KYBERNETIKA — VOLUME 39 (2003), NUMBER 3, PAGES 369 – 385

COMPLEMENTARY MATRICES IN THE INCLUSION PRINCIPLE FOR DYNAMIC CONTROLLERS

LUBOMÍR BAKULE, JOSÉ RODELLAR AND JOSEP M. ROSSELL

A generalized structure of complementary matrices involved in the input-state-output Inclusion Principle for linear time-invariant systems (LTI) including contractibility conditions for static state feedback controllers is well known. In this paper, it is shown how to further extend this structure in a systematic way when considering contractibility of dynamic controllers. Necessary and sufficient conditions for contractibility are proved in terms of both unstructured and block structured complementary matrices for general expansion/contraction transformation matrices. Explicit sufficient conditions for blocks of complementary matrices ensuring contractibility are proved for general expansion/contraction transformation matrices. Moreover, these conditions are further specialized for a particular class of transformation matrices. The results are derived in parallel for two important cases of the Inclusion Principle namely for the case of expandability of controllers and the case of extensions.

Keywords: linear time-invariant continuous-time systems, dynamic controllers, inclusion principle, large scale systems, overlapping, decomposition, decentralization

AMS Subject Classification: 93B17, 93A15, 93A14, 34A30, 34H05, 15A04

1. INTRODUCTION

The *Inclusion Principle* proposed in the context of analysis and control of complex and large scale systems in [11, 14, 15, 17] establishes essentially a mathematical framework for two dynamic systems with different dimensions, in which solutions of the system with larger dimension include solutions of the system with smaller dimension. The relation between both systems is constructed usually on the base of appropriate linear transformations between the corresponding systems in the original and expanded spaces, where a key role in the selection of appropriate structure of all matrices in the expanded space is played by the so called *complementary matrices* [12, 17]. The standard forms of complementary matrices such as *aggregations* and *restrictions* have been used in fact as the only well known forms for many years because the conditions for their selection did not allowed to derive some other more flexible structures of these matrices. A contribution to this issue has been presented in [1, 2, 3, 4, 5] giving a new procedure for a flexible selection of complementary matrices based on appropriate changes of basis in the systems.

When considering control, the following problem arises: give conditions to ensure that a controller designed for one of the systems can be transformed to be implemented in the other system in such a way that the Inclusion Principle holds for the closed-loops systems. A typical case in the literature is when an original system **S** with overlapped components is expanded to a bigger one with a number of disjoint subsystems. Then, decentralized controllers are designed in the expanded system $\tilde{\mathbf{S}}$ and then contracted for implementation in the original system **S**. This scheme leads to the concept of contractibility. Also, in a reverse direction, controllers can be designed in the original system **S** and transformed for implementation in the bigger system $\tilde{\mathbf{S}}$. This direction leads naturally to the concept of expandability.

Early work on contractibility was done for static state controllers in [10, 11, 15] and for dynamic controllers (including estimators) in [6, 13], but only with the use of standard complementary matrices in the context of aggregations and restrictions. Contractibility conditions of dynamic controllers were also derived in [8, 9] for the particular expansion/contraction process referred to as extension, without using complementary matrices. Recently, contractibility of dynamic controllers has been revisited in a more general framework, in which a broader definition of contractibility is proposed to include the specific cases of restrictions, aggregations and extensions [16, 18]. However, the conditions presented in [16] involve complicated matrix products without using complementary matrices. Thus, they are difficult to apply for control design.

In this paper structural properties of contractibility for dynamic controllers are given for expansion/contraction processes by using complementary matrices. The concept of contractibility given in [16], [18] is used to follow two parallel lines to develop contractibility conditions in this paper: The first case considers expandability of controllers, i.e. the control is designed without any restriction in the small system **S** and then expanded into the big system $\tilde{\mathbf{S}}$. The second case considers extensions, i.e. the control is designed without any restriction in the big system $\tilde{\mathbf{S}}$ and contracted for implementation in the small system **S**. This case is important for decentralized control design.

Briefly, the contribution of the paper for continuous-time linear time-invariant systems can be summarized as follows:

- Necessary and sufficient conditions for contractibility are stated for general expansion/contraction transformation matrices in terms of both unstructured and block structured complementary matrices.
- Sufficient conditions for contractibility of dynamic controllers at this general level are given in the form of explicit conditions on complementary matrices. These conditions specify possible choices of these matrices for feasible control design.
- Sufficient conditions for contractibility of dynamic controllers are given for a particular standard selection of transformation matrices. These conditions offer the possibility of an easy and flexible choice of complementary matrices.

2. PROBLEM STATEMENT

To formulate the problem, a minimum of necessary preliminaries is introduced now.

2.1. Preliminaries

Consider a linear time-invariant systems

$$\mathbf{S}: \quad \dot{x} = Ax + Bu, \ x(0) = x_0, \qquad \tilde{\mathbf{S}}: \quad \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}, \ \tilde{x}(0) = \tilde{x}_0,$$
$$y = Cx, \qquad \tilde{y} = \tilde{C}\tilde{x}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^l$ are the state, input and output of $\mathbf{S}$ at time $t \in \mathbb{R}^+$, and $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$, $\tilde{u}(t) \in \mathbb{R}^{\tilde{m}}$, $\tilde{y}(t) \in \mathbb{R}^{\tilde{l}}$ are those ones of $\tilde{\mathbf{S}}$. A, B, C and $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are constant matrices of dimensions $n\times n$, $n \times m$, $l\times n$ and $\tilde{n} \times\tilde{n}$, $\tilde{n}\times \tilde{m}$, $\tilde{l} \times \tilde{n}$, respectively. Suppose that the dimensions of the state, input and output vectors x, u, y of $\mathbf{S}$ are smaller than (or at most equal to) those of $\tilde{x}$, $\tilde{u}$, $\tilde{y}$ of $\tilde{\mathbf{S}}$. Denote $x(t; x_0, u)$ and $y[x(t)]$ the state behaviour and the corresponding output of $\mathbf{S}$ for a fixed input $u(t)$ and for an initial state $x(0) = x_0$, respectively. Similar notations $\tilde{x}(t; \tilde{x}_0, \tilde{u})$ and $\tilde{y}[\tilde{x}(t)]$ are used for the state behaviour and output of $\tilde{\mathbf{S}}$.

Let us consider the linear time-invariant dynamic controllers

$$\mathbf{C}: \ \dot{z} = Fz + Pu + Gy, \ z(0) = z_0, \qquad \tilde{\mathbf{C}}: \ \dot{\tilde{z}} = \tilde{F}\tilde{z} + \tilde{P}\tilde{u} + \tilde{G}\tilde{y}, \ \tilde{z}(0) = \tilde{z}_0,$$
$$u = Hz + Ky + v, \qquad \tilde{u} = \tilde{H}\tilde{z} + \tilde{K}\tilde{y} + \tilde{v}, \tag{2}$$

for the systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$, respectively, where $z(t)\in\mathbb{R}^p$ is the state of $\mathbf{C}$ at time $t\in\mathbb{R}^+$ and $\tilde{z}(t)\in\mathbb{R}^{\tilde{p}}$ is this one of $\tilde{\mathbf{C}}$. The vectors $v(t)\in\mathbb{R}^m$, $\tilde{v}(t)\in\mathbb{R}^{\tilde{m}}$ are new inputs to the corresponding closed-loop systems. The matrices F, P, G, H, K, $\tilde{F}$, $\tilde{P}$, $\tilde{G}$, $\tilde{H}$, $\tilde{K}$ are constant with appropriate dimensions.

Let us consider the following transformations:

$$\begin{aligned} V &: \mathbb{R}^n \longrightarrow \mathbb{R}^{\tilde{n}}, & U &: \mathbb{R}^{\tilde{n}} \longrightarrow \mathbb{R}^n, \\ R &: \mathbb{R}^m \longrightarrow \mathbb{R}^{\tilde{m}}, & Q &: \mathbb{R}^{\tilde{m}} \longrightarrow \mathbb{R}^m, \\ T &: \mathbb{R}^l \longrightarrow \mathbb{R}^{\tilde{l}}, & S &: \mathbb{R}^{\tilde{l}} \longrightarrow \mathbb{R}^l, \\ E &: \mathbb{R}^p \longrightarrow \mathbb{R}^{\tilde{p}}, & D &: \mathbb{R}^{\tilde{p}} \longrightarrow \mathbb{R}^p, \end{aligned} \tag{3}$$

where $\mathrm{rank}(V) = n$, $\mathrm{rank}(R) = m$, $\mathrm{rank}(T) = l$, $\mathrm{rank}(E) = p$ and such that $UV = I_n$, $QR = I_m$, $ST = I_l$, $DE = I_p$, where I_n, I_m, I_l, I_p are identity matrices of indicated dimensions.

Definition 1. (*Inclusion Principle*) A system $\tilde{\mathbf{S}}$ is an expansion of the system $\mathbf{S}$ or $\mathbf{S}$ is included in $\tilde{\mathbf{S}}$, $\tilde{S} \supset \mathbf{S}$, if there exists a quadruplet of transformations (U, V, R, S) such that, for any initial state x_0 and any fixed input $u(t)$ of $\mathbf{S}$, the choice $\tilde{x}_0 = Vx_0$, $\tilde{u}(t) = Ru(t)$ for all $t \geq 0$ of the initial state $\tilde{x}_0$ and input $\tilde{u}(t)$ of the system $\tilde{\mathbf{S}}$, implies $x(t; x_0, u) = U\tilde{x}(t; \tilde{x}_0, \tilde{u})$ and $y[x(t)] = S\tilde{y}[\tilde{x}(t)]$ for all $t \geq 0$.

Definition 2. Suppose $\tilde{\mathbf{S}} \supset \mathbf{S}$ by Definition 1. A controller $\mathbf{C}$ for $\mathbf{S}$ is expandable to the controller $\tilde{\mathbf{C}}$ of $\tilde{\mathbf{S}}$, if there exist transformations (U, V, R, S, D, E) such that, for any initial state x_0, any fixed input $u(t)$ of $\mathbf{S}$ and any initial state z_0 of $\mathbf{C}$, the choice $\tilde{z}_0 = Ez_0$ implies $z(t; z_0, u, y) = D\tilde{z}(t; \tilde{z}_0, \tilde{u}, \tilde{y})$ and $R\,(Hz(t) + Ky(t)) = \tilde{H}\tilde{z}(t) + \tilde{K}\tilde{y}(t)$ for all $t \geq 0$.

Definitions 1 and 2 characterize the inclusion of the closed-loop system $(\mathbf{S}, \mathbf{C})$ into the closed-loop system $(\tilde{\mathbf{S}}, \tilde{\mathbf{C}})$ when the control $u(t)$ is designed as a free control for the system $\mathbf{S}$, that is $(\tilde{\mathbf{S}}, \tilde{\mathbf{C}}) \supset (\mathbf{S}, \mathbf{C})$.

Definition 3. (Extension) A system $\tilde{\mathbf{S}}$ is an extension of $\mathbf{S}$ if there exist transformations (V, Q, T) such that, for any initial state x_0 of $\mathbf{S}$ and any fixed input $\tilde{u}(t)$ of $\tilde{\mathbf{S}}$, the choice $\tilde{x}_0 = Vx_0$ and $u(t) = Q\tilde{u}(t)$ for all $t \geq 0$ implies $\tilde{x}(t; \tilde{x}_0, \tilde{u}) = Vx(t; x_0, u)$ and $\tilde{y}[\tilde{x}(t)] = Ty[x(t)]$ for all $t \geq 0$.

Definition 4. Suppose $\tilde{\mathbf{S}} \supset \mathbf{S}$ by Definition 3. A controller $\tilde{\mathbf{C}}$ for $\tilde{\mathbf{S}}$ is contractible to the controller $\mathbf{C}$ of $\mathbf{S}$, if there exist transformations (V, Q, T, D) such that, for any initial state x_0 of $\mathbf{S}$, any initial state $\tilde{z}_0$ of $\tilde{\mathbf{C}}$ and any fixed input $\tilde{u}(t)$ of $\tilde{\mathbf{S}}$, the choice $z_0 = D\tilde{z}_0$ implies $z(t; z_0, u, y) = D\tilde{z}(t; \tilde{z}_0, \tilde{u}, \tilde{y})$ and $Hz(t) + Ky(t) = Q(\tilde{H}\tilde{z}(t) + \tilde{K}\tilde{y}(t))$ for all $t \geq 0$.

Definitions 3 and 4 correspond to the particular but important case of extensions [8, 9, 10].

Now, suppose that the pairs of matrices (U, V), (Q, R), (S, T) and (D, E) are given. Then, the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{F}$, $\tilde{P}$, $\tilde{G}$, $\tilde{H}$ and $\tilde{K}$ can be expressed as

$$\begin{aligned} &\tilde{A} = VAU + M, &&\tilde{B} = VBQ + N, &&\tilde{C} = TCU + L, \\ &\tilde{F} = EFD + M_F, &&\tilde{P} = EPQ + Y_P, &&\tilde{G} = EGS + N_G, \\ &\tilde{H} = RHD + L_H, &&\tilde{K} = RKS + J_K, && \end{aligned} \tag{4}$$

where M, N, L, M_F, Y_P, N_G, L_H and J_K are complementary matrices of appropriate dimensions. The relations between the systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ in terms of complementary matrices are given by the following theorems [9, 10, 13, 14, 15, 17, 18].

Theorem 1. A system $\tilde{\mathbf{S}}$ is an expansion of $\mathbf{S}$ by Definition 1 if and only if

$$UM^{i}V = 0, \quad UM^{i-1}NR = 0, \quad SLM^{i-1}V = 0, \quad SLM^{i-1}NR = 0 \tag{5}$$

hold for all $i = 1, \ldots, \tilde{n}$.

Theorem 2. A system $\tilde{\mathbf{S}}$ is an extension of $\mathbf{S}$ by Definition 3 if and only if $MV = 0$, $N = 0$, $LV = 0$.

Theorem 2 implies Theorem 1. In both cases, the system $\tilde{\mathbf{S}} \supset \mathbf{S}$ and the Inclusion Principle given by Definition 1 holds. We can observe from Theorem 2 that the extensions are rather restrictive because the complementary matrix $N = 0$, and the other matrices M and L have a limited structure. Therefore, it can be more useful to consider Definition 1 to achieve higher freedom in the design of controllers.

Necessary and sufficient conditions for contractibility by using Definitions 2 and 4 are given now by the following theorems [16, 18].

Theorem 3. A controller $\mathbf{C}$ for $\mathbf{S}$ is expandable to the controller $\tilde{\mathbf{C}}$ of $\tilde{\mathbf{S}}$ by Definition 2 if and only if

$$\begin{array}{llll} \text{a)} & D\tilde{F}^i E = F^i, & \text{b)} & D\tilde{F}^i \tilde{G}\tilde{C}\tilde{A}^j V = F^i GCA^j, \\ \text{c)} & D\tilde{F}^i \tilde{G}\tilde{C}\tilde{A}^j \tilde{B}R = F^i GCA^j B, & \text{d)} & D\tilde{F}^i \tilde{P}R = F^i P, \\ \text{e)} & \tilde{H}\tilde{F}^i E = RHF^i, & \text{f)} & \tilde{H}\tilde{F}^i \tilde{P}R = RHF^i P, \\ \text{g)} & \tilde{H}\tilde{F}^i \tilde{G}\tilde{C}\tilde{A}^j V = RHF^i GCA^j, & \text{h)} & \tilde{H}\tilde{F}^i \tilde{G}\tilde{C}\tilde{A}^j \tilde{B}R = RHF^i GCA^j B, \\ \text{i)} & \tilde{K}\tilde{C}\tilde{A}^i V = RKCA^i, & \text{j)} & \tilde{K}\tilde{C}\tilde{A}^i \tilde{B}R = RKCA^i B \end{array} \tag{6}$$

hold for all $i, j = 0, 1, 2, \ldots$.

Theorem 4. A controller $\tilde{\mathbf{C}}$ for $\tilde{\mathbf{S}}$ is contractible to the controller $\mathbf{C}$ of $\mathbf{S}$ by Definition 4 if and only if

$$\begin{array}{llll} \text{a)} & D\tilde{F}^i = F^i D, & \text{b)} & D\tilde{F}^i \tilde{G}\tilde{C}\tilde{A}^j V = F^i GCA^j, \\ \text{c)} & D\tilde{F}^i \tilde{G}\tilde{C}\tilde{A}^j \tilde{B} = F^i GCA^j BQ, & \text{d)} & D\tilde{F}^i \tilde{P} = F^i PQ, \\ \text{e)} & Q\tilde{H}\tilde{F}^i = HF^i D, & \text{f)} & Q\tilde{H}\tilde{F}^i \tilde{P} = HF^i PQ, \\ \text{g)} & Q\tilde{H}\tilde{F}^i \tilde{G}\tilde{C}\tilde{A}^j V = HF^i GCA^j, & \text{h)} & Q\tilde{H}\tilde{F}^i \tilde{G}\tilde{C}\tilde{A}^j \tilde{B} = HF^i GCA^j BQ, \\ \text{i)} & Q\tilde{K}\tilde{C}\tilde{A}^i V = KCA^i, & \text{j)} & Q\tilde{K}\tilde{C}\tilde{A}^i \tilde{B} = KCA^i BQ \end{array} \tag{7}$$

hold for all $i, j = 0, 1, 2, \ldots$

Theorem 4 reduces to the following theorem when considering the conditions $\tilde{A}V = VA$, $\tilde{B} = VBQ$, $\tilde{C}V = TC$ [8, 10, 18].

Theorem 5. A controller $\tilde{\mathbf{C}}$ for $\tilde{\mathbf{S}}$ is contractible to the controller $\mathbf{C}$ of $\mathbf{S}$ by Definition 4 if and only if

$$\begin{array}{lll} \text{a)}\ D\tilde{F} = FD, & \text{b)}\ D\tilde{G}TC = GC, & \text{c)}\ D\tilde{P} = PQ, \\ \text{d)}\ Q\tilde{H} = HD, & \text{e)}\ Q\tilde{K}TC = KC. & \end{array} \tag{8}$$

Remark. The requirements given in Theorems 1 and 2 directly follow from the imposition of the conditions given by Definitions 1 and 3, respectively. Theorems 3 and 4 are obtained through the contractibility conditions from Definitions 2 and 4, respectively, considering $z(t) = e^{Ft}z_0 + \int_0^t e^{F(t-\tau)}[Pu(\tau) + Gy(\tau)]\,\mathrm{d}\tau$ and $\tilde{z}(t) = e^{\tilde{F}t}\tilde{z}_0 + \int_0^t e^{\tilde{F}(t-\tau)}[\tilde{P}\tilde{u}(\tau) + \tilde{G}\tilde{y}(\tau)]\,\mathrm{d}\tau$ with $y(t) = C\left[e^{At}x_0 + \int_0^t e^{A(t-\sigma)}Bu(\sigma)\,\mathrm{d}\sigma\right]$ and $\tilde{y}(t) = \tilde{C}\left[e^{\tilde{A}t}\tilde{x}_0 + \int_0^t e^{\tilde{A}(t-\sigma)}\tilde{B}\tilde{u}(\sigma)\,\mathrm{d}\sigma\right]$. The direct comparison of elements between the Taylor series expansions of e^{Ft}, $e^{\tilde{F}t}$, e^{At}, $e^{\tilde{A}t}$ and taking into account the relations (4) result in the assertions of the above theorems.

2.2. The problem

The usage of the Inclusion Principle depends essentially on the choice of the transformation matrices and complementary matrices in the expansion-contraction process [12]. A recent effort has been concentrated on deriving conditions to get generalized structures of complementary matrices for different systems [1, 2, 3, 4, 5]. These results include only the contractibility conditions for static state controllers. The necessary and sufficient conditions given by Theorems 3 and 4 have been derived for dynamic controllers without considering complementary matrices [16, 18]. However, these conditions are difficult to be verified in controller design because they all include complicated matrix products. The way to overcome this problem is by introducing the complementary matrices defined in (4) and expressing the contractibility conditions in terms of these matrices. These new conditions are much more simple and flexible than those (6) and (7). In this way, the contractibility conditions rely on the appropriate selection of complementary matrices. To the authors knowledge, there is no systematic procedure for the selection of complementary matrices in the case of dynamic controllers.

Therefore, the motivation of this work is to provide a systematic generalization of the structure of complementary matrices for contractibility of dynamic controllers for continuous-time LTI systems to obtain a more flexible computational tool, mainly for decentralized control design. Contractibility means that a controller is designed in one of the systems in such a way that it is guaranteed that the closed-loop system $(\tilde{\mathbf{S}}, \tilde{\mathbf{C}})$ includes the closed-loop system $(\mathbf{S}, \mathbf{C})$. The Problem is formulated as follows:

- To derive necessary and sufficient conditions for contractibility of dynamic controllers for general expansion-contraction transformation matrices given in the form of unstructured complementary matrices.
- To derive necessary and sufficient conditions for contractibility of dynamic controllers for general expansion-contraction transformation matrices given in the form of block structured complementary matrices.
- To derive sufficient conditions for contractibility of dynamic controllers for general expansion-contraction transformation matrices given in the form of explicit conditions on blocks of the structured complementary matrices, thus enabling feasible flexible choices of such matrices.
- To specialize the above sufficient explicit conditions of contractibility for a particular standard selection of transformation matrices thus illustrating the possibility of an easy and flexible choice of complementary matrices.

3. MAIN RESULTS

The results included in this section cover the expansion-contraction process of dynamic controllers in parallel for two cases of the Inclusion Principle characterized by the pairs of Definitions 1 – 2 and 3 – 4.

Subsection 3.1 includes necessary and sufficient conditions for contractibility of dynamic controllers given in the form of globally structured complementary matrices. Subsection 3.2 summarizes the expansion-contraction process for systems by using the change of basis within the Inclusion Principle. Subsection 3.3 presents necessary and sufficient conditions for contractibility of dynamic controllers in the form of block structured complementary matrices in the new basis. Subsection 3.4 presents explicit conditions of contractibility, when applying minimal sets of sufficient requirements within theorems of previous Subsection 3.3. Subsection 3.5 presents propositions resulting from the explicit conditions on contractibility in the original basis, when using a particular selection of transformation matrices. They are important mainly for decentralized control design.

3.1. Contractibility of dynamic controllers

The complementary matrices play a fundamental role in the design of controllers and estimators. Theorems 3 and 4 give contractibility conditions in terms of implicit relations involving matrices of both systems $(\mathbf{S},\mathbf{C})$ and $(\tilde{\mathbf{S}},\tilde{\mathbf{C}})$. However, it is necessary to give the above conditions in explicit form by using the complementary matrices M, N, L, M_F, Y_P, N_G, L_H, J_K because this choice allows consequently to select the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{F}$, $\tilde{P}$, $\tilde{G}$, $\tilde{H}$, $\tilde{K}$, respectively, with a higher degree of freedom as required by the control design.

Theorem 6. The controller $\mathbf{C}$ for $\mathbf{S}$ is expandable to the controller $\tilde{\mathbf{C}}$ of $\tilde{\mathbf{S}}$ by Definition 2 if and only if

$$\begin{aligned}
&\text{a)}\quad DM_F^{i+1}E = 0,\\
&\text{b)}\quad DM_F^{i}N_G(TC+LV) = 0, \quad DM_F^{i}N_G LM^{j+1}V = 0,\\
&\text{c)}\quad DM_F^{i}N_G LM^{j}NR = 0,\\
&\text{d)}\quad DM_F^{i}Y_P R = 0,\\
&\text{e)}\quad L_H M_F^{i}E = 0,\\
&\text{f)}\quad L_H M_F^{i}Y_P R = 0,\\
&\text{g)}\quad L_H M_F^{i}N_G(TC+LV) = 0, \quad L_H M_F^{i}N_G LM^{j+1}V = 0,\\
&\text{h)}\quad L_H M_F^{i}N_G LM^{j}NR = 0,\\
&\text{i)}\quad J_K(TC+LV) = 0, \quad J_K LM^{i+1}V = 0,\\
&\text{j)}\quad J_K LM^{i}NR = 0
\end{aligned} \tag{9}$$

hold for all i, $j = 0,1,2,\ldots$

Proof. The proof starts from the expressions (6) that assure the expandability of the controller in the sense of Definition 2. We will prove only the relations a) and b) because the remaining conditions follow a similar process. Proof of part a): Consider the relation a) given in (6), that is, $D\tilde{F}^i E = F^i$ together with (4). We obtain $DE = I_p$ for $i = 0$ which holds by hypothesis. We get $D\,(EFD + M_F)\,E = F$ for $i = 1$, that is, $DM_F E = 0$ since $DE = I_p$. In general, we get $DM_F^i E = 0$ for $i \geq 1$. Then, $DM_F^{i+1} E = 0$ for all $i \geq 0$. This proves a).

Proof of part b): Consider the relation b) given in (6), i.e. $D\tilde{F}^i \tilde{G}\tilde{C}\tilde{A}^j V = F^i GCA^j$. We obtain $DN_G LM^j V = 0$ for $i = 0, j \geq 1$. We get $DM_F^i N_G\,(TC + LV) = 0$ for $i \geq 0$, $j = 0$. We obtain $DM_F^i N_G LM^j V = 0$ for $i \geq 1$, $j \geq 1$. Summarizing these relations, we get $DM_F^i N_G\,(TC + LV) = 0$ and $DM_F^i N_G LM^{j+1} V = 0$ for all $i, j \geq 0$. □

Theorem 7. A controller $\tilde{\mathbf{C}}$ for $\tilde{\mathbf{S}}$ is contractible to the controller $\mathbf{C}$ of $\mathbf{S}$ by Definition 4 if and only if

$$\begin{array}{lll} \text{a)}\ DM_F = 0, & \text{b)}\ DN_G TC = 0, & \text{c)}\ DY_P = 0, \\ \text{d)}\ QL_H = 0, & \text{e)}\ QJ_K TC = 0 & \end{array} \tag{10}$$

hold.

Proof. The proof is straightforward from the corresponding relations a) – e) given by Theorem 5 together with relations (4). □

3.2. Expansion–contraction process of systems

In order to simplify the notation, consider the system $\mathbf{S}$:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{matrix} n_1 \\ n_2 \\ n_3 \end{matrix} \overset{\begin{matrix} n_1 & n_2 & n_3 \end{matrix}}{\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{matrix} n_1 \\ n_2 \\ n_3 \end{matrix} \overset{\begin{matrix} m_1 & m_2 & m_3 \end{matrix}}{\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{matrix} l_1 \\ l_2 \\ l_3 \end{matrix} \overset{\begin{matrix} n_1 & n_2 & n_3 \end{matrix}}{\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{11}$$

where n_i, m_i and l_i indicate the dimensions of the corresponding matrices with $n_1 + n_2 + n_3 = n$, $m_1 + m_2 + m_3 = m$, $l_1 + l_2 + l_3 = l$ and $n + n_2 = \tilde{n}$, $m + m_2 = \tilde{m}$, $l + l_2 = \tilde{l}$. Suppose subsystems $\mathbf{S_1}$ and $\mathbf{S_2}$ defined by x_i, u_i, $(\cdot)_{ij}$ for i, $j = 1, 2$ and

i, $j = 2, 3$, respectively. $(\cdot)_{ij}$ denotes simultaneously A_{ij}, B_{ij}, C_{ij} in (11). Therefore, overlapping appears in x_2, u_2, $(\cdot)_{22}$. This system overlapping structure defined by these blocks of matrices has been extensively adopted as prototype in the literature. We summarize the most important results about the structure and properties of the complementary matrices such that the Inclusion Principle is guaranteed. These results will be necessary later on in the derivation of contractibility conditions. The expansion-contraction process between the systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ can be schematically illustrated in the form

$$\begin{array}{ccccc} \mathbf{S} & \longrightarrow & \tilde{\mathbf{S}} & \longrightarrow & \mathbf{S} \\ \mathbb{R}^n & \xrightarrow{V} & \mathbb{R}^{\tilde{n}} & \xrightarrow{U} & \mathbb{R}^n, \\ \mathbb{R}^m & \xrightarrow{R} & \mathbb{R}^{\tilde{m}} & \xrightarrow{Q} & \mathbb{R}^m, \\ \mathbb{R}^l & \xrightarrow{T} & \mathbb{R}^{\tilde{l}} & \xrightarrow{S} & \mathbb{R}^l. \end{array} \tag{12}$$

As considered in [1, 2, 3, 4, 5] convenient changes of basis can be introduced in $\tilde{\mathbf{S}}$, so that this scheme is modified in the form

$$\begin{array}{ccccccccc} \mathbf{S} & \longrightarrow & \tilde{\mathbf{S}} & \longrightarrow & \bar{\tilde{\mathbf{S}}} & \longrightarrow & \tilde{\mathbf{S}} & \longrightarrow & \mathbf{S} \\ \mathbb{R}^n & \xrightarrow{V} & \mathbb{R}^{\tilde{n}} & \xrightarrow{T_A^{-1}} & \bar{\mathbb{R}}^{\tilde{n}} & \xrightarrow{T_A} & \mathbb{R}^{\tilde{n}} & \xrightarrow{U} & \mathbb{R}^n, \\ \mathbb{R}^m & \xrightarrow{R} & \mathbb{R}^{\tilde{m}} & \xrightarrow{T_B^{-1}} & \bar{\mathbb{R}}^{\tilde{m}} & \xrightarrow{T_B} & \mathbb{R}^{\tilde{m}} & \xrightarrow{Q} & \mathbb{R}^m, \\ \mathbb{R}^l & \xrightarrow{T} & \mathbb{R}^{\tilde{l}} & \xrightarrow{T_C^{-1}} & \bar{\mathbb{R}}^{\tilde{l}} & \xrightarrow{T_C} & \mathbb{R}^{\tilde{l}} & \xrightarrow{S} & \mathbb{R}^l, \end{array} \tag{13}$$

where $\bar{\tilde{\mathbf{S}}}$ denotes the expanded system in the new basis. Suppose given the matrices V, R and T. Define

$$U = (V^tV)^{-1}V^t, \quad Q = (R^tR)^{-1}R^t, \quad S = (T^tT)^{-1}T^t \tag{14}$$

as the pseudoinverses of V, R and T, respectively. Let us consider the change of basis

$$T_A = (V \ W_A), \qquad T_B = (R \ W_B), \qquad T_C = (T \ W_C), \tag{15}$$

where the matrices W_A, W_B, W_C are chosen such that $\operatorname{Im} W_A = \operatorname{Ker} U$, $\operatorname{Im} W_B = \operatorname{Ker} Q$, $\operatorname{Im} W_C = \operatorname{Ker} S$.

Consider the relations

$$\bar{V} = T_A^{-1}V, \quad \bar{R} = T_B^{-1}R, \quad \bar{T} = T_C^{-1}T, \quad \bar{U} = UT_A, \quad \bar{Q} = QT_B, \quad \bar{S} = ST_C. \tag{16}$$

Consider a pair of linear time-invariant systems

$$\begin{aligned} \mathbf{S}: \quad & \dot{x} = Ax + Bu, \ x(0) = x_0, & \bar{\tilde{\mathbf{S}}}: \quad & \dot{\bar{\tilde{x}}} = \bar{\tilde{A}}\bar{\tilde{x}} + \bar{\tilde{B}}\bar{\tilde{u}}, \ \bar{\tilde{x}}(0) = \bar{\tilde{x}}_0, \\ & y = Cx, & & \bar{\tilde{y}} = \bar{\tilde{C}}\bar{\tilde{x}}, \end{aligned} \tag{17}$$

where the matrices $\tilde{\bar{A}}$, $\tilde{\bar{B}}$, $\tilde{\bar{C}}$ have appropriate dimensions. The vectors $\tilde{\bar{x}}$, $\tilde{\bar{u}}$, $\tilde{\bar{y}}$ are defined as $\tilde{\bar{x}} = \bar{V}x$, $\tilde{\bar{u}} = \bar{R}u$, $\tilde{\bar{y}} = \bar{T}y$. Denote the relations (4) for the open-loop system as

$$\tilde{\bar{A}} = \bar{V}A\bar{U} + \bar{M}, \qquad \tilde{\bar{B}} = \bar{V}B\bar{Q} + \bar{N}, \qquad \tilde{\bar{C}} = \bar{T}C\bar{U} + \bar{L}, \tag{18}$$

where new complementary matrices are

$$\bar{M} = T_A^{-1} M T_A, \qquad \bar{N} = T_A^{-1} N T_B, \qquad \bar{L} = T_C^{-1} L T_A. \tag{19}$$

The conditions (5) for the Inclusion Principle by Definition 1 are now as follows:

$$\bar{U}\bar{M}^i\bar{V} = 0, \quad \bar{U}\bar{M}^{i-1}\bar{N}\bar{R} = 0, \quad \bar{S}\bar{L}\bar{M}^{i-1}\bar{V} = 0, \quad \bar{S}\bar{L}\bar{M}^{i-1}\bar{N}\bar{R} = 0 \tag{20}$$

hold for all $i = 1, 2, \ldots, \tilde{n}$.

Consider in $\tilde{\mathbf{S}}$ the matrices $M = (M_{ij})$, $N = (N_{ij})$, $L = (L_{ij})$, $i, j = 1, \ldots, 4$, where each submatrix has appropriate dimensions. Denote now the matrices $\bar{M}$, $\bar{N}$ and $\bar{L}$ as follows:

$$\bar{M} = \begin{pmatrix} \bar{M}_{11} & \bar{M}_{12} \\ \bar{M}_{21} & \bar{M}_{22} \end{pmatrix}, \quad \bar{N} = \begin{pmatrix} \bar{N}_{11} & \bar{N}_{12} \\ \bar{N}_{21} & \bar{N}_{22} \end{pmatrix}, \quad \bar{L} = \begin{pmatrix} \bar{L}_{11} & \bar{L}_{12} \\ \bar{L}_{21} & \bar{L}_{22} \end{pmatrix} \tag{21}$$

such that $\bar{M}_{11}$, $\bar{M}_{22}$ are $n \times n$, $n_2 \times n_2$ matrices, respectively. $\bar{N}_{11}$, $\bar{N}_{22}$ are $n \times m$, $n_2 \times m_2$ matrices, respectively. $\bar{L}_{11}$, $\bar{L}_{22}$ are $l \times n$, $l_2 \times n_2$ matrices, respectively.

The conditions on the blocks $\bar{M}_{ij}$, $\bar{N}_{ij}$ and $\bar{L}_{ij}$, $i, j = 1, 2$ to satisfy (20) have been proved in [1, 2]. They are finally reduced to the conditions on submatrices in the form:

$$\begin{aligned} \bar{M}_{12}\bar{M}_{22}^{i-2}\bar{M}_{21} &= 0, \ \text{for } i = 2, \ldots, \tilde{n}, \\ \bar{M}_{12}\bar{M}_{22}^{i-2}\bar{N}_{21} &= 0, \ \text{for } i = 2, \ldots, \tilde{n}, \\ \bar{L}_{12}\bar{M}_{22}^{i-2}\bar{M}_{21} &= 0, \ \text{for } i = 2, \ldots, \tilde{n}, \\ \bar{L}_{12}\bar{M}_{22}^{i-2}\bar{N}_{21} &= 0, \ \text{for } i = 2, \ldots, \tilde{n} + 1, \end{aligned} \tag{22}$$

where $\bar{M} = \begin{pmatrix} 0 & \bar{M}_{12} \\ \bar{M}_{21} & \bar{M}_{22} \end{pmatrix}$, $\bar{N} = \begin{pmatrix} 0 & \bar{N}_{12} \\ \bar{N}_{21} & \bar{N}_{22} \end{pmatrix}$ and $\bar{L} = \begin{pmatrix} 0 & \bar{L}_{12} \\ \bar{L}_{21} & \bar{L}_{22} \end{pmatrix}$.

Similarly, the requirements given in Theorem 2 imply in the new basis that the complementary matrices $\bar{M}$, $\bar{N}$ and $\bar{L}$ have the following structure:

$$\bar{M} = \begin{pmatrix} 0 & 0 \\ \bar{M}_{21} & \bar{M}_{22} \end{pmatrix}, \qquad \bar{N} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \bar{L} = \begin{pmatrix} 0 & 0 \\ \bar{L}_{21} & \bar{L}_{22} \end{pmatrix}. \tag{23}$$

3.3. Expansion-contraction process of dynamic controllers

Analogously to the expansion-contraction of systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$, consider the following schemes for expansion/contraction of controllers $\mathbf{C}$ and $\tilde{\mathbf{C}}$:

$$\begin{array}{ccccc} \mathbf{C} & \longrightarrow & \tilde{\mathbf{C}} & \longrightarrow & \mathbf{C}, \\ \mathbb{R}^p & \xrightarrow{E} & \mathbb{R}^{\tilde{p}} & \xrightarrow{D} & \mathbb{R}^p \end{array} \tag{24}$$

and

$$\begin{array}{ccccccccc} \mathbf{C} & \longrightarrow & \tilde{\mathbf{C}} & \longrightarrow & \tilde{\tilde{\mathbf{C}}} & \longrightarrow & \tilde{\mathbf{C}} & \longrightarrow & \mathbf{C}, \\ \mathbb{R}^{p} & \xrightarrow{E} & \mathbb{R}^{\tilde{p}} & \xrightarrow{T_F^{-1}} & \bar{\mathbb{R}}^{\tilde{p}} & \xrightarrow{T_F} & \mathbb{R}^{\tilde{p}} & \xrightarrow{D} & \mathbb{R}^{r}, \end{array} \tag{25}$$

where $T_F = (E \;\; W_D)$ and W_D satisfies $\operatorname{Im} W_D = \operatorname{Ker} D$ and where $p+p_2 = \tilde{p}$. Consider the complementary matrices $M_F = \left(M_{F_{ij}}\right)$, $Y_P = \left(Y_{P_{ij}}\right)$, $N_G = \left(N_{G_{ij}}\right)$, $L_H = \left(L_{H_{ij}}\right)$, $J_K = \left(J_{K_{ij}}\right)$ for $i, j = 1, \ldots, 4$ in $\tilde{\mathbf{C}}$. Consider the block matrices in $\tilde{\tilde{\mathbf{C}}}$ as follows:

$$\begin{aligned} &\bar{M}_F = \begin{pmatrix} \bar{M}_{F_{11}} & \bar{M}_{F_{12}} \\ \bar{M}_{F_{21}} & \bar{M}_{F_{22}} \end{pmatrix}, \quad \bar{Y}_P = \begin{pmatrix} \bar{Y}_{P_{11}} & \bar{Y}_{P_{12}} \\ \bar{Y}_{P_{21}} & \bar{Y}_{P_{22}} \end{pmatrix}, \quad \bar{N}_G = \begin{pmatrix} \bar{N}_{G_{11}} & \bar{N}_{G_{12}} \\ \bar{N}_{G_{21}} & \bar{N}_{G_{22}} \end{pmatrix}, \\ &\bar{L}_H = \begin{pmatrix} \bar{L}_{H_{11}} & \bar{L}_{H_{12}} \\ \bar{L}_{H_{21}} & \bar{L}_{H_{22}} \end{pmatrix}, \quad \bar{J}_K = \begin{pmatrix} \bar{J}_{K_{11}} & \bar{J}_{K_{12}} \\ \bar{J}_{K_{21}} & \bar{J}_{K_{22}} \end{pmatrix}, \end{aligned} \tag{26}$$

where $\bar{M}_{F_{11}}$, $\bar{M}_{F_{22}}$ are $p \times p$, $p_2 \times p_2$ matrices, respectively. $\bar{Y}_{P_{11}}$, $\bar{Y}_{P_{22}}$ are $p \times m$, $p_2 \times m_2$ matrices, respectively. $\bar{N}_{G_{11}}$, $\bar{N}_{G_{22}}$ are $p \times l$, $p_2 \times l_2$ matrices, respectively. $\bar{L}_{H_{11}}$, $\bar{L}_{H_{22}}$ are $m \times p$, $m_2 \times p_2$ matrices, respectively. $\bar{J}_{K_{11}}$, $\bar{J}_{K_{22}}$ are $m \times l$, $m_2 \times l_2$ matrices, respectively.

Theorem 8. A controller $\mathbf{C}$ for $\mathbf{S}$ is expandable to the controller $\tilde{\tilde{\mathbf{C}}}$ of $\tilde{\tilde{\mathbf{S}}}$ by Definition 2 if and only if

$$\begin{aligned}
&\text{a)} \;\; \bar{M}_{F_{11}} = 0, \quad \bar{M}_{F_{12}} \bar{M}^{i}_{F_{22}} \bar{M}_{F_{21}} = 0, \\
&\text{b)} \;\; \bar{N}_{G_{11}} C + \bar{N}_{G_{12}} \bar{L}_{21} = 0, \quad \bar{M}_{F_{12}} \bar{M}^{i}_{F_{22}} \left(\bar{N}_{G_{21}} C + \bar{N}_{G_{22}} \bar{L}_{21}\right) = 0, \\
&\qquad \left(\bar{N}_{G_{11}} \bar{L}_{12} + \bar{N}_{G_{12}} \bar{L}_{22}\right) \bar{M}^{j}_{22} \bar{M}_{21} = 0, \quad \bar{M}_{F_{12}} \bar{M}^{i}_{F_{22}} \bar{N}_{G_{22}} \bar{L}_{22} \bar{M}^{j}_{22} \bar{M}_{21} = 0, \\
&\text{c)} \;\; \bar{N}_{G_{12}} \bar{L}_{22} \bar{M}^{j}_{22} \bar{N}_{21} = 0, \quad \bar{M}_{F_{12}} \bar{M}^{i}_{F_{22}} \bar{N}_{G_{22}} \bar{L}_{22} \bar{M}^{j}_{22} \bar{N}_{21} = 0, \\
&\text{d)} \;\; \bar{Y}_{P_{11}} = 0, \quad \bar{M}_{F_{12}} \bar{M}^{i}_{F_{22}} \bar{Y}_{P_{21}} = 0, \\
&\text{e)} \;\; \bar{L}_{H_{11}} = 0, \quad \bar{L}_{H_{21}} = 0, \quad \bar{L}_{H_{12}} \bar{M}^{i}_{F_{22}} \bar{M}_{F_{21}} = 0, \quad \bar{L}_{H_{22}} \bar{M}^{i}_{F_{22}} \bar{M}_{F_{21}} = 0, \\
&\text{f)} \;\; \bar{L}_{H_{12}} \bar{M}^{i}_{F_{22}} \bar{Y}_{P_{21}} = 0, \quad \bar{L}_{H_{22}} \bar{M}^{i}_{F_{22}} \bar{Y}_{P_{21}} = 0, \\
&\text{g)} \;\; \bar{L}_{H_{12}} \bar{M}^{i}_{F_{22}} \left(\bar{N}_{G_{21}} C + \bar{N}_{G_{22}} \bar{L}_{21}\right) = 0, \quad \bar{L}_{H_{22}} \bar{M}^{i}_{F_{22}} \left(\bar{N}_{G_{21}} C + \bar{N}_{G_{22}} \bar{L}_{21}\right) = 0, \\
&\qquad \bar{L}_{H_{12}} \bar{M}^{i}_{F_{22}} \bar{N}_{G_{22}} \bar{L}_{22} \bar{M}^{j}_{22} \bar{M}_{21} = 0, \quad \bar{L}_{H_{22}} \bar{M}^{i}_{F_{22}} \bar{N}_{G_{22}} \bar{L}_{22} \bar{M}^{j}_{22} \bar{M}_{21} = 0, \\
&\text{h)} \;\; \bar{L}_{H_{12}} \bar{M}^{i}_{F_{22}} \bar{N}_{G_{22}} \bar{L}_{22} \bar{M}^{j}_{22} \bar{N}_{21} = 0, \quad \bar{L}_{H_{22}} \bar{M}^{i}_{F_{22}} \bar{N}_{G_{22}} \bar{L}_{22} \bar{M}^{j}_{22} \bar{N}_{21} = 0, \\
&\text{i)} \;\; \bar{J}_{K_{11}} C + \bar{J}_{K_{12}} \bar{L}_{21} = 0, \quad \bar{J}_{K_{21}} C + \bar{J}_{K_{22}} \bar{L}_{21} = 0, \\
&\qquad \bar{J}_{K_{12}} \bar{L}_{22} \bar{M}^{i}_{22} \bar{M}_{21} = 0, \quad \bar{J}_{K_{22}} \bar{L}_{22} \bar{M}^{i}_{22} \bar{M}_{21} = 0, \\
&\text{j)} \;\; \bar{J}_{K_{12}} \bar{L}_{22} \bar{M}^{i}_{22} \bar{N}_{21} = 0, \quad \bar{J}_{K_{22}} \bar{L}_{22} \bar{M}^{i}_{22} \bar{N}_{21} = 0
\end{aligned} \tag{27}$$

hold for all i, $j \geq 0$, where the matrices $\bar{M}_F$, $\bar{Y}_P$, $\bar{N}_G$, $\bar{L}_H$, $\bar{J}_K$ have the structure given in (26).

Proof. The proof follows a similar way for all items a)–j). Because of this, we prove only the conditions a) and b). Proof of part a): Consider the relation a) given in (9) in the new basis, that is, $\bar{D}\bar{M}_F^{i+1}\bar{E} = 0$. We obtain $\bar{M}_{F_{11}} = 0$ for $i = 0$. We get $\bar{M}_{F_{12}}\bar{M}_{F_{21}} = 0$ for $i = 1$. In general, $\bar{M}_{F_{12}}\bar{M}_{F_{22}}^{i-1}\bar{M}_{F_{21}} = 0$ holds for $i \geq 1$. Then, $\bar{M}_{F_{12}}\bar{M}_{F_{22}}^{i}\bar{M}_{F_{21}} = 0$ for all $i \geq 0$. This proves a). Proof of part b): Consider the first relation of b) given in (9) when consider the new basis, $\bar{D}\bar{M}_F^{i}\bar{N}_G\left(\bar{T}C + \bar{L}\bar{V}\right) = 0$. We obtain $\bar{N}_{G_{11}}C + \bar{N}_{G_{12}}\bar{L}_{21} = 0$ for $i = 0$. We get $\bar{M}_{F_{12}}\bar{M}_{F_{22}}^{i-1}\left(\bar{N}_{G_{21}}C + \bar{N}_{G_{22}}\bar{L}_{21}\right) = 0$ for all $i \geq 1$, that is, $\bar{M}_{F_{12}}\bar{M}_{F_{22}}^{i}\left(\bar{N}_{G_{21}}C + \bar{N}_{G_{22}}\bar{L}_{21}\right) = 0$ for all $i \geq 0$. From the second condition of b) in (9), $\bar{D}\bar{M}_F^{i}\bar{N}_G\bar{L}\bar{M}^{j+1}\bar{V} = 0$, we obtain $\left(\bar{N}_{G_{11}}\bar{L}_{12} + \bar{N}_{G_{12}}\bar{L}_{22}\right)$ $\bar{M}_{22}^{j}\bar{M}_{21} = 0$ for $i = 0$, $j \geq 0$ and $\bar{M}_{F_{12}}\bar{M}_{F_{22}}^{i-1}\bar{N}_{G_{22}}\bar{L}_{22}\bar{M}_{22}^{j}\bar{M}_{21} = 0$ for all $i \geq 1$, $j \geq 0$. Therefore, $\bar{M}_{F_{12}}\bar{M}_{F_{22}}^{i}\bar{N}_{G_{22}}\bar{L}_{22}\bar{M}_{22}^{j}\bar{M}_{21} = 0$ holds for all i, $j \geq 0$. This proves part b). □

Theorem 9. A controller $\tilde{\bar{\mathbf{C}}}$ for $\tilde{\bar{\mathbf{S}}}$ is contractible to the controller $\mathbf{C}$ of $\mathbf{S}$ by Definition 4 if and only if

$$\text{a) } \bar{M}_{F_{11}} = 0,\ \bar{M}_{F_{12}} = 0, \quad \text{b) } \bar{N}_{G_{11}}C = 0, \quad \text{c) } \bar{Y}_{P_{11}} = 0,\ \bar{Y}_{P_{12}} = 0,$$
$$\text{d) } \bar{L}_{H_{11}} = 0,\ \bar{L}_{H_{12}} = 0, \quad \text{e) } \bar{J}_{K_{11}} = 0 \tag{28}$$

hold, where the matrices $\bar{M}_F$, $\bar{Y}_P$, $\bar{N}_G$, $\bar{L}_H$, $\bar{J}_K$ have the structure given in (26).

Proof. The proof is made directly by using Theorem 7. □

3.4. Explicit sufficient conditions for contractibility

There exist, of course, infinite number of possibilities how to select the complementary matrices satisfying Theorems 8 and 9. The above conditions on the complementary submatrices are more flexible than the corresponding relations given by Theorems 6 and 7. However, when designing control laws, the designer needs to know explicit conditions assuring the contractibility of the controllers. The particular conditions stated in the following propositions give us packs of sufficient requirements for blocks of complementary matrices satisfying Theorems 8 and 9, respectively.

Proposition 1. A controller $\mathbf{C}$ for $\mathbf{S}$ is expandable to the controller $\tilde{\bar{\mathbf{C}}}$ of $\tilde{\bar{\mathbf{S}}}$ by Definition 2 if $\bar{L}_{21} = 0$, $\bar{M}_{F_{11}} = 0$, $\bar{M}_{F_{21}} = 0$, $\bar{Y}_{P_{11}} = 0$, $\bar{Y}_{P_{21}} = 0$, $\bar{N}_{G_{11}} = 0$, $\bar{N}_{G_{21}} = 0$, $\bar{L}_{H_{11}} = 0$, $\bar{L}_{H_{21}} = 0$, $\bar{J}_{K_{11}} = 0$, $\bar{J}_{K_{21}} = 0$ and either

$$\text{a) } \bar{L}_{22} = 0 \quad \text{or} \quad \text{b) } \bar{M}_{21} = 0,\ \bar{N}_{21} = 0 \tag{29}$$

hold, where the matrices $\bar{M}_F$, $\bar{Y}_P$, $\bar{N}_G$, $\bar{L}_H$, $\bar{J}_K$ have the structure given in (26).

Proof. The proof is made by substituting the above conditions into the relations a) – j) given by Theorem 8. □

Proposition 2. A controller $\tilde{\tilde{\mathbf{C}}}$ for $\tilde{\tilde{\mathbf{S}}}$ is contractible to the controller $\mathbf{C}$ of $\mathbf{S}$ by Definition 4 if $\bar{M}_{F_{11}} = 0$, $\bar{M}_{F_{12}} = 0$, $\bar{Y}_{P_{11}} = 0$, $\bar{Y}_{P_{12}} = 0$, $\bar{N}_{G_{11}} = 0$, $\bar{L}_{H_{11}} = 0$, $\bar{L}_{H_{12}} = 0$, $\bar{J}_{K_{11}} = 0$ hold, where the matrices $\bar{M}_F$, $\bar{Y}_P$, $\bar{N}_G$, $\bar{L}_H$, $\bar{J}_K$ have the structure given in (26).

Proof. The proof is made by substituting the above conditions into the relations a) – e) given by Theorem 9. □

These conditions are better readable in the block matrix form as follows.

Proposition 3. A controller $\mathbf{C}$ for $\mathbf{S}$ is expandable to the controller $\tilde{\tilde{\mathbf{C}}}$ of $\tilde{\tilde{\mathbf{S}}}$ by Definition 2 if $\bar{M}_F = \begin{pmatrix} 0 & \bar{M}_{F_{12}} \\ 0 & \bar{M}_{F_{22}} \end{pmatrix}$, $\bar{Y}_P = \begin{pmatrix} 0 & \bar{Y}_{P_{12}} \\ 0 & \bar{Y}_{P_{22}} \end{pmatrix}$, $\bar{N}_G = \begin{pmatrix} 0 & \bar{N}_{G_{12}} \\ 0 & \bar{N}_{G_{22}} \end{pmatrix}$, $\bar{L}_H = \begin{pmatrix} 0 & \bar{L}_{H_{12}} \\ 0 & \bar{L}_{H_{22}} \end{pmatrix}$, $\bar{J}_K = \begin{pmatrix} 0 & \bar{J}_{K_{12}} \\ 0 & \bar{J}_{K_{22}} \end{pmatrix}$ and either

$$\text{a)}\ \bar{L} = \begin{pmatrix} 0 & \bar{L}_{12} \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \text{b)}\ \bar{M} = \begin{pmatrix} 0 & \bar{M}_{12} \\ 0 & \bar{M}_{22} \end{pmatrix}, \bar{N} = \begin{pmatrix} 0 & \bar{N}_{12} \\ 0 & \bar{N}_{22} \end{pmatrix}, \bar{L} = \begin{pmatrix} 0 & \bar{L}_{12} \\ 0 & \bar{L}_{22} \end{pmatrix} \tag{30}$$

hold, where the matrices $\bar{M}_F$, $\bar{Y}_P$, $\bar{N}_G$, $\bar{L}_H$, $\bar{J}_K$ have the structure given in (26).

Proof. The proof is straightforward when using Proposition 1. □

Proposition 4. A controller $\tilde{\tilde{\mathbf{C}}}$ for $\tilde{\tilde{\mathbf{S}}}$ is contractible to the controller $\mathbf{C}$ of $\mathbf{S}$ by Definition 4 if $\bar{M}_F = \begin{pmatrix} 0 & 0 \\ \bar{M}_{F_{21}} & \bar{M}_{F_{22}} \end{pmatrix}$, $\bar{Y}_P = \begin{pmatrix} 0 & 0 \\ \bar{Y}_{P_{21}} & \bar{Y}_{P_{22}} \end{pmatrix}$, $\bar{N}_G = \begin{pmatrix} 0 & \bar{N}_{G_{12}} \\ \bar{N}_{G_{21}} & \bar{N}_{G_{22}} \end{pmatrix}$, $\bar{L}_H = \begin{pmatrix} 0 & 0 \\ \bar{L}_{H_{21}} & \bar{L}_{H_{22}} \end{pmatrix}$, $\bar{J}_K = \begin{pmatrix} 0 & \bar{J}_{K_{12}} \\ \bar{J}_{K_{21}} & \bar{J}_{K_{22}} \end{pmatrix}$ hold, where the matrices $\bar{M}_F$, $\bar{Y}_P$, $\bar{N}_G$, $\bar{L}_H$, $\bar{J}_K$ have the structure given in (26).

Proof. The proof is straightforward when using Proposition 2. □

3.5. Particular selection of complementary matrices

Consider the overlapping structure defined by blocks in (11). Let us use the following specific transformation matrices V, R, T and E to define the expansion/contraction process:

$$V = \begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \end{pmatrix}, \quad R = \begin{pmatrix} I_{m_1} & 0 & 0 \\ 0 & I_{m_2} & 0 \\ 0 & I_{m_2} & 0 \\ 0 & 0 & I_{m_3} \end{pmatrix},$$
$$T = \begin{pmatrix} I_{l_1} & 0 & 0 \\ 0 & I_{l_2} & 0 \\ 0 & I_{l_2} & 0 \\ 0 & 0 & I_{l_3} \end{pmatrix}, \quad E = \begin{pmatrix} I_{p_1} & 0 & 0 \\ 0 & I_{p_2} & 0 \\ 0 & I_{p_2} & 0 \\ 0 & 0 & I_{p_3} \end{pmatrix}. \tag{31}$$

The changes of basis (15) for the expanded system and (25) for the controller are given by the matrices

$$T_A = \begin{pmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & I_{n_2} & 0 & I_{n_2} \\ 0 & I_{n_2} & 0 & -I_{n_2} \\ 0 & 0 & I_{n_3} & 0 \end{pmatrix}, \quad T_A^{-1} = \begin{pmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & \frac{1}{2}I_{n_2} & \frac{1}{2}I_{n_2} & 0 \\ 0 & 0 & 0 & I_{n_3} \\ 0 & \frac{1}{2}I_{n_2} & -\frac{1}{2}I_{n_2} & 0 \end{pmatrix}. \tag{32}$$

I_{n_i} denote the identity matrices of orders n_i, $i = 1, 2, 3$. Analogously for T_B, T_B^{-1}, T_C, T_C^{-1} and T_F, T_F^{-1} in schemes (13) and (25), respectively. The complementary matrix M in $\tilde{\mathbf{S}}$ by using Definition 1 has the following form:

$$M = \begin{pmatrix} 0 & M_{12} & -M_{12} & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} \\ -M_{21} & -(M_{22}+M_{23}+M_{33}) & M_{33} & -M_{24} \\ 0 & M_{42} & -M_{42} & 0 \end{pmatrix}. \tag{33}$$

The complementary matrices N and L have the same structure as the matrix M in (33). The complementary matrices M, N and L in $\tilde{\mathbf{S}}$ by using Definition 3 are as follows [1, 2]:

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} \\ -M_{21} & -M_{22} & -M_{23} & -M_{24} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N = 0, \quad L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ L_{21} & L_{22} & L_{23} & L_{24} \\ -L_{21} & -L_{22} & -L_{23} & -L_{24} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{34}$$

The propositions derived in the previous subsection have the following form in the initial basis when selecting the transformation matrices (31).

Proposition 5. A controller $\mathbf{C}$ for $\mathbf{S}$ is expandable to the controller $\tilde{\mathbf{C}}$ of $\tilde{\mathbf{S}}$ by Definition 2 if

$$M_F = \begin{pmatrix} 0 & M_{F_{12}} & -M_{F_{12}} & 0 \\ 0 & M_{F_{22}} & -M_{F_{22}} & 0 \\ 0 & M_{F_{32}} & -M_{F_{32}} & 0 \\ 0 & M_{F_{42}} & -M_{F_{42}} & 0 \end{pmatrix}, \quad Y_P = \begin{pmatrix} 0 & Y_{P_{12}} & -Y_{P_{12}} & 0 \\ 0 & Y_{P_{22}} & -Y_{P_{22}} & 0 \\ 0 & Y_{P_{32}} & -Y_{P_{32}} & 0 \\ 0 & Y_{P_{42}} & -Y_{P_{42}} & 0 \end{pmatrix},$$

$$N_G = \begin{pmatrix} 0 & N_{G_{12}} & -N_{G_{12}} & 0 \\ 0 & N_{G_{22}} & -N_{G_{22}} & 0 \\ 0 & N_{G_{32}} & -N_{G_{32}} & 0 \\ 0 & N_{G_{42}} & -N_{G_{42}} & 0 \end{pmatrix}, \quad L_H = \begin{pmatrix} 0 & L_{H_{12}} & -L_{H_{12}} & 0 \\ 0 & L_{H_{22}} & -L_{H_{22}} & 0 \\ 0 & L_{H_{32}} & -L_{H_{32}} & 0 \\ 0 & L_{H_{42}} & -L_{H_{42}} & 0 \end{pmatrix}, \quad J_K = \begin{pmatrix} 0 & J_{K_{12}} & -J_{K_{12}} & 0 \\ 0 & J_{K_{22}} & -J_{K_{22}} & 0 \\ 0 & J_{K_{32}} & -J_{K_{32}} & 0 \\ 0 & J_{K_{42}} & -J_{K_{42}} & 0 \end{pmatrix}$$

and either

$$\text{a)}\ L = \begin{pmatrix} 0 & L_{12} & -L_{12} & 0 \\ 0 & L_{22} & -L_{22} & 0 \\ 0 & L_{22} & -L_{22} & 0 \\ 0 & L_{42} & -L_{42} & 0 \end{pmatrix} \quad \text{or}$$

$$\text{b)}\ M = \begin{pmatrix} 0 & M_{12} & -M_{12} & 0 \\ 0 & M_{22} & -M_{22} & 0 \\ 0 & M_{32} & -M_{32} & 0 \\ 0 & M_{42} & -M_{42} & 0 \end{pmatrix},\ N = \begin{pmatrix} 0 & N_{12} & -N_{12} & 0 \\ 0 & N_{22} & -N_{22} & 0 \\ 0 & N_{32} & -N_{32} & 0 \\ 0 & N_{42} & -N_{42} & 0 \end{pmatrix},\ L = \begin{pmatrix} 0 & L_{12} & -L_{12} & 0 \\ 0 & L_{22} & -L_{22} & 0 \\ 0 & L_{32} & -L_{32} & 0 \\ 0 & L_{42} & -L_{42} & 0 \end{pmatrix} \tag{35}$$

hold.

Proof. The proof is straightforward when using Proposition 3. □

Proposition 6. A controller $\tilde{\mathbf{C}}$ for $\tilde{\mathbf{S}}$ is contractible to $\mathbf{C}$ of $\mathbf{S}$ by Definition 4 if the matrices M_F, Y_P, N_G, L_H and J_K have the following form:

$$M_F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ M_{F21} & M_{F22} & M_{F23} & M_{F24} \\ -M_{F21} & -M_{F22} & -M_{F23} & -M_{F24} \\ 0 & 0 & 0 & 0 \end{pmatrix},\ Y_P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ Y_{P21} & Y_{P22} & Y_{P23} & Y_{P24} \\ -Y_{P21} & -Y_{P22} & -Y_{P23} & -Y_{P24} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$N_G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ N_{G21} & N_{G22} & N_{G23} & N_{G24} \\ -N_{G21} & -N_{G22} & -N_{G23} & -N_{G24} \\ 0 & 0 & 0 & 0 \end{pmatrix},\ L_H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ L_{H21} & L_{H22} & L_{H23} & L_{H24} \\ -L_{H21} & -L_{H22} & -L_{H23} & -L_{H24} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{36}$$

$$J_K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ J_{K21} & J_{K22} & J_{K23} & J_{K24} \\ -J_{K21} & -J_{K22} & -J_{K23} & -J_{K24} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Proof. The proof is straightforward when using Proposition 4. □

Remark. Suppose that the controller $\mathbf{C}$ is not a dynamic controller but a simple static output feedback. In this situation, the control laws u and $\tilde{u}$ given in (2) have been reduced to $u = Ky + v$ and $\tilde{u} = \tilde{K}\tilde{y} + \tilde{v}$, respectively, and $\tilde{F}$, $\tilde{P}$, $\tilde{G}$, $\tilde{H}$ are zero matrices. Then, the conditions a) – j) in (6) given by Theorem 3 have been reduced only to conditions i) – j), that is, the control law $\tilde{u} = \tilde{K}\tilde{y} + \tilde{v}$ is contractible to $u = Ky + v$ if and only if $\tilde{K}\tilde{C}\tilde{A}^i V = RKCA^i$ and $\tilde{K}\tilde{C}\tilde{A}^i\tilde{B}R = RKCA^iB$ [7, 13].

4. CONCLUSION

The main result contributed by this paper is a systematic presentation of a set of contractibility conditions for dynamic controllers for linear time-invariant systems in terms of the complementary matrices involved in the expansion/contraction framework of the Inclusion Principle. Contractibility means that a controller is designed in one of the systems in such a way that it is guaranteed that the closed-loop system $(\tilde{\mathbf{S}}, \tilde{\mathbf{C}})$ is an expansion of the closed-loop system $(\mathbf{S}, \mathbf{C})$ in the sense of the Inclusion Principle. For general expansion/contraction transformations, necessary and sufficient conditions for contractibility are proved. These conditions are twofold: first, they involve unstructured complementary matrices; second, they involve complementary matrices with certain block structure. The block structure offers a higher degree of freedom in selection of complementary matrices as compared with previous well known results. Further, this block structure is exploited to obtain explicit sufficient requirements for blocks of complementary matrices ensuring contractibility. This is useful for enabling flexible choices of such matrices. Specific choices are finally given for a particular class of expansion/contraction transformation matrices. The results are derived in parallel for two important cases of the Inclusion Principle. The first case considers expandable controllers, i. e. the control is designed without any restriction for the small system and then expanded into bigger system. The second case considers extensions; i. e. the control is designed without any restrictions for bigger system and then contracted into small system for implementation. This case is important for overlapping decentralized control design.

ACKNOWLEDGEMENTS

This work was supported in part by the DURSI under Grant PIV, by the AS CR under Grant A2075802 and by the CICYT under Grant DPI2002-04018-C02-01.

(Received August 6, 2002.)

REFERENCES

[1] L. Bakule, J. Rodellar, and J. M. Rossell: Generalized selection of complementary matrices in the inclusion principle. IEEE Trans. Automat. Control *45* (2000), 6, 1237–1243.

[2] L. Bakule, J. Rodellar, and J. M. Rossell: Structure of expansion–contraction matrices in the inclusion principle for dynamic systems. SIAM J. Matrix Anal. Appl. *21* (2000), 4, 1136–1155.

[3] L. Bakule, J. Rodellar, and J. M. Rossell: Controllability–observability of expanded composite systems. Linear Algebra Appl. *332–334* (2001), 381–400.

[4] L. Bakule, J. Rodellar, and J. M. Rossell: Overlapping quadratic optimal control of linear time–varying commutative systems. SIAM J. Control Optim. *40* (2002), 5, 1611–1627.

[5] L. Bakule, J. Rodellar, J. M. Rossell, and P. Rubió: Preservation of controllability–observability in expanded systems. IEEE Trans. Automat. Control *46* (2001), 7, 1155–1162.

[6] M. Hodžić and D. D. Šiljak: Decentralized estimation and control with overlapping information sets. IEEE Trans. Automat. Control *31* (1986), 1, 83–86.

[7] A. İftar: Decentralized optimal control with extension. In: Proc. First IFAC Symposium on Design Methods of Control Systems, Zurich 1991, pp. 747–752.

[8] A. İftar: Decentralized estimation and control with overlapping input, state, and output decomposition. Automatica *29* (1993), 2, 511–516.

[9] A. İftar: Overlapping decentralized dynamic optimal control. Internat. J. Control *58* (1993), 1, 187–209.

[10] A. İftar and Ü. Özgüner: Contractible controller design and optimal control with state and input inclusion. Automatica *26* (1990), 3, 593–597.

[11] M. Ikeda and D. D. Šiljak: Overlapping decompositions, expansions and contractions of dynamic systems. Large Scale Systems *1* (1980), 1, 29–38.

[12] M. Ikeda and D. D. Šiljak: Generalized decompositions of dynamic systems and vector Lyapunov functions. IEEE Trans. Automat. Control *26* (1981), 5, 1118–1125.

[13] M. Ikeda and D. D. Šiljak: Overlapping decentralized control with input, state, and output inclusion. Control–Theory and Advanced Technology *2* (1986), 2, 155–172.

[14] M. Ikeda, D. D. Šiljak, and D. E. White: Decentralized control with overlapping information sets. J. Optim. Theory App. *34* (1981), 2, 279–310.

[15] M. Ikeda, D. D. Šiljak, and D. E. White: An inclusion principle for dynamic systems. IEEE Trans. Automat. Control *29* (1984), 3, 244–249.

[16] S. S. Stanković and D. D. Šiljak: Contractibility of overlapping decentralized control. Systems Control Lett. *44* (2001), 189–200.

[17] D. D. Šiljak: Decentralized Control of Complex Systems. Academic Press, New York 1991.

[18] D. D. Šiljak, S. M. Mladenović, and S. S. Stanković: Overlapping decentralized observation and control of a platoon of vehicles. In: Proc. American Control Conference, San Diego 1999, pp. 4522–4526.

Lubomír Bakule, Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 08 Praha 8. Czech Republic.
e-mail: bakule@utia.cas.cz

José Rodellar, Department of Applied Mathematics III, Technical University of Catalonia (UPC), Campus Nord, C-2, 08034-Barcelona. Spain.
e-mail: jose.rodellar@upc.es

Josep M. Rossell, Department of Applied Mathematics III, Technical University of Catalonia (UPC), Campus Manresa, 08240-Manresa (Barcelona). Spain.
e-mail: josep.maria.rossell@upc.es