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ON THE REGULATOR PROBLEM FOR A CLASS OF LTI SYSTEMS WITH DELAYS[1]

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This paper deals with the problem of tracking a reference signal while maintaining the stability of the closed loop system for linear time invariant systems with delays in the states. We show that conditions for the existence of a solution to this problem (the so-called regulation problem), similar to those known for the case of delay-free linear systems, may be given. We propose a solution for both the state and error feedback regulation.

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1. INTRODUCTION

Tracking a reference signal is a typical problem in control theory. In particular, for time invariant systems, the problem of designing a controller in such a way that a plant follows a reference signal as near as possible while maintaining the stability of the plant-controller system, the so-called *regulation problem*, has been addressed either in the linear case (see, for example [4], or [13], in which a complete solution has been given) and in the nonlinear setting [8]. However, for dynamical systems that include time delays, the regulation problem has been only partially addressed. Typical examples of such systems are economic fluctuations, mass transportation, mixing, etc. Time delays appear also when the time between measurements, computation of the control strategy and application of the input signals are significant.

Several works dealing with systems with delays can be found in the literature, for example [1, 11] for the problem of optimal control and [9, 12] for the problem of stabilization. [3] has addressed the problem of tracking a polynomial signal for a linear system with delays, while [14] presents a method for designing a hybrid structure for a class of linear delay systems. More recently, [10] deals with robust tracking and model following controller for a class of uncertain time-delay systems.

In this paper, we study the regulation problem of linear time invariant (LTI) systems with delays in the states for both state and error feedback. Using similar arguments to those given in the delay-free linear case, we show that under additional

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assumptions, it is possible to find conditions under which the respective regulation problem for the system with delays has a solution.

The paper is organized as follows. In Section 2, we state the regulation problem for systems with delays for both the state and error feedback. In Section 3 we derive conditions for the solvability of the state feedback regulation problem and in Section 4 we present the same for the error feedback regulation problem. In Section 5 an illustrative example show the performance of the proposed scheme. Finally, some conclusions are presented in Section 6. In the Appendix we have some results about the solvability of linear matrix equations that we use in the proofs of some theorems.

2. THE REGULATION PROBLEM FOR A CLASS OF LTI SYSTEMS WITH DELAYS

Let us consider a linear time invariant system with delays subject to an external disturbance signal, described by

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^{N} A_i x(t-\tau_i) + Bu(t) + Pw(t) & (1)\\ \dot{w}(t) &= Sw(t) & (2)\\ e(t) &= Cx(t) - Rw(t) & (3)\\ x(t_0+s) &= \varphi(s) & (4) \end{aligned}$$

where $x \in I\!R^n$, $u \in I\!R^m$, $e \in I\!R^p$, $w \in I\!R^q$ and τ_i are real positive time delays, with $0 = \tau_0 < \tau_1 < \ldots < \tau_N$; and the initial condition $\varphi(s)$ is a continuous function defined on $[-\tau_N, 0]$. Equation (2) describes an external reference and/or disturbance signal generator, while equation (3) describes the output tracking error signal between the system output and a reference signal.

The problem we face here is designing a feedback controller to guarantee the tracking performance. The structure of this controller depends of the amount of information available from the system. The most convenient situation is when we have all the information available, namely, the complete states $x(t)$ and $w(t)$ of the system and the exosystem respectively. This is the case of a *state feedback regulator*. However, normally only a partial set of state variables is available, usually in the form of outputs $y(t)$ of the system and some reference signals. This is defined as the *error feedback regulator problem.*

More formally, the **State Feedback Regulation Problem with Delays (SFRPD)** can be stated as the problem of finding, if possible, a feedback

$$u(t) = \sum_{i=0}^{N} K_i x(t-\tau_i) + Tw(t) \qquad (5)$$

such that:

$\mathbf{S}_{SD}$. The equilibrium point $x(t) = 0$ of the closed loop system without perturbations

$$\dot{x}(t) = (A_0+BK_0)x(t)+(A_1+BK_1)x(t-\tau_1)+\cdots+(A_N+BK_N)x(t-\tau_N) \quad (6)$$

is asymptotically stable. That means, for any initial condition $\varphi : [t_0 - \tau_N, t_0] \to I\!R^n$ the corresponding solution $x(t)$ of (6), satisfies $x(t) \to 0$ as $t \to \infty$.

$\mathbf{R}_{SD}$. $\lim_{t\to\infty} e(t) = 0$.

In the same way, the **Error Feedback Regulation Problem with Delays (EFRPD)** can be then formulated as the problem of finding a dynamical controller

$$\begin{aligned} \dot{\xi}(t) &= \sum_{i=0}^{N} \tilde{F}_i\xi(t-\tau_i) + \sum_{i=0}^{N} L_i[Cx(t-\tau_i) - Rw(t-\tau_i)] \\ u(t) &= \sum_{i=0}^{N} \tilde{K}_i\xi(t-\tau_i) \end{aligned} \quad (7)$$

such that:

$\mathbf{S}_{ED}$. The equilibrium point of the composed system

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^{N} A_i x(t-\tau_i) + \sum_{i=0}^{N} B\tilde{K}_i\xi(t-\tau_i) \\ \dot{\xi}(t) &= \sum_{i=0}^{N} \tilde{F}_i\xi(t-\tau_i) + \sum_{i=0}^{N} L_iCx(t-\tau_i) \end{aligned}$$

is asymptotically stable.

$\mathbf{R}_{ED}$. For any initial condition $(x(0), \xi(0), w(0))$ the solution $(x(t), \xi(t), w(t))$ of the closed loop system (1), (3), (7) is such that

$$\lim_{t\to\infty} e(t) = 0.$$

3. THE STATE FEEDBACK REGULATOR PROBLEM

To propose a solution to this problem, the following instrumental assumptions are considered:

H1. All the eigenvalues of S lie on the closed right side of the complex plane.

H2. There exist matrices $K_0, \cdots, K_N$ and symmetric matrices $Q > 0$ and M associated with the Lyapunov equation

$$(A_0 + BK_0)^T Q + Q(A_0 + BK_0) + (N+1)M = 0$$

satisfying

$$M > 0$$
$$M - \sum_{i=1}^{N} Q(A_i + BK_i)M^{-1}(A_i + BK_i)^T Q > 0.$$

It can be shown that if assumption H2 holds, then the equilibrium point of system (6) is asymptotically stable [15]. This property will be used in the following to design the controller.

Remark 1. It should be noted that Assumption H2 implies that the stabilization properties do not depend on the delay values. This condition, which can be also expressed in terms of equivalent LMI's as in [2], could be in some sense conservative, since in many systems the magnitude of the delay may influence directly the stability properties.

We state first the following result.

Lemma 2. Assume that H1 and H2 hold. Then, for $u(t)$ given by (5), condition R_{SD}) is satisfied if there exists a solution X to the matrix equations

$$XS = \sum_{i=0}^{N}(A_i + BK_i)Xe^{-\tau_i S} + P_c \tag{8}$$

$$0 = CX - R, \tag{9}$$

where $P_c = BT + P$.

Proof. Let us consider the function $\hat{x}(t) = x(t) - Xw(t)$. Thus

$$\begin{aligned}\dot{\hat{x}}(t) &= (A_0 + BK_0)(\hat{x}(t) + Xw(t)) + (A_1 + BK_1)(\hat{x}(t-\tau_1) + Xw(t-\tau_1)) \\ &\quad + \cdots + (A_N + BK_N)(\hat{x}(t-\tau_N) + Xw(t-\tau_N)) \\ &\quad + BTw(t) + Pw(t) - XSw(t)\end{aligned}$$

Since $\dot{w} = Sw$, we obtain

$$w(t-\alpha) = e^{-\alpha S}w(t). \tag{10}$$

Rearranging the terms, we have that

$$\dot{\hat{x}}(t) = \sum_{i=0}^{N}(A_i + BK_i)\hat{x}(t-\tau_i) + \left(\sum_{i=0}^{N}(A_i + BK_i)Xe^{-\tau_i S} + P + BT - XS\right)w.$$

Since $P_c = BT + P$ then, by (8), we get

$$\dot{\hat{x}}(t) = \sum_{i=0}^{N}(A_i + B_0K_i)\hat{x}(t-\tau_i).$$

Thus, from H2 it follows that $\hat{x}(t) \to 0$ as $t \to \infty$.

For the output tracking error, we thus have now that

$$e(t) = C\hat{x}(t) + (CX - R)w$$

and since $CX - R = 0$, S_{RD} is satisfied, i.e., $\lim_{t\to\infty} e(t) = 0$. □

As in the delay-free case, based on Lemma 2, we may now state sufficient conditions for the existence of a solution of the SFRPD.

Theorem 3. Assume H1, H2 hold. Then, the SFRPD has a solution if the matrix equations

$$\Pi S = \sum_{i=0}^{N} A_i \Pi e^{-\tau_i S} + B\Gamma + P \tag{11}$$

$$0 = C\Pi - R \tag{12}$$

have a solution Π, Γ.

Proof. The proof can be carried out along the following steps:

Step 1. Find, if possible, matrices $K_0, \cdots, K_N$ such that condition H2 holds.

Step 2. Set $T = \Gamma - \sum_{i=0}^{N} K_i \Pi e^{-\tau_i S}$.
Therefore

$$\begin{aligned} \Pi S &= \sum_{i=0}^{N} A_i \Pi e^{-\tau_i S} + B\left(T + \sum_{i=0}^{N} K_i \Pi e^{-\tau_i S}\right) + P \\ 0 &= C\Pi - R, \end{aligned}$$

from where

$$\Pi S = \sum_{i=0}^{N} (A_i + BK_i)\Pi e^{-\tau_i S} + BT + P$$

or equivalently

$$\Pi S = \sum_{i=0}^{N} (A_i + B_0 K_i)\Pi e^{-\tau_i S} + P_c.$$

By the previous equation and equation (12), the theorem follows from Lemma 2 if we define $X = \Pi$. □

Remark 4. Equations (11) – (12) express the existence of a subspace $x = \Pi w$ that is rendered invariant by $u = \Gamma w$, on which the output tracking error is zeroed. In fact, we have that

$$\dot{x}(t) = \Pi S w(t) = \sum_{i=0}^{N} A_i \Pi w(t - \tau_i) + B\Gamma w(t) + P w(t) \tag{13}$$

$$0 = C\Pi w(t) - R w(t). \tag{14}$$

Using (10), (13) – (14) implies, for all $w(t)$,

$$\begin{aligned} \Pi S w &= \left(\sum_{i=0}^{N} A_i \Pi e^{-\tau_i S} + B\Gamma + P\right) w \\ 0 &= (C\Pi - R)w. \end{aligned}$$

A necessary and sufficient condition for the existence of universal solvability of equations (11) – (12) is given by the following result.

Theorem 5. Equations (11) – (12) are universally solvable, i. e., have a solution Π and Γ for every P and R given, if and only if

$$\operatorname{rank}\begin{bmatrix} \sum_{i=0}^{N} A_i e^{-\tau_i\lambda} - \lambda I & B \\ C & 0 \end{bmatrix} = n+p \qquad \forall \lambda \in \sigma(S). \tag{15}$$

Proof. In order to prove this theorem we use Theorem 10 of the Appendix. Equations (11) – (12) can be written as

$$\begin{aligned} \begin{pmatrix} -P \\ R \end{pmatrix} &= \begin{pmatrix} A_0 & B \\ C & 0 \end{pmatrix}\begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} + \sum_{i=1}^{N}\begin{pmatrix} A_i & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} e^{-\tau_i S} \\ &\quad - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} S \end{aligned}$$

which clearly has the form (35) if we define $D_1 = I$, $D_2 = S$, $D_i = e^{-\tau_{i-2}S}$, for $3 \le i \le N+2$.

Since $e^{-\tau_i S}e^{-\tau_j S} = e^{-\tau_j S}e^{-\tau_i S}$, the assumption of Theorem 10 holds. Now, let us take an eigenvector v of S, i. e., $Sv = \lambda v$ for $\lambda \in \sigma(S)$, then, $S^k v = \lambda^k v$ and $e^{\alpha S}v = e^{\alpha\lambda}v$. Thus, the eigentuples of $(D_1, \dots, D_k)$ have the form

$$\widehat{\lambda} = (1, \lambda, e^{-\tau_1\lambda}, \cdots, e^{-\tau_N\lambda}),$$

from where the equation (36) is

$$G(\widehat{\lambda}) = \begin{pmatrix} A_0 & B \\ C & 0 \end{pmatrix} + \sum_{i=1}^{N}\begin{pmatrix} A_i & 0 \\ 0 & 0 \end{pmatrix} e^{-\tau_i\lambda} - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\lambda$$

for all $\lambda \in \sigma(S)$. Finally, since $G(\widehat{\lambda})$ has full row rank iff (15) is satisfied, the proof of the theorem follows from Theorem 10. □

4. THE ERROR FEEDBACK REGULATOR PROBLEM

For this case, in a similar way to the state feedback we prove first a result which will be used later to provide a solution to the problem considered.

Lemma 6. Assume H1. Suppose there exists a feedback law (7) for which condition S_{ED} holds. Then, condition R_{ED} also holds if and only if there exist matrices Π and Ψ which solve the linear equations

$$\Pi S = \sum_{i=0}^{N} A_i \Pi e^{-\tau_i S} + \sum_{i=0}^{N} B\tilde{K}_i \Psi e^{-\tau_i S} + P \tag{16}$$

$$\Psi S = \sum_{i=0}^{N} \tilde{F}_i \Psi e^{-\tau_i S} \tag{17}$$

$$0 = C\Pi - R. \tag{18}$$

P r o o f. First we consider the closed loop system

$$\begin{aligned}\dot{x}(t) &= \sum_{i=0}^{N} A_i x(t-\tau_i) + \sum_{i=0}^{N} B\tilde{K}_i \xi(t-\tau_i) + Pw(t)\\ \dot{\xi}(t) &= \sum_{i=0}^{N} \tilde{F}_i \xi(t-\tau_i) + \sum_{i=0}^{N} L_i C x(t-\tau_i) - \sum_{i=0}^{N} L_i R w(t-\tau_i).\end{aligned}$$

Consider now the coordinates transformation $\tilde{x} = x - \Pi w$, $\tilde{\xi} = \xi - \Psi w$. In the new coordinates thus defined, the equations which describe the closed loop system assume the form

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{\tilde{\xi}}(t) \end{bmatrix} = \sum_{i=0}^{N} \begin{bmatrix} A_i & B\tilde{K}_i \\ L_i C & \tilde{F}_i \end{bmatrix} \begin{bmatrix} \tilde{x}(t-\tau_i) \\ \tilde{\xi}(t-\tau_i) \end{bmatrix} + \Delta w(t)$$

where

$$\Delta = \begin{bmatrix} \sum_{i=0}^{N} A_i \Pi e^{-\tau_i S} + \sum_{i=0}^{N} B\tilde{K}_i \Psi e^{-\tau_i S} + P - \Pi S \\ \sum_{i=0}^{N} \tilde{F}_i \Psi e^{-\tau_i S} + \sum_{i=0}^{N} L_i C \Pi e^{-\tau_i S} - \sum_{i=0}^{N} L_i R e^{-\tau_i S} - \Psi S \end{bmatrix}.$$

Rearranging the terms of matrix Δ from equation (4) we have the equation

$$\begin{bmatrix} \Pi \\ \Psi \end{bmatrix} S = \sum_{i=0}^{N} \begin{bmatrix} A_i & B\tilde{K}_i \\ L_i C & \tilde{F}_i \end{bmatrix} \begin{bmatrix} \Pi \\ \Psi \end{bmatrix} e^{-\tau_i S} + \begin{bmatrix} P \\ -\sum_{i=0}^{N} L_i R e^{-\tau_i S} \end{bmatrix}. \tag{19}$$

In order to obtain the conditions under which (19) has a solution we apply Theorem 10 of the Appendix, where the equation (35) takes the form

$$\begin{aligned}\begin{bmatrix} -P \\ \sum_{i=0}^{N} L_i R e^{-\tau_i S} \end{bmatrix} &= \sum_{i=0}^{N} \begin{bmatrix} A_i & B\tilde{K}_i \\ L_i C & \tilde{F}_i \end{bmatrix} \begin{bmatrix} \Pi \\ \Psi \end{bmatrix} e^{-\tau_i S} \\ &\quad - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Pi \\ \Psi \end{bmatrix} S\end{aligned}$$

where

$$G_1 = \begin{bmatrix} A_0 & B\tilde{K}_0 \\ L_0 C & \tilde{F}_0 \end{bmatrix}, \quad G_2 = -\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

$$G_i = \begin{bmatrix} A_{i-2} & B\tilde{K}_{i-2} \\ L_{i-2} C & \tilde{F}_{i-2} \end{bmatrix}, \quad i = 3, \ldots, N+2, \quad Z = \begin{bmatrix} \Pi \\ \Psi \end{bmatrix}, \quad D_1 = I,$$

$$D_2 = S, D_i = e^{-\tau_{i-2} S}, \quad i = 3, \ldots, N+2, \quad Q = \begin{bmatrix} -P \\ \sum_{i=0}^{N} L_i R e^{-\tau_i S} \end{bmatrix}$$

the corresponding joint eigentuple is

$$\hat{\lambda} = (1, \lambda, e^{-\tau_1 \lambda}, \ldots, e^{-\tau_N \lambda}).$$

Then from Theorem 10 the equation (19) has a solution and is unique if the matrix (36) has full row rank, $2n+q, \forall\lambda \in \Psi(S)$, for every joint eigentupla $\hat{\lambda} = (\lambda_1, \ldots, \lambda_{N+2})$ of $(D_1, \ldots, D_{N+2})$, where in this case

$$G(\hat{\lambda}) = \begin{bmatrix} A_0 & B\tilde{K}_0 \\ L_0 C & \tilde{F}_0 \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \lambda + \sum_{i=1}^{N} \begin{bmatrix} A_i & B\tilde{K}_i \\ L_i C & \tilde{F}_i \end{bmatrix} e^{-\tau_i \lambda}.$$

Now the ESD condition implies that

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{\tilde{\xi}}(t) \end{bmatrix} = \sum_{i=0}^{N} \begin{bmatrix} A_i & B\tilde{K}_i \\ L_i C & \tilde{F}_i \end{bmatrix} \begin{bmatrix} \tilde{x}(t-\tau_i) \\ \tilde{\xi}(t-\tau_i) \end{bmatrix}$$

is stable.

If the matrices Π and Ψ that satisfy (19) exist, then $\Delta = 0$ and from equation (4.) it follows that $\tilde{x}(t) \to 0$ and $\tilde{\xi}(t) \to 0$, therefore $x(t) \to \Pi w(t)$ and $\xi(t) \to \Psi w(t)$ as $t \to \infty$. The error mapping $e(t) = Cx(t) - Rw(t)$ tends in the limit to

$$e(t) = (C\Pi - R)w(t).$$

Since the exosystem is antistable, then $e(t) \to 0$ if and only if

$$C\Pi - R = 0.$$

Thus ERD holds if and only if the unique solution Π and Ψ of (19) satisfies also (18).

From the first block of the equation (19) we obtain

$$\Pi S = \sum_{i=0}^{N} A_i \Pi e^{-\tau_i S} + \sum_{i=0}^{N} B\tilde{K}_i \Psi e^{-\tau_i S} + P$$

that coincides with equation (16).

From the second block of the equation (19) we take the following

$$\begin{aligned} \Psi S &= \sum_{i=0}^{N} L_i C \Pi e^{-\tau_i S} + \sum_{i=0}^{N} \tilde{F}_i \Psi e^{-\tau_i S} - \sum_{i=0}^{N} L_i R e^{-\tau_i S} \\ &= \sum_{i=0}^{N} L_i [C\Pi - R] e^{-\tau_i S} + \sum_{i=0}^{N} \tilde{F}_i \Psi e^{-\tau_i S}. \end{aligned}$$

Taking into account that $C\Pi - R = 0$ yields

$$\Psi S = \sum_{i=0}^{N} \tilde{F}_i \Psi e^{-\tau_i S}$$

that coincides with equation (17). □

We have seen in the previous section that the problem of output regulation can be solved by a feedback law (5), which depends on the states of the system and those of the exosystem. For the error feedback case, since these states are not completely available, we will use a state observer to provide this information.

We recall the extended system described by

$$\begin{aligned}\dot{x}(t) &= \sum_{i=0}^{N} A_i x(t-\tau_i) + Bu(t) + Pw(t) \\ \dot{w}(t) &= Sw(t) \\ e(t) &= Cx(t) - Rw(t)\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\dot{x}^e(t) &= \sum_{i=0}^{N} A_i^e x^e(t-\tau_i) + B^e u(t) \\ e(t) &= C^e x^e(t)\end{aligned} \tag{20}$$

where

$$x^e(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \quad A_0^e = \begin{bmatrix} A_0 & P \\ 0 & S \end{bmatrix}, \quad A_i^e = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 1, \ldots, N,$$

$$B^e = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad C^e = \begin{bmatrix} C & -R \end{bmatrix}.$$

Now, for the system (20) an observer can be proposed. This observer in the present case may take the form

$$\dot{\xi}(t) = \sum_{i=0}^{N} [A_i^e - L_i C^e]\xi(t-\tau_i) + B^e u(t) + \sum_{i=0}^{N} L_i e(t-\tau_i). \tag{21}$$

In order to give conditions of stability of the proposed observer we have the following theorem.

Theorem 7. Let L_0 be a matrix such that $(A_0^e - L_0 C^e)$ is a stable matrix. Let R and M be symmetric matrices associated with the Lyapunov equation

$$(A_0^e - L_0 C^e)^T R + R(A_0^e - L_0 C^e) + (N+1)M = 0$$

satisfying

$$M > 0$$

$$M - \sum_{i=1}^{N} R(A_i^e - L_i C^e) M^{-1} (A_i^e - L_i C^e)^T R > 0.$$

Then the observer described by (21) is asymptotically stable and $\xi(t) \to x^e(t)$ as $t \to \infty$.

Proof. Defining the observation error as

$$z(t) = x^e(t) - \xi(t) \tag{22}$$

and taking the derivative of the error we have

$$\dot{z}(t) = \sum_{i=0}^{N}(A_i^e - L_iC^e)z(t-\tau_i) \tag{23}$$

or equivalently

$$\dot{z}(t) = (A_0^e - L_0C^e)z(t) + (A_1^e - L_1C^e)z(t-\tau_1) + \cdots + (A_N^e - L_NC^e)z(t-\tau_N) \tag{24}$$

and thus, by Theorem 6 in Chapter 4 of [15] we derive the result. □

From the above discussion the procedure to construct the error feedback controller may be stated as follows

Step 1. Calculate matrices $K_0, \ldots, K_N$ such that the system

$$\dot{x}(t) = \sum_{i=0}^{N}[A_i + BK_i]x(t-\tau_i)$$

is stable.

Step 2. Calculate, if possible, a solution Π and Γ of the equations (11)–(12).

Step 3. Calculate the observer (21) for the state $x^e(t)$, where

$$\dot{\xi}(t) = \sum_{i=0}^{N}[A_i^e - L_iC^e]\xi(t-\tau_i)$$

is stable.

Finally, the controller will have the structure

$$\begin{aligned} \dot{\xi}(t) &= \sum_{i=0}^{N}\tilde{F}_i\xi(t-\tau_i) + \sum_{i=0}^{N}L_ie(t-\tau_i) \\ &= \sum_{i=0}^{N}F_i^0\xi^0(t-\tau_i) + \sum_{i=0}^{N}F_i^1\xi^1(t-\tau_i) + \sum_{i=0}^{N}L_ie(t-\tau_i) \end{aligned} \tag{25}$$

$$\begin{aligned} u(t) &= \sum_{i=0}^{N}\tilde{K}_i\xi(t-\tau_i) \\ &= \begin{bmatrix} K_0 & T \end{bmatrix}\begin{bmatrix} \xi^0(t) \\ \xi^1(t) \end{bmatrix} + \sum_{i=1}^{N}\begin{bmatrix} K_i & 0 \end{bmatrix}\begin{bmatrix} \xi^0(t-\tau_i) \\ \xi^1(t-\tau_i) \end{bmatrix} \\ &= \sum_{i=0}^{N}K_i\xi^0(t-\tau_i) + \mathrm{T}\xi^1(t) \end{aligned} \tag{26}$$

where

$$\mathrm{T} = \Gamma - \sum_{i=0}^{N} K_i \Pi e^{-\tau_i S} \tag{27}$$

and

$$F_i^0 = \begin{bmatrix} A_i - L_i^0 C + BK_i \\ -L_i^1 C \end{bmatrix}, \quad i = 0, \ldots, N \tag{28}$$

$$F_0^1 = \begin{bmatrix} P + L_0^0 R + B\Gamma - BK_0\Pi \\ S + L_0^1 R \end{bmatrix}, \quad F_i^1 = \begin{bmatrix} L_i^0 R - BK_i\Pi \\ L_i^1 R \end{bmatrix}, \quad i = 1, \ldots, N \tag{29}$$

$$\xi(t) = \begin{bmatrix} \xi^0(t) \\ \xi^1(t) \end{bmatrix}, \xi^0(t) \in R^n, \xi^1(t) \in R^q, F_i^0 \in R^{(n+q)\times n}, F_i^1 \in R^{(n+q)\times q}.$$

The fact that this controller solves the problem under consideration is proved in the following result.

Theorem 8. Assume H1, H2 and condition of Theorem 7 hold. Then the EFRPD can be solved if and only if there exist matrices Π and Γ which solve the linear matrix equations (11) and (12).

Proof. *(Necessity)*

Suppose that exist a solution, then by Lemma 6 the equations (11) and (12) hold with

$$\Gamma = \sum_{i=0}^{N} \tilde{K}_i \Psi e^{-\tau_i S}$$

$$\Pi S = \sum_{i=0}^{N} A_i \Pi e^{-\tau_i S} + \sum_{i=0}^{N} B\tilde{K}_i \Psi e^{-\tau_i S} + P.$$

(Sufficiency)

If there exist matrices Π and Γ, then by Theorem 7 we can design the controller (25). We will show now that this controller satisfy the condition ESD).

The subsystem

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{\tilde{\xi}}(t) \end{bmatrix} = \sum_{i=0}^{N} \begin{bmatrix} A_i & B\tilde{K}_i \\ L_i C & \tilde{F}_i \end{bmatrix} \begin{bmatrix} \tilde{x}(t-\tau_i) \\ \tilde{\xi}(t-\tau_i) \end{bmatrix} \tag{30}$$

by substituting (25), (26), (28) and (29), takes the form

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{\tilde{\xi}}(t) \end{bmatrix} = \begin{bmatrix} A_0 & BK_0 & B\mathrm{T} \\ L_0 C & F_0^0 & F_0^1 \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \tilde{\xi}^0(t) \\ \tilde{\xi}^1(t) \end{bmatrix} + \sum_{i=1}^{N} \begin{bmatrix} A_i & BK_i & 0 \\ L_i C & F_i^0 & F_i^1 \end{bmatrix} \begin{bmatrix} \tilde{x}(t-\tau_i) \\ \tilde{\xi}^0(t-\tau_i) \\ \tilde{\xi}^1(t-\tau_i) \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{\tilde{\xi}}(t) \end{bmatrix} = \begin{bmatrix} A_0 & BK_0 & BT \\ L_0^0 C & A_0 - L_0^0 C + BK_0 & P + L_0^0 R + BT \\ L_0^1 C & -L_0^1 C & S + L_0^1 R \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \tilde{\xi}^0(t) \\ \tilde{\xi}^1(t) \end{bmatrix} + \sum_{i=1}^{N} \begin{bmatrix} A_i & BK_i & 0 \\ L_i^0 C & A_i - L_i^0 C + BK_i & L_i^0 R \\ L_i^1 C & -L_i^1 C & L_i^1 R \end{bmatrix} \begin{bmatrix} \tilde{x}(t-\tau_i) \\ \tilde{\xi}^0(t-\tau_i) \\ \tilde{\xi}^1(t-\tau_i) \end{bmatrix}.$$

Using the matrix

$$\Xi = \begin{bmatrix} I & 0 & 0 \\ -I & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

for a coordinate transformation, the subsystem takes the form

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{\zeta}}(t) \\ \dot{\tilde{\phi}}(t) \end{bmatrix} &= \begin{bmatrix} A_0 + BK_0 & BK_0 & BT \\ 0 & A_0 - L_0^0 C & P + L_0^0 R \\ 0 & -L_0^1 C & S + L_0^1 R \end{bmatrix} \begin{bmatrix} \tilde{\zeta}(t) \\ \tilde{\phi}^0(t) \\ \tilde{\phi}^1(t) \end{bmatrix} \\ &+ \sum_{i=1}^{N} \begin{bmatrix} A_i + BK_i & BK_i & 0 \\ 0 & A_i - L_i^0 C & L_i^0 R \\ 0 & -L_i^1 C & L_i^1 R \end{bmatrix} \begin{bmatrix} \tilde{\zeta}(t-\tau_i) \\ \tilde{\phi}^0(t-\tau_i) \\ \tilde{\phi}^1(t-\tau_i) \end{bmatrix} \\ &= \begin{bmatrix} A_0 + BK_0 & \begin{bmatrix} BK_0 & BT \end{bmatrix} \\ 0 & A_0^e - L_0 C^e \end{bmatrix} \begin{bmatrix} \tilde{\zeta}(t) \\ \tilde{\phi}(t) \end{bmatrix} \\ &+ \sum_{i=1}^{N} \begin{bmatrix} A_i + BK_i & \begin{bmatrix} BK_i & 0 \end{bmatrix} \\ 0 & A_i^e - L_i C^e \end{bmatrix} \begin{bmatrix} \tilde{\zeta}(t-\tau_i) \\ \tilde{\phi}(t-\tau_i) \end{bmatrix} \end{aligned}$$

or equivalently

$$\dot{\tilde{\zeta}}(t) = \sum_{i=0}^{N} [A_i + BK_i] \tilde{\zeta}(t-\tau_i) + \begin{bmatrix} BK_0 & BT \end{bmatrix} \tilde{\phi}(t) + \sum_{i=1}^{N} + \begin{bmatrix} BK_i & 0 \end{bmatrix} \tilde{\phi}(t-\tau_i)$$

$$\dot{\tilde{\phi}}(t) = \sum_{i=0}^{N} [A_i^e - L_i C^e] \tilde{\phi}(t-\tau_i).$$

By Theorem 7 we have that $\tilde{\phi}(t) \to 0$ as $t \to \infty$. Then, since H2 holds we have $\tilde{\zeta}(t) \to 0$ as $t \to \infty$, therefore the condition ESD) holds.

In order to prove the condition ERD) we use Lemma 6. If there exist Π and Γ, then setting

$$\Psi = \begin{bmatrix} \Pi \\ I \end{bmatrix} = \begin{bmatrix} \Psi^0 \\ \Psi^1 \end{bmatrix} \tag{31}$$

where

$$\Psi^0 \in R^{n \times q}, \Psi^1 \in R^{q \times q}$$

we have

$$\Pi S = \sum_{i=0}^{N} A_i \Pi e^{-\tau_i S} + \sum_{i=0}^{N} BK_i \Psi^0 e^{-\tau_i S} + B\mathrm{T}\Psi^1 + P \tag{32}$$

which, using (27) and (31) the equation (32) takes the form

$$\Pi S = \sum_{i=0}^{N} A_i \Pi e^{-\tau_i S} + B\Gamma + P$$

that is equal to (11).

From (17) and (25), (28) and (29) we have

$$\begin{aligned}
\begin{bmatrix} \Pi \\ I \end{bmatrix} S &= \sum_{i=0}^{N} \begin{bmatrix} A_i - L_i^0 C + BK_i \\ -L_i^1 C \end{bmatrix} \Psi^0 e^{-\tau_i S} + \begin{bmatrix} P + L_0^0 R + B\mathrm{T} \\ S + L_0^1 R \end{bmatrix} \Psi^1 \\
&\quad + \sum_{i=1}^{N} \begin{bmatrix} L_i^0 R \\ L_i^1 R \end{bmatrix} \Psi^1 e^{-\tau_i S} \\
&= \begin{bmatrix} \text{term} \\ -\sum_{i=0}^{N} L_i^1 C \Pi e^{-\tau_i S} + S + \sum_{i=0}^{N} L_i^1 R e^{-\tau_i S} \end{bmatrix}
\end{aligned}$$

where

$$\begin{aligned}
\text{term} &= \sum_{i=0}^{N} A_i \Pi e^{-\tau_i S} - \sum_{i=0}^{N} L_i^0 C \Pi e^{-\tau_i S} + \sum_{i=0}^{N} BK_i \Pi e^{-\tau_i S} + P \\
&\quad + \sum_{i=0}^{N} L_i R e^{-\tau_i S} + B\Gamma - \sum_{i=0}^{N} BK_i \Pi e^{-\tau_i S}
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} \Pi \\ I \end{bmatrix} S &= \begin{bmatrix} \sum_{i=0}^{N} A_i \Pi e^{-\tau_i S} - \sum_{i=0}^{N} L_i^0 [C\Pi - R] e^{-\tau_i S} + B\Gamma + P \\ S - \sum_{i=0}^{N} L_i^1 [C\Pi - R] e^{-\tau_i S} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=0}^{N} A_i \Pi e^{-\tau_i S} + B\Gamma + P \\ S \end{bmatrix}
\end{aligned}$$

which satisfies (11) and (12). Finally (18) also coincides with (12). □

5. EXAMPLE

In this section we take an example and we implement the output regulation by state feedback and by error feedback. We consider the following system

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -3 & 5 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x(t-\tau_1) + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x(t-\tau_2) \\
&\quad + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t) & (33) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t). & (34)
\end{aligned}$$

In this case the reference signal comes from an exosystem described by:

$$\begin{aligned} \dot{w}_1 &= \alpha w_2(t) \\ \dot{w}_2 &= -\alpha w_1(t) \\ y_r(t) &= w_1(t). \end{aligned}$$

5.1. State feedback regulation

Using the Matlab LMI Toolbox, we get

$$\Pi = \begin{bmatrix} 1.0000 & 0 \\ 0.7149 & -0.0394 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -3.3623 & 1.0490 \end{bmatrix}$$

$$K_0 = \begin{bmatrix} -6.4602 & -1.6379 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & -2.0000 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & -1.0000 \end{bmatrix},$$
$$F_2 = \begin{bmatrix} 5.9555 & -0.3102 \end{bmatrix}.$$

We simulated the system with $\tau_1 = 0.5$, $\tau_2 = 0.8$ and $\alpha = 1$. In Figure 1 we show the output $y(t)$ and the reference signal $w_1(t)$. Figure 2 shows the tracking error.

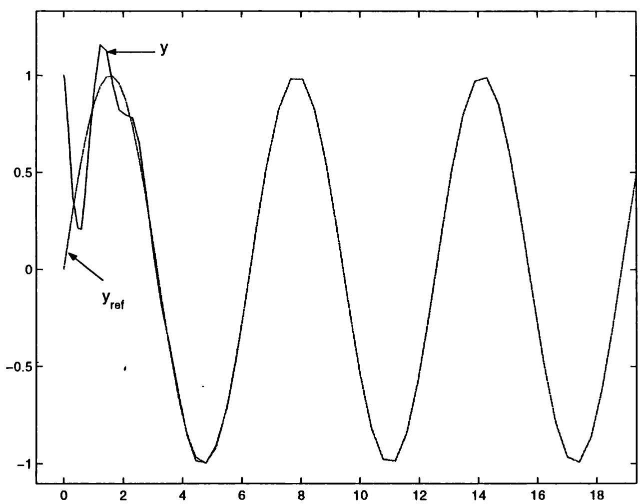

Fig. 1. SFRPD. Output $y(t) = x_1(t)$ vs reference $w_1(t)$.

5.2. Error feedback regulation

For this case, we obtain

$$L_0 = \begin{bmatrix} 7.5994 \\ 18.3244 \\ 2.0094 \\ 0.6173 \end{bmatrix}, \quad L_1 = \begin{bmatrix} -0.5000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.5000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \end{bmatrix}$$

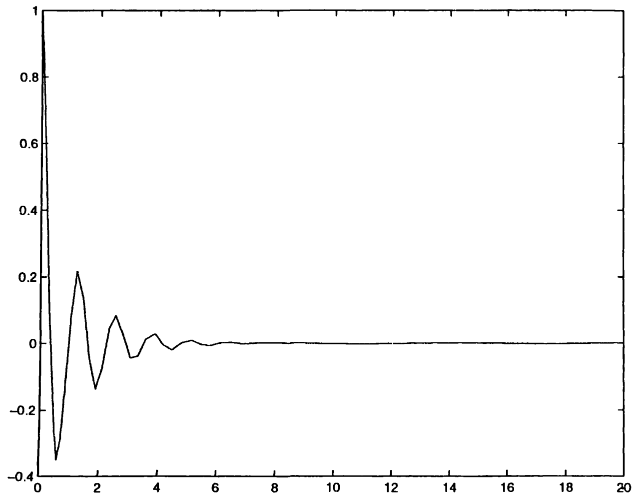

Fig. 2. SFRPD. Tracking error $e(t) = y(t) - w_1(t)$.

$$F_0^0 = \begin{bmatrix} -10.5994 & 5.0000 \\ -23.7845 & -0.6379 \\ -2.0094 & 0 \\ -0.6173 & 0 \end{bmatrix}, \quad F_1^0 = \begin{bmatrix} -0.5000 & 0 \\ 0.0000 & 0.0000 \\ 0.0000 & 0 \\ 0.0000 & 0 \end{bmatrix},$$

$$F_2^0 = \begin{bmatrix} -0.5000 & 0 \\ 0.0000 & 0.0000 \\ 0.0000 & 0 \\ 0.0000 & 0 \end{bmatrix}, \quad F_0^1 = \begin{bmatrix} 8.5994 & 0 \\ 22.5931 & 1.9845 \\ 2.0094 & 1.0000 \\ -0.3827 & 0 \end{bmatrix},$$

$$F_1^1 = \begin{bmatrix} -0.5000 & 0 \\ 1.4297 & -0.0787 \\ 0.0000 & 0 \\ 0.0000 & 0 \end{bmatrix}, \quad F_2^1 = \begin{bmatrix} -0.5000 & 0 \\ 0.7149 & -0.0394 \\ 0.0000 & 0 \\ 0.0000 & 0 \end{bmatrix}.$$

In Figure 3 and 4 we show the reference vs. the output of the system and the tracking error respectively.

6. CONCLUSIONS

The regulation problem for a class of LTI system has been addressed for both state and error feedback. We have shown that if the problem of stabilization of the delay system can be solved, then the regulation condition can be also satisfied on the basis of the existence of a solution of certain linear matrix equations. An illustrative example shows the performance of the proposed structure.

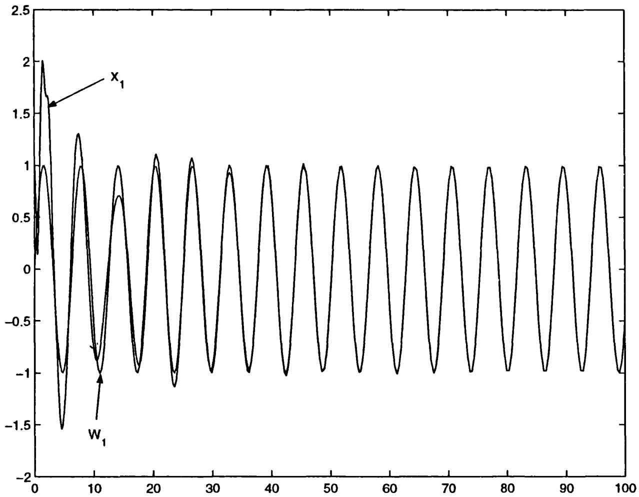

Fig. 3. EFRPD. Output $y(t) = x_1(t)$ vs reference $w_1(t)$.

The extension to LTI systems with delays both in the states and the inputs is also possible under additional considerations of the stabilizability properties of such systems.

APPENDIX: ON THE SOLVABILITY OF LINEAR MATRIX EQUATIONS

Definition 9. Let $(D_1, \ldots, D_k\ )$ be a set of commutative $m \times m$ matrices. A vector $\widehat{\lambda} = (\lambda_1, \cdots, \lambda_k)$ is a joint eigentuple of $(D_1, \ldots, D_k)$ if there exists a common corresponding eigenvector, i. e., if there exists $v \neq 0$ such that

$$D_i v = \lambda_i v \qquad i = 1, \ldots, k$$

Theorem 10. [7] Let $G_i \in I\!R^{n \times m}$, $D_i \in I\!R^{p \times p}$ and suppose $D_i D_j = D_j D_i$ for all $i, j = 1, \ldots, k$. Then the equation

$$\sum_{i=1}^{k} G_i Z D_i = Q \tag{35}$$

is universally solvable (i. e. has a solution Z for every Q) iff $G(\widehat{\lambda})$ has full row rank for every joint eigentuple $\widehat{\lambda} = (\lambda_1, \ldots, \lambda_k)$ of $(D_1, \cdots, D_k)$, where

$$G(\widehat{\lambda}) = \sum_{i=1}^{k} G_i \lambda_i. \tag{36}$$

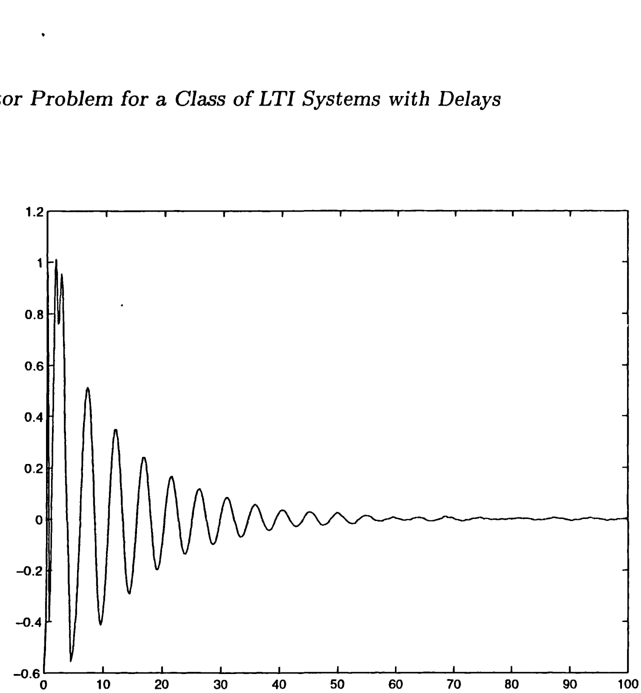

Fig. 4. EFRPD. Tracking error $e(t) = y(t) - w_1(t)$.

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