \in \Pi(\sigma)$. Define an X_2-investment process C_2 by

$$C_2(t) := q_{g,T} + \int_0^t p(X_2,u)\,\mathrm{d}X_2(u)$$

and apply Theorem 5.2 (5.10) to prove that $\mathcal{L}(C_{X_1}, X_1|P^1_{X_1}) = \mathcal{L}(C_2, X_2|P^2_{X_2})$, hence $\mathcal{L}(C_{X_1}, X_1, g(X_1,T)) = \mathcal{L}(C_2, X_2, g(X_2,T))$. It follows that

$$C_2 \geq 0,\ \ C_2(0) = q_{g,T},\ \ C_2(T) = g(X_2,T)\ \text{-almost surely}$$

and therefore, by Theorem 5.4, C_2 equals to C_{X_2} outside a $P^2_{X_2}$-null set. We have proved that the p is a hedging portfolio against g and the second equality in (5.21), since $\mathcal{L}(C_2|P^2_{X_2}) = \mathcal{L}(C_{X_1}|P^1_{X_1})$. The first equality in (5.21) follows by Theorem 5.2 (ii) because both C_{X_1} and C_{X_2} are controlled by the same portfolio $p \in \Pi(\sigma)$. □

Remark 5.6. Up to now, we have considered only an investor who is interested only in a stock-market and ignores the parallel financial market driven by an interest rate $r(x,t)$, i.e. by a $C(\mathbb{R}^+)$-progressive process $r \in B$. A self-financing investment process is in this case defined by

$$\begin{aligned} R(X,t)C(t) &= c + \int_0^t p(X,u)X(u)(b(X,u) - r(X,u))\,\mathrm{d}u \\ &+ \int_0^t p(X,u)X(u)\sigma(X,u)\,\mathrm{d}W(u), \end{aligned}$$

where $(b,\sigma) \in BS_M$, X is a (b,σ)-price, p a portfolio in $\Pi(\sigma), c > 0$ and

$$R(x,t) := e^{-\int_0^t r(x,u)\,\mathrm{d}u},\ t \geq 0,\ x \in C(\mathbb{R}^+)$$

defines the discount factor born by the interest rate r. (See [6, 3.1.5]). The same reasoning, as we have performed in this section, applies that having a financial claim w in (5.12), we define its price q_X (see [6, 3.2.10]) by

$$q_X := E_{Q_X}[R(X,T)w],$$

where

$$\mathrm{d}Q_X = \exp\left\{-\int_0^T \frac{b(X,t)-r(X,t)}{\sigma(X,t)}\,\mathrm{d}W(t) - \frac{1}{2}\int_0^T \left(\frac{b(x,t)-r(X,t)}{\sigma(X,t)}\right)^2 \mathrm{d}t\right\}\mathrm{d}P.$$

Further assume that r is a *deterministic process* in B, denote by $Y = RX$ the discounted price X and compute that

$$\begin{aligned}\mathrm{d}Y(t) &= Y(t)(b(X,t)-r(X,t))\,\mathrm{d}t + Y(t)\sigma(X,t)\,\mathrm{d}W(t)\\ &= Y(t)(\overline{b}(Y,t)-\overline{r}(Y,t))\,\mathrm{d}t + Y(t)\overline{\sigma}(Y,t)\,\mathrm{d}W(t),\end{aligned}$$

where $\overline{p}(x,t) := p(R^{-1}x,t)$. It is obvious that $(\overline{b}-\overline{r},\overline{\sigma}) \in BS_M$ (provided that r is a deterministic process) and that Y is a $(\overline{b}-\overline{r},\overline{\sigma})$-stock price. Thus, $Q_X = P_Y$ and we compute that

$$q_X = e^{-\int_0^T r(u)\,\mathrm{d}u} E_{P_Y} w, \quad Y = R\cdot X.$$

If

$$w = (X(T)-K)^+ = e^{\int_0^T r(u)\,\mathrm{d}u}\left(Y(T) - Ke^{-\int_0^T r(u)\,\mathrm{d}u}\right)^+,$$

then

$$q_X = e^{\int_0^T r(u)\,\mathrm{d}u}\int_{C(\mathbb{R}+)}\left(y(T) - Ke^{-\int_0^T r(u)\,\mathrm{d}u}\right)^+ \mu_{\overline{\sigma}}(\mathrm{d}y).$$

We do not know yet whether the above duality between (r,b,σ)-stock/financial markets and (b,σ)-stock markets can be extended to a more general interest rates $r(x,t)$.

6. TECHNICALITIES

Recall and introduce our notations and definitions. If X is a process on $(\Omega,\mathcal{F},P)$, we write $\mathcal{N}_P = \{N \in \mathcal{F}, P(F) = 0\}$ and

$$\begin{aligned}&\sigma_t(X) = \sigma(X(s), s \le t), && \sigma_\infty(X) = \sigma(X(s), s < \infty),\\ &\mathcal{G}_t^X := \sigma(\sigma_t(X) \cup \mathcal{N}_P), && \mathcal{F}_t^X := \mathcal{G}_{t+}^X := \cap_{h>0}\mathcal{G}_{t+h}^X.\end{aligned}$$

Calling $(\sigma_t(X)), (\mathcal{G}_t^X)$ and $(\mathcal{F}_t^X)$ the canonical filtration of X, the P-completion of $(\sigma_t(X))$ and the P-augmentation of $(\sigma_t(X))$, respectively. If $\mathbb{X}$ is the canonical process on $C(\mathbb{R}^+)$ and μ a Borel probability measure on $C(\mathbb{R}^+)$, we write $(\mathcal{G}_t^\mu)$ and $(\mathcal{F}_t^\mu)$ for the μ-completion and μ-augmentation of $\sigma_t(\mathbb{X})$ in $(C(\mathbb{R}^+), \mathcal{B}^\mu, \mu)$, respectively if $\mathcal{B}^\mu = \mathcal{B}(C(\mathbb{R}^+))^\mu$ denotes the μ-completion of $\mathcal{B}(C(\mathbb{R}^+))$. We need to have cleared the relation between the filtrations $(\mathcal{G}_t^X)$ and $(\mathcal{F}_t^X)$. The following is obvious:

Lemma 6.1. Let X be a process on $(\Omega, \mathcal{F}, P)$ with $X(0) \in \mathbb{R}$. Then

(i) $\mathcal{G}_\infty^X = \mathcal{F}_\infty^X$, where $\mathcal{F}_\infty := \sigma(\cup_{t\geq 0}\mathcal{F}_t)$ if $(\mathcal{F}_t)$ is a filtration, though the inclusion $\mathcal{G}_0^X \subseteq \mathcal{G}_{0+}^X$ may be proper even for a continuous X.

(ii) If Y is a left continuous process, $Y(0) \in \mathbb{R}$, then Y is an $\mathcal{G}_t^X$-adapted process iff it is $\mathcal{F}_t^X$-adapted.

(iii) If G is an $\mathcal{F}_t^X$-progressive process with $G(0) \in \mathbb{R}$ such that

$$\int_0^t |G(u)|\, du < \infty \text{ almost surely, } t \geq 0, \tag{6.1}$$

then there is a $\mathcal{G}_t^X$-progressive process F such that $F = G$ $\lambda \otimes P$-almost everywhere. To prove (iii), combine (ii) and the following standard procedure.

(iv) If $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ is a complete filtration and G an $\mathcal{F}_t$-progressive process such that (6.1) holds, then $F_n(t) := G(0) + n\int_{(t-\frac{1}{n})^+}^t G(u)\,du$ defines a sequence of continuous $\mathcal{F}_t$-adapted processes such that $G = \lim_{n\to\infty} F_n$ $\lambda \otimes P$-almost everywhere.

More difficult is

Lemma 6.2. Let X be a process on $(\Omega, \mathcal{F}, P)$ with $X(0) \in \mathbb{R}$ such that any $\mathcal{G}_t^X$-martingale has a continuous modification. Then arbitrary $\mathcal{F}_t^X$-local martingale G with $G(0) \in \mathbb{R}$ has a continuous $\mathcal{G}_t^X$-adapted modification.

P r o o f. Assume without loss of generality that G is an $\mathcal{F}_t^X$-martingale. Since $(\mathcal{F}_t^X)$ is a right-continuous and complete filtration, it follows by Doob Regularization Theorem ([12, 67.7, p. 173]) that G has a modification with RCLL-trajectories (right-continuous with finite left limits). Assume without loss of generality that the G itself is a process with RCLL, hence locally bounded trajectories. It follows by (iii) in 6.1 that $G = F$ $\lambda \otimes P$-almost everywhere, F being an $\mathcal{G}_t^X$-progressive process. In particular,

$$F(t) = G(t) \;\; P\text{-almost surely for } t \in D \text{ and } \overline{D} = \mathbb{R}^+. \tag{6.2}$$

Fix $0 < T \in D$ and put $F'(t) = E^{\mathcal{G}_t^X} F(T)$ for $t \geq 0$ and note that F' is a $\mathcal{G}_t^X$-martingale such that (6.2) yields

$$F'(t) = E^{\mathcal{G}_t^X} G(T) = E^{\mathcal{G}_t^X} E^{\mathcal{F}_t^X} G(T) = E^{\mathcal{G}_t^X} G(t) = E^{\mathcal{G}_t^X} F(t) = F(t) \tag{6.3}$$

P-almost surely for $t \leq T$ and $t \in D$. Our hypothesis on $(\mathcal{G}_t^X)$ says that F' has a continuous $\mathcal{G}_t^X$-adapted modification F'' such that, according to (6.3),

$$F''(t) = F'(t) = F(t) = G(t) \;\; P\text{-almost surely, } t \leq T,\; t \in D$$

holds. Hence P-almost surely

$$F''(t) = \lim_{n\to\infty} F''(t_n) = \lim_{n\to\infty} G(t_n) = G(t) \quad \text{if} \quad t_n \leq T,\;\; t_n \in D,\;\; t_n \downarrow t$$

and therefore F'' is a continuous $\mathcal{G}_t^X$-adapted modification of G on $[0,T)$. Letting $T \uparrow \infty$, we conclude the proof. □

In what follows, we shall need the following slight improvement of Lemma 1.25 in [8, p. 13]. (The σ-algebra $\mathcal{G}_\infty^X$ is not exactly the standard P-completion of the σ-algebra $\sigma_\infty(X)$):

Lemma 6.3. If X is a process on $(\Omega, \mathcal{F}, P)$ and S Polish space, then

$$H : (\Omega, \mathcal{G}_\infty^X) \to (S, \mathcal{B}(S)) \text{ is a measurable map}$$

if and only if there is a measurable map $H' : (\Omega, \sigma_\infty(X)) \to (S, \mathcal{B}(S))$ such that $H = H'$ P-almost surely.

Just observe that

$$\mathcal{G}_\infty^X = \{F \subseteq \Omega, F \triangle B \in \mathcal{N}_P \text{ for some } B \in \sigma_\infty(X)\} \tag{6.4}$$

and repeat the proof of 12.25 in [8] via $S = [0,1]$ and the Borel isomorphism theorem ([4, 8.3.6]).

Lemma 6.4. Let X and Y be continuous processes on an $(\Omega, \mathcal{F}, P)$ with $X(0), Y(0) \in \mathbb{R}$. Denote by μ the completed Borel probability $\mathcal{L}(X)$. Then

(i) $\mathcal{F}_\infty^Y \subseteq \mathcal{F}_\infty^X$ iff $Y = g(X)$ P-almost surely, $g : C(\mathbb{R}^+) \to C(\mathbb{R}^+)$ Borel.

(ii) If $g : C(\mathbb{R}^+) \to C(\mathbb{R}^+)$ is a Borel transformation such that $Y = g(X)$ holds P-almost surely, then g is a continuous $\mathcal{F}_t^\mu$-adapted process provided that Y is an $\mathcal{F}_t^X$-adapted process.

Proof. Since $\mathcal{G}_\infty^X = \mathcal{F}_\infty^X$ and $\mathcal{G}_\infty^Y = \mathcal{F}_\infty^Y$ by (i) in Lemma 6.1, Lemma 6.3 applies to prove ($\Leftarrow$) in (i) directly. If $\mathcal{F}_\infty^Y \subseteq \mathcal{F}_\infty^X$, then, again by Lemma 6.1 (i) and Lemma 6.3, there is

$$Y' : (\Omega, \sigma_\infty(X)) \to (C(\mathbb{R}^+), \mathcal{B}(C(\mathbb{R}^+))) \text{ such that } Y' = Y$$

holds almost surely. Lemma 1.13 in [8, p. 7] exhibits a Borel $g : C(\mathbb{R}^+) \to C(\mathbb{R}^+)$ such that $Y' = g(X)$ everywhere on Ω.

If Y is an $\mathcal{F}_t^X$-adapted process, it follows by (ii) in 6.1 that

$$p_t \circ Y : (\Omega, \mathcal{G}_t^X) \to (C[0,t], \mathcal{B}(C[0,t])) \text{ for any } t \geq 0,$$

where $p_t : C(\mathbb{R}^+) \to C[0,t]$ denotes the projection. Fix $t \geq 0$ and apply our Lemma 6.3 and Lemma 1.13 in [8] again (this time for Polish space $S = C[0,t]$) to exhibit a Borel transformation g_t of $C[0,t]$ such that $p_t \circ Y = g_t(p_t \circ X)$ holds almost surely. If $Y = g(X)$ almost surely and $g : C(\mathbb{R}^+) \to C(\mathbb{R}^+)$ is a Borel map, we conclude that $(p_t \circ g)(X) = (g_t \circ p_t)(X)$ outside a P-null set and $p_t \circ g = g_t \circ p_t$ outside a μ-null set. Since

$$(C(\mathbb{R}^+), \sigma_t(\mathbb{X})) \xrightarrow{p_t} (C[0,t], \mathcal{B}(C[0,t])) \xrightarrow{g_t} (C[0,t], \mathcal{B}(C[0,t])),$$

we prove that

$$p_t \circ g : (C(\mathbb{R}^+), \mathcal{G}_t^\mu) \to (C[0,t], \mathcal{B}(C[0,t]))$$

by Lemma 6.3. This is exactly as to say that g is a $\mathcal{G}_t^\mu$ and therefore $\mathcal{F}_t^\mu$-adapted process. □

Lemma 6.5. Let X be a continuous process on $(\Omega, \mathcal{F}, P)$ with $X(0) \in \mathbb{R}$ and μ the completed probability distribution $\mathcal{L}(X)$.

(i) If g is an $\mathcal{F}_t^\mu$-progressive process, then $g(X)$ is an $\mathcal{F}_t^X$-progressive process.

(ii) If G is an $\mathcal{F}_t^X$-progressive process with $G(0) \in \mathbb{R}$ such that (6.1) holds, then there is an $\mathcal{F}_t^\mu$-progressive process g such that $G = g(X)$ $\lambda \otimes P$-almost everywhere.

For (i) we do not assume that $X(0) \in \mathbb{R}$.

Proof. (i) Fix $t \geq 0$ and note that $X : (\Omega, \sigma_t(X)) \to (C(\mathbb{R}^+), \sigma_t(\mathbb{X}))$ implies that $X : (\Omega, \mathcal{G}_t^X) \to (C(\mathbb{R}^+), \mathcal{G}_t^\mu)$ and finally that $X : (\Omega, \mathcal{F}_t^X) \to (C(\mathbb{R}^+), \mathcal{F}_t^\mu)$. Putting $T(\omega, s) = (X(\omega), s)$, the latter measurability proves that

$$T : (\Omega \times [0,t], \mathcal{F}_t^X \otimes \mathcal{B}[0,t]) \to (C(\mathbb{R}^+) \times [0,t], \mathcal{F}_t^\mu \otimes \mathcal{B}[0,t]),$$

while the $\mathcal{F}_t^\mu$-progressivity of g yields

$$g : (C(\mathbb{R}^+) \times [0,t], \mathcal{F}_t^\mu \otimes \mathcal{B}[0,t]) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})).$$

Thus,

$$g(X) = g(T) : (\Omega \times [0,t], \mathcal{F}_t^X \otimes \mathcal{B}[0,t]) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

holds for all t and $g(X)$ is an $\mathcal{F}_t^X$-progressive process.

(ii) According to (iv) in Lemma 6.1 $G = \lim_{n\to\infty} F_n$ $\lambda \otimes P$-almost everywhere, where F_n's are continuous $\mathcal{F}_t^X$-adapted processes with $F_n(0) \in \mathbb{R}$. Apply both (i) and (ii) in Lemma 6.4 to exhibit continuous $\mathcal{F}_t^\mu$-adapted processes g_n such that $F_n = g_n(X)$ holds $\lambda \otimes P$-almost everywhere for arbitrary $n \in \mathbb{N}$. Putting $g := \overline{\lim}_{n\to\infty} g_n$ if the $\overline{\lim}$ is finite and $g := 0$ if not, we define an $\mathcal{F}_t^\mu$-progressive process such that $G = g(X)$ $\lambda \otimes P$-almost everywhere. □

Lemma 6.6. For $i = 1, 2$ consider a continuous semimartingale

$$N_i = (\Omega^i, \mathcal{F}^i, P^i, \mathcal{F}_t^i, N_i), \quad \mathrm{d}N_i = \mathrm{d}B_i + \mathrm{d}M_i \quad \text{with} \quad \mathrm{d}\|B_i\| + \mathrm{d}\langle M\rangle \ll \lambda$$

P^i-almost surely and a continuous $\mathcal{F}_t^i$-adapted process X_i such that

$$\mathcal{L}(N_1, X_1 | P^1) = \mathcal{L}(N_2, X_2 | P^2). \tag{6.5}$$

Let μ be the completed distribution $\mathcal{L}(X_1|P^1) = \mathcal{L}(X_2|P^2)$. Then

$$\mathcal{L}\left(X_1, \int c(X_1)\,\mathrm{d}N_1 \middle| P^1\right) = \mathcal{L}\left(X_2, \int c(X_2)\,\mathrm{d}N_2 \middle| P^2\right) \tag{6.6}$$

for any $\mathcal{F}_t^\mu$-progressive process c such that

$$\int_0^t |c(X_i)| + |c(X_i)|^2 (d||B_i|| + \mathrm{d}\langle M\rangle) < \infty \quad P^i\text{-almost surely, } t \geq 0, \quad i = 1,2 \tag{6.7}$$

hold. By $||B_i||$ we mean the variation of B_i.

Note that $c(X_i)$ is an $\mathcal{F}_t^{X_i}$-progressive, hence $\mathcal{F}_t^i$-progressive process by (i) in 6.5 and that (6.7) guarantees that the stochastic integral $\int c(X_i)\,\mathrm{d}N_i$ in (6.6) is defined correctly.

Proof. By $\mathcal{C}$ denote the set of all $\mathcal{F}_t^\mu$-progressive processes c with the property (6.7) and by $\mathcal{C}_0$ the set of all $c \in \mathcal{C}$ such that (6.6) is true. Observe that $\mathcal{C}_0$ is a set that is closed in $\mathcal{C}$ with respect to the convergence on $\mathcal{C}$ defined by

$$c_n \to c \ \equiv \ c_n \to c \ \text{ almost everywhere, } |c_n| \leq d \in \mathcal{C}. \tag{6.8}$$

Indeed, if (6.8) is assumed for a sequence $c_n \in \mathcal{C}_0$ and $c \in \mathcal{C}$, then $\mathrm{d}||B_i|| \ll \lambda$, $\mathrm{d}\langle M\rangle \ll \lambda$ almost surely implies that outside a P^i-null set

$$c_n(X_i) \to c(X_i) \quad \mathrm{d}||B_i|| + \mathrm{d}\langle M\rangle\text{-almost everywhere on } \mathbb{R}^+, \ i = 1,2$$

and therefore, by the Dominated Convergence Theorem,

$$\int_0^t |c_n(X_i) - c(X_i)|\,\mathrm{d}||B_i|| \to 0, \quad \int_0^t |c_n(X_i) - c(X_i)|^2\,\mathrm{d}\langle M_i\rangle \to 0, \quad t \geq 0, i = 1,2$$

is also true outside a P^i-null set. Hence, according to 2.1.12 in [6, part III]

$$\max_{s\leq t}\left|\int_0^s c_n(X_i)\,\mathrm{d}N_i - \int_0^s c(X_i)\,\mathrm{d}N_i\right| \to 0 \text{ in } P^i\text{-probability, } t \geq 0, \quad i = 1,2,$$

which is as to say that $\int c_n(X_i)\,\mathrm{d}N^i \to \int c(X_i)\,\mathrm{d}N^i$ in P^i-probability as $C(\mathbb{R}^+)$-random variables. Since

$$\mathcal{L}\left(X_1, \int c_n(X_1)\,\mathrm{d}N_1 \middle| P^1\right) = \mathcal{L}\left(X_2, \int c_n(X_2)\,\mathrm{d}N_2 \middle| P^2\right), \quad n \in \mathbb{N},$$

we conclude that $c \in \mathcal{C}_0$.

Now, if c is a continuous $\mathcal{F}_t^\mu$-adapted process, then it belongs to $\mathcal{C}$ and it belongs to $\mathcal{C}_0$ because for arbitrary $t \geq 0$

$$\sum_{k=0}^{n-1} c\left(X_i, \frac{kt}{n}\right)\left(N_i\left(\frac{(k+1)t}{n}\right) - N_i\left(\frac{kt}{n}\right)\right) \to \int_0^t c(X_i)\,\mathrm{d}N_i$$

in P^i-probability for $i = 1,2$. If c is a bounded $\mathcal{F}_t^\mu$-progressive process, then it is in $\mathcal{C}$ and it is in $\mathcal{C}_0$ because Lemma 6.1 (iv) provides a sequence c_n of continuous $\mathcal{F}_t^\mu$-adapted processes that converge to c in sense of (6.8). Since bounded $\mathcal{F}_t^\mu$-adapted processes are easily seen to be a dense set in $\mathcal{C}$ w.r.t. the convergence (6.8), the proof is complete. □

Lemma 6.7. Let X in (2.5) be a weak solution to the Engelbert–Schmidt equation (2.8) that is unique in law and Q a probability measure on $\sigma_\infty(X)$. Then $Q = P$ on $\sigma_\infty(X)$ provided that X is an $(\Omega, \sigma_\infty(X), Q, \sigma_t(X))$-local martingale and $Q \sim P$ on $\sigma_\infty(X)$.

Proof. X is also $(\Omega, \sigma_\infty(X), Q, \mathcal{G}_t^X)$-local martingale with

$$\langle X\rangle(t) = \int_0^t S^2(X,u)\,\mathrm{d}u, \quad t \geq 0, \quad Q\text{-almost surely.}$$

Hence, $\mathcal{L}(X|Q)$ is a solution to the local martingale problem for $(S^2, 0)$. By Stroock–Varadhan Theorem (18.7 in [8, p. 341]), we get that there is a weak solution $X' = (\Omega', \mathcal{F}', P', \mathcal{F}_t', W', X')$ to (2.8) such that $\mathcal{L}(X'|P') = \mathcal{L}(X|Q)$. It follows that $\mathcal{L}(X|P) = \mathcal{L}(X|Q)$, hence $Q = P$ on $\sigma_\infty(X)$. □

Lemma 6.8. Let X in (2.5) be a weak solution to the equation (2.8) that is unique in law. Assume, moreover, that X is a (true) P-martingale. Then arbitrary $(P, \mathcal{F}_t^X)$-local martingale G with $G(0) = y \in \mathbb{R}$ is stochastic integral

$$G(t) = y + \int_0^t H(u)\,\mathrm{d}X(u), \quad P\text{-almost surely}, \ t \geq 0, \tag{6.9}$$

where

$$H \text{ is an } \mathcal{F}_t^X\text{-progressive process, and } H(0) \in \mathbb{R}, \int_0^t H^2(u)S^2(X,u)\,\mathrm{d}u < \infty \tag{6.10}$$

holds P-almost surely for all $t \geq 0$.

In particular, any $(P, \mathcal{F}_t^X)$-local martingale has a continuous modification.

Note that the stochastic integral in (6.9) is well defined, since X is a continuous $(P, \mathcal{F}_t^X)$-martingale and

$$\int_0^t H^2(u)\,d\langle X\rangle(u) = \int_0^t H^2(u)S^2(X,u)\,\mathrm{d}u < \infty$$

holds P-almost surely for all $t \geq 0$.

Proof. Denote by $\mathrm{MM}(X)$ the set of all probability measures Q defined on $\sigma_\infty(X)$ such that X is an $(\Omega, \sigma_\infty(X), Q)$-martingale and apply Lemma 6.7 to verify that

$$P|\sigma_\infty(X) \text{ is an extremal point of the convex set } \mathrm{MM}(X). \tag{6.11}$$

According to Yor Theorem (in the form of 2.5.7 in [6, part III]) (6.11) implies that any $(P, \mathcal{G}_t^X)$-martingale has a continuous modification. Thus, Lemma 6.2 applies to prove that G can be modified to a continuous $(P, \mathcal{G}_t^X)$-local martingale denoted again by G. According to Yor Theorem again, (this time in the form of 2.5.12 in [6, part III]) applying (6.11) once more, there exists a process H with the properties (6.10) such that (6.9) holds. □

7. PROBLEMS

Some of the problems listed bellow will be treated in Part II of the present paper also published in this issue.

(A) In connection with Corollary 3.3, one should find some other volatilities σ (outside of the Lipschitz and diffusion case (3.9) and (3.10), respectively) such that the Engelbert–Schmidt equation (3.7) has a weak solution and it is unique in law.

(B) Theorems 4.2 and 4.3 exhibit $(0,\sigma)$-price X as a transformed exponential of the Wiener process if σ is a diffusion volatility. In case that X is a homogenous Markov process one should try to establish its Kolmogorov equation, perhaps by Volkonski method (See, III.21 in [12]).

(C) Corollary 4.5 says how to compute $Ef(X(t))$ if X is $(0,\sigma)$-price, σ a diffusion volatility and $f \in C^2$. This, unfortunately, does not cover even the standard cases in the financial mathematics. The continuation of the present paper will offer a method of computing $E(X(t)-K)^+$ for some simple volatilities σ as in Example 4.7.

(D) If σ is a two valued volatility as in Example 4.7, we are able to establish one dimensional distributions $\mathcal{L}(X(t))$ in a complicated but an explicite form and therefore also $E(X(t)-K)^+$. Multivariate volatilities, not symmetric about the initial price X would, of course, provide a more realistic model for a $(0,\sigma)$-stock price.

(E) Another method how to compute the quantities as $E(X(T)-K)^+$ is that one very succesful in the standard Black–Scholes model with a constant σ : Consider a continuous volatility coefficient $\sigma(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ constructed by means of a function $F(x,t) \in C^{21}(\mathbb{R} \times \mathbb{R}^+)$

$$\frac{\partial F}{\partial t}(x,t) = -\frac{1}{2}x^2\sigma^2(x,t)\frac{\partial^2 F}{\partial x^2}(x,t), \quad F(x,T) = (x-K)^+$$

for $(x,t) \in \mathbb{R}^+ \times [0,T)$ such that the corresponding Engelbert–Schmidt equation (3.7) has a weak solution X and it is unique in law. If

$$\max_{t\le T} |F(x,t)| \le c(1+|x|^p) \text{ for some } p \ge 1,$$

then, by Itô formula, $(F(X(t)), t \ge 0)$ is an $\mathcal{F}_t^X$-martingale and therefore $E(X(T)-K)^+ = F(x,0)$ if $X(0)=x$. See 3.3.9 in [6], or more generally, the PDE stochastic representation theory presented, for example in [9, 6.7 section] or in [16, Chapter 15].

(F) What are the properties of $\mu_\sigma = \mathcal{L}(X)$, where X is a $(0,\sigma)$-price with a diffusion coefficient σ, what properties of σ would imply that $\mu_\sigma \sim \mu_1$ or that $\mathcal{L}(X(t)) \sim \mathcal{L}(Y(t))$, where Y is the exponential of a Wiener process? This seems to be important when trying to apply the formulas as those given by Remark 4.6.

(G) What happens in Example 4.7, putting $\sigma_1 = 1$ and letting $\sigma_2 = n \to \infty$, denoting the corresponding two-valued volatility by σ_n and by X_n the $(0,\sigma_n)$-stock price. We believe that

$$X_n \to Z \text{ in distribution in } C(\mathbb{R}^+) \text{ where } Z(t) = x\exp\{|W(t) - t/2|\}.$$

(H) Remark 5.6 suggests the following problem: Which $C(\mathbb{R}^+)$-progressive processes $r(x,t)$ are such that the map $x \in C(\mathbb{R}^+) \mapsto R(x)x \in C(\mathbb{R}^+)$ is μ_σ-almost surely invertible if $\sigma \in S_M$ and $R(x,t) = \exp\{-\int_0^t r(x,u)\,du\}$.

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