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ERLANG DISTRIBUTED ACTIVITY TIMES IN STOCHASTIC ACTIVITY NETWORKS

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It is assumed that activity times in stochastic activity networks (SANs) are independent Erlang random variable (r.v.). A recurrence method of determining the kth moments of the completion time is presented. Applications are provided for illustration and are used to evaluate the applicability and appropriateness of the Erlang model to represent activity network.

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1. INTRODUCTION

Stochastic activity networks (SANs) can be represented by a priori cumulative distribution function (c.d.f.). Normally distributed is proposed by Sculli [15] and Kamburowski [7], exponentially distributed is assumed by Kamburowski [8], Kulkarni and Adlakha [10], and by Magott and Skudlarski [12]. Bendell et al [1] considered the problem of using the Erlang distribution as a representation of activity times. Their method based on the moments approach which is a more practical alternative to both the analytical and the numerical integration. However, they derived the first four central moments of $\max(X_1, X_2)$ and $X_1 + X_2$ only where X_1 and X_2 are independent r.v.'s. A survey of recent developments and complexity in SANs can be found in Elmaghraby [4] and [5]. This paper generalized the work of Bendell et al [1]. It presents the kth moments of the $\max(X_1, X_2, \ldots, X_n)$ and the c.d.f. of the sum of n independent r.v.'s is also given.

SANs are defined as $(N, A, F(\cdot))$, where N is the set of nodes $N = \{1, 2, \ldots, n\}$, A is the set of arcs $A = \{a_1, a_2, \ldots, a_m\}$ and $F(t) = \mathsf{P}(T_r \le t)$, for $t > 0$ is the c.d.f., where T_r is a r.v. which describes the duration of the arc a_r. The network has one starting and one ending node and is acyclic, i.e., the nodes are numbered in such a way that whenever there exists an $\mathrm{arc}(i, j)$, then $i < j$. We shall use the names "activity" and "arc", "event" and "node", "project" and "network" synonymously.

The main problem in SANs (largest and shortest path) is divided into two categories. The first category is to find the distribution of

$$Y = X_1 + X_2 + \cdots + X_n \tag{1}$$

where the r.v.'s $X_1, X_2, \cdots, X_n$ represent the time of the project activities. The distribution of this sum can be achieved through the convolution operation.

The second category is to find the distribution of

$$X_{n:n} = \max(X_1, X_2, \cdots, X_n) \quad \text{and} \quad X_{1:n} = \min(X_1, X_2, \cdots, X_n) \tag{2}$$

that is, finding $H_n(t) = \mathsf{P}(X_{n:n} \le t)$ and $L_n(t) = \mathsf{P}(X_{1:n} \le t)$ or finding the kth moments of $X_{n:n}$ and $X_{1:n}$ or approximating them.

We shall derive the distribution of (1) and the kth moments of (2) that is, $\mu_{1:n}^{(k)} = \mathsf{E}(X_{1:n}^k)$ and $\mu_{n:n}^{(k)} = \mathsf{E}(X_{n:n}^k)$, of the minimum and the maximum of a sample of size n from the Erlang distribution. Two applications are examined in light of the Erlang model. These applications are provided for illustration and are used to evaluate the applicability and appropriateness of the Erlang model to represent activity network.

We finish this section with some definitions, see Ross [14]. Let $F(x)$ be the c.d.f. of the r.v. X, $\overline{F}(x) = 1 - F(x)$ denotes the survival function.

Definition 1. A c.d.f. $F(x)$ is New Better than Used in Expectation (NBUE) if $F(x)$ has a finite mean μ and $\int_t^\infty \overline{F}(x)\,dx \le \mu\overline{F}(t)$ for $t \ge 0$.

Definition 2. Let X and Y are two r.v.'s with c.d.f.'s $F(x)$ and $F(y)$ respectively, Y is stochastically larger than X, written $Y \succ X$, if $\overline{F}(y) \ge \overline{F}(x)$.

Definition 3. The r.v. X is said to be more variable than Y, written $X \ge_v Y$ with $\mathsf{E}(X) = \mathsf{E}(Y)$, when $\mathsf{E}[\phi(X)] \ge \mathsf{E}[\phi(Y)]$ for all increasing and convex functions ϕ.

Proposition 1. If $F(x)$ is an NBUE distribution having mean μ, then

$$F \le_v \exp(\mu),$$

where $\exp(\mu)$ is the exponential distribution with mean μ.

The above proposition states that the exponential distribution is maximal variability in the class of NBUE distributions.

2. MAIN RESULTS

Let X_i be a r.v. that obeys the Erlang distribution with the probability density function (p.d.f.) given by

$$f_i(x) = \frac{\lambda_i^{m_i} x^{m_i-1} e^{-\lambda_i x}}{\Gamma(m_i)} \quad x \ge 0,\ \lambda_i > 0 \tag{3}$$

and the cumulative distribution function

$$F_i(x) = 1 - \sum_{t=0}^{m_i-1} \frac{(\lambda_i x)^t}{t!} e^{-\lambda_i x} \quad x \ge 0,\ \lambda_i > 0, \tag{4}$$

for $i = 1, 2, \ldots, n$, where m_i denotes to the shape parameter, λ_i denotes to the scale parameter and $\Gamma(\cdot)$ is the gamma function. The sum on the right-hand side involves the first m_i terms of the Poisson probability mass function. There is a strong relation between the exponential, the Erlang, and the Gamma d.f.'s. The Erlang distribution is a special case of the Gamma distribution with p.d.f., $f(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$ $x \geq 0$, λ and $\alpha > 0$, when $\alpha = m$ and m is a positive integer. The exponential distribution is obtained by letting $m = 1$. In addition, the Erlang r.v. results from the sum of m independent, exponentially distributed r.v.'s with parameter λ.

The following lemma gives the c.d.f. of (1) and Theorem 1 gives the kth moments of (2).

Lemma 1. The cumulative distribution function of (1) is given by

$$F(y) = \prod_{i=1}^{n} (\lambda_i)^{m_i} \sum_{i=1}^{n} \sum_{j=1}^{m_i} B_{ij} (\frac{1}{\lambda_i})^{m_i - j + 1} \left(1 - e^{-\lambda_i y} \sum_{r=0}^{m_i - j} \frac{(\lambda_i y)^r}{r!} \right) \tag{5}$$

where

$$B_{ij} = \frac{1}{(j-1)!} \frac{\mathrm{d}^{j-1}}{\mathrm{d}s^{j-1}} \left(\frac{(s + \lambda_i)^{m_i}}{\prod_{k=1}^{n} (s + \lambda_k)^{m_k}} \right) \tag{6}$$

evaluated at $s = -\lambda_i$ and $\frac{\mathrm{d}}{\mathrm{d}s}(\cdot)$ stands for the derivative.

Proof. The Laplace transform of the p.d.f. of (1) can be written as

$$f^*(s) = \prod_{i=1}^{n} \left(\frac{\lambda_i}{s + \lambda_i} \right)^{m_i}.$$

By using partial fraction expansion Kleinrock [9] (formulae (I40) and (I41)) showed that

$$f^*(s) = \prod_{i=1}^{n} (\lambda_i)^{m_i} \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{B_{ij}}{(s + \lambda_i)^{m_i - j + 1}}$$

where B_{ij} is defined in (6). Then the p.d.f. of (1) is given by

$$f(y) = \prod_{i=1}^{n} (\lambda_i)^{m_i} \sum_{i=1}^{n} \sum_{j=1}^{m_i} B_{ij} \frac{y^{m_i - j}}{(m_i - j)!} e^{-\lambda_i y}.$$

In consequence, the c.d.f. can be written as in (5). □

Theorem 1. Let X_i's be independent and non-identically r.v.'s that obey the Erlang distribution. For $n = 1, 2, \ldots$ and $k = 1, 2, \ldots$

$$\mu_{n:n}^{(k)} = k \sum_{j=1}^{n} (-1)^{j+1} I_j \tag{7}$$

where

$$I_j = \sum_{1\le i_1<i_2\ldots\le i_j\le n}\sum \sum_{t_1=0}^{m_{i_1}-1}\cdots\sum_{t_j=0}^{m_{i_j}-1}\frac{\Gamma(\sum_{s=1}^{j}t_s+k)}{(\sum_{s=1}^{j}\lambda_{i_s})^{\sum_{s=1}^{j}t_s+k}}\prod_{s=1}^{j}\frac{\lambda_{i_s}^{t_s}}{t_s!} \tag{8}$$

and

$$\mu_{1:n}^{(k)} = I_n. \tag{9}$$

Proof. By definition

$$\mu_{n:n}^{(k)} = k\int_0^\infty x^{k-1}(1-F_{n:n}(x))\mathrm{d}x$$

$$= k\int_0^\infty x^{k-1}\left(1-\prod_{i=1}^{n}(1-\sum_{t=0}^{m_i-1}\frac{(\lambda_i x)^t}{t!}e^{-\lambda_i x})\right)\mathrm{d}x$$

$$= k\int_0^\infty x^{k-1}\left(\sum_{i_1=1}^{n}\sum_{t_1=0}^{m_{i_1}-1}\frac{(\lambda_{i_1}x)^{t_1}}{t_1!}e^{-\lambda_{i_1}x}\right.$$

$$- \sum_{1\le i_1<i_2\le n}\sum\sum_{t_1=0}^{m_{i_1}-1}\sum_{t_2=0}^{m_{i_2}-1}\frac{\lambda_{i_1}^{t_1}\lambda_{i_2}^{t_2}}{t_1!t_2!}x^{t_1+t_2}\,e^{-(\lambda_{i_1}+\lambda_{i_2})x}$$

$$+ \sum_{1\le i_1<i_2\le i_3\le n}\sum\sum_{t_1=0}^{m_{i_1}-1}\sum_{t_2=0}^{m_{i_2}-1}\sum_{t_3=0}^{m_{i_3}-1}\frac{\lambda_{i_1}^{t_1}\lambda_{i_2}^{t_2}\lambda_{i_3}^{t_3}}{t_1!t_2!t_3!}x^{t_1+t_2+t_3}e^{-(\lambda_{i_1}+\lambda_{i_2}+\lambda_{i_3})x}$$

$$\left.+\cdots+(-1)^{n-1}\sum_{t_1=0}^{m_{i_1}-1}\cdots\sum_{t_n=0}^{m_{i_n}-1}\prod_{s=1}^{n}\frac{\lambda_{i_s}^{t_s}}{t_s!}x^{\sum_{s=1}^{n}t_s}e^{-x\sum_{s=1}^{n}\lambda_s}\right)\mathrm{d}x$$

where $F_{n:n}(x) = \prod_{i=1}^{n}F_i(x)$ and $F_{1:n}(x) = 1-\prod_{i=1}^{n}(1-F_i(x))$ are the c.d.f.'s of the $\max(X_1,X_2,\cdots,X_n)$ and the d.f. of the $\min(X_1,X_2,\cdots,X_n)$ respectively.

Upon using

$$\int_0^\infty x^{\beta+k-1}e^{-\alpha x}x = \frac{\Gamma(k+\beta)}{\alpha^{k+\beta}} \tag{10}$$

we get (7). The proof of (9) follows from the relation

$$\mu_{1:n}^{(k)} = k\int_0^\infty x^{k-1}\prod_{i=1}^{n}(1-F_i(x))\mathrm{d}x$$

$$= k\int_0^\infty x^{k-1}\prod_{i=1}^{n}\left(\sum_{t=0}^{m_i-1}\frac{(\lambda_i x)^t}{t!}e^{-\lambda_i x}\right)\mathrm{d}x$$

and using (10). □

Bounds for Erlang Estimate (EE). The Increasing Failure Rate (IFR) distribution class (recall that $F(x)$ belongs to the IFR class if $r(t) = \frac{f(t)}{1-F(t)}$ is increasing) contains all distributions proposed for describing the activity duration in SANs, see Kamburowski [8]. This class includes the exponential, the Weibull with parameter $\alpha \geq 1$, the Erlang, and all strongly unimodal distributions of non-negative r.v.'s. So, it is easy to argue that EE is located between a lower bound (LB) based on the critical path method (CPM) with deterministic model, see Devroye [2] and the upper bound (UB) with exponentially distributed times of activities which is given by Kamburowski [8]. Better lower bounds have been discussed by Fulkerson [6], Elmaghraby [3], and Robillard [13]. However, the computational effort for these procedures are increasing. A survey of the above methods, recent developments and complexity can be found in Elmaghraby [4] and [5], Tavares [16] and the papers cited herein.

Lemma 2. LB $\leq$ EE.

Proof. The relation follows by using Jensen's inequality

$$\psi[\mathsf{E}(X)] \leq \mathsf{E}[\psi(X)]$$

for each convex function ψ and each r.v. X, the higher moments are also follow from Jensen's inequality. □

Lemma 3. EE $\leq$ UB.

Proof. Since the Erlang distribution belongs to the IFR class and the exponential distribution is the maximal one on it, then the proof follows. □

3. APPLICATIONS

The Erlang distribution is used in this paper to represent the input distribution of activity duration. The thrust, the appropriateness and the advantages of using the Erlang distribution are discussed in Bendell et al [1] by providing some numerical examples to demonstrate the accuracy of the method. This paper devoted to compare the Erlang estimate (when $m = 2$ and 3) with the other distribution estimates such as: the exponential and the normal distribution.

In order to perform the comparison between EE and other distribution estimates, two example networks are given in Figures 1 and 2 with mean indicated on each arc. The comparison of the estimates is given in Tables 1 and 2, where C.V. denotes to the coefficient of variation $(=\frac{\sqrt{\text{variance}}}{\text{mean}})$.

To compute the mean completion time of the project, let X_i's $i = 1, 2, \ldots, n$ be n r.v.'s with

$$h(X_1, X_2, \cdots, X_n) = \mathsf{E}\max(X_1, X_2, \cdots, X_n). \tag{11}$$

The mean completion time of the node j is determined recursively by

$$\begin{aligned} W_1 &= 0 \\ W_j &= h(W_i + X_{ij},\ i \in B_j) \quad j = 2, 3, \ldots \end{aligned} \tag{12}$$

where B_j denotes the set of predecessors of arcs connected with the node j and X_{ij} denotes the random duration of the arc (i,j) which is assumed to be Erlang distributed. The other moments can be easily computed by using equations (7) and (12) with a little bit modifications. Some special cases, for the maximum of two and three r.v.'s, are given in the Appendix.

To sum up the computations for obtaining the kth moment completion time of the project, we present the following algorithm:

INPUT: k, m and $\mathsf{E}(X_{ij})$

OUTPUT: The completion time of node j, $j = 2, 3, \ldots, n$.

Step 1. Set $W_1 = 0$ (starting node).

Step 2. Compute W_j, for $j = 2, 3, 4, \ldots, n$ by using equations (12), (7) and (8).

Step 3. Go to Step 2.

Example 1. The network in Figure 1, for which the duration of each activity is exponentially distributed, is taken from Kamburowski [7]. Each activity duration is discretized by three values of equal probability. This was done in order to achieve the exact mean completion time through enumeration all possible realizations. The normal approach which is given by Sculli [15] is also presented. Table 1 summarizes the completion time for these estimates as well as the Erlang estimate for two different values ($m = 2$ and 3).

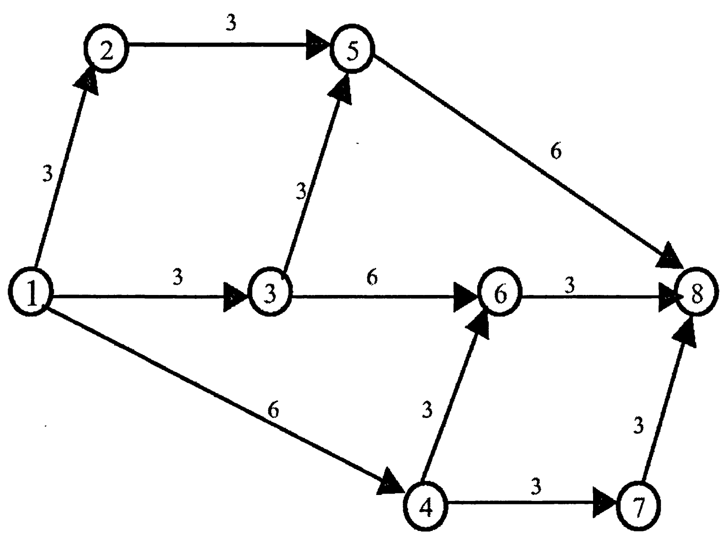

Fig. 1. Example network with mean duration is shown next to each activity.

Example 2. The network in Figure 2, for which the activity times are assumed to be exponentially distributed, was first introduced by Loulou and Beale [11] and used by Kulkarni and Adlakha [10] who developed analytical procedure known as Markov PERT networks (MPNs). They estimated the mean completion time of this network to be 40.985. But their method suffers from the exponential state explosion. Table 2 summarizes the completion time for different estimates.

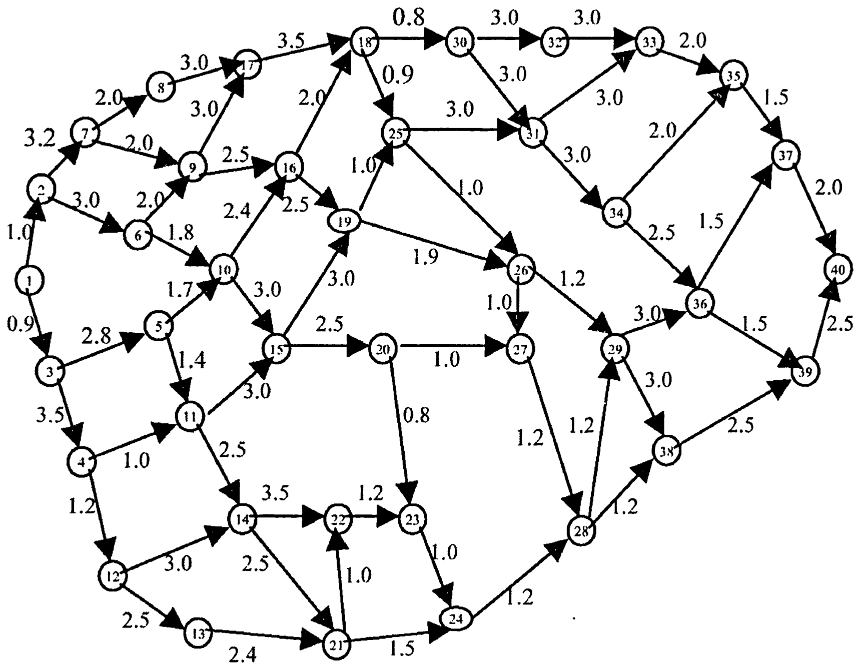

Fig. 2. Network (Loulou and Beale). The numbers on the arcs represent mean activity duration.

Table 1. Comparison between different estimates.

Type of activity duration	Mean Estimates	Variance Estimates	C.V.	Appr. Error (%)
Exact	18.257	21.078	0.2514	
Sculli [15]	20.064	25.512	0.2517	9.89
Exponential (UB) [8]	26.777	301.69	0.6486	46.67
Erlang ($m = 2$)	22.446	112.81	0.4731	22.94
Erlang ($m = 3$)	20.455	93.59	0.4729	10.27
PERT(LB) [2]	12	54	0.6123	−34.27

Table 2. Comparison between different estimates.

Type of activity duration	Mean Estimates	Variance Estimates	C.V.	Appr. Error (%)
Kulkarni and Adlakha [10]	40.985	58.803	0.187	
Sculli [15]	42.61	12.03	0.081	3.96
Exponential (UB) [8]	315.75	56144	0.75	670.40
Erlang ($m = 2$)	167.05	8137.3	0.54	307.58
Erlang ($m = 3$)	122.30	2972.6	0.445	198.40
PERT(LB) [2]	26.1	70.05	0.321	−36.32

The values of the percentage errors for the estimates are included in Tables 1 and 2. By considering the approximated exact values, which give good approximations for the mean completion time, for the networks in Example 1 and 2 we get 18.257 and 40.985. We note that from the Tables 1 and 2 the percentage errors of the Erlang estimate are smaller than the exponential upper bound given in [8].

Remark 1. Exponential (UB) is the Erlang distribution with $m = 1$.

Remark 2. From Tables 1 and 2 we see that the Erlang estimate is an upper bound much better than the best upper bound which is given by Kamburowski [8].

4. COMMENTS

Kamburowski [8] proposed a method to determine an upper bound on the mean completion time by replacing the individual activity durations with exponentially distributed r.v.'s with the same mean, and proceeding iteratively over the nodes. He computed the mean completion time of the node j using equation (12) by replacing the distribution of W_j (which is not exponentially distributed) with an exponential of the same mean as $\mathsf{E}(W_j)$. This compounded the error committed and resulted in a very loose UB. We follow the same procedure by considering the distribution of W_j is the Erlang distribution. This also compounded the error committed and resulted in a little loose UB compared with Kamburowski [8]. There is also another reason caused the error to be less than in Kamburowski [8], that is the Erlang distribution has less extreme compared with the exponential distribution.

Kulkarni and Adlakha [10] do not follow the above strategy. They represented the PERT network by a finite-state, absorbing, continuous-time Markov chain (CTMC) with a single absorbing state. The state space of this chain is dependent on the structure of the PERT network. The special structure of the chain allows them to develop an exact analysis of the network. These assumptions led them to avoid using the maximum operation to calculate the project completion time and hence there is no error committed of assuming that the distribution of W_j is again exponential (and the independent assumption in this case may disappeared). Consequently, their estimate for the moments completion time is getting tighter than the UB given by Kamburowski [8] and the Erlang estimate which is located between them.

However, their method suffers from the exponential state explosion and requires excessive computational effort to find the mean and the variance. Also one must use a computer software to find these parameters even in a small network, e. g., in the examples examined by them the in-degree of any node is at most two. If it is assumed that the in-degree of some nodes are three or more, the computational efforts could be impossible. On the other hand, these parameters can be derived manually using the Erlang distribution. There is another important remark on their work. They claimed that their work is stable. It seems that this is not always true, e.g. consider Example 1 which is also cited in their paper. They estimated the mean and the variance to be (14.7787, 46.6981) whereas the estimation by Kamburowski [8] are (10.13, 60.46), i.e., Kamburowski's [8] mean estimate is less than that of Kulkarni and Adlakha [10].

5. CONCLUDING REMARKS

The advantages of using the Erlang distribution as a representation of activity times are stated in Bendell et al [10]. They have derived only the four central moments of the sum and the maximum of two independent Erlang r.v.'s. This paper generalized the work of Bendell et al [10]. The distribution of the sum of n independent non-identically distributed r.v.'s as well as the kth moments of the maximum and the minimum are derived. Bounds on the mean completion time of the project are also given. The higher moments can be obtained easily by using equations (7), (8) and (12). Applications show that the Erlang estimate is an upper bound which is less than that obtained by Kamburowski [8]. Similar results for stochastic shortest-route networks can be obtained by using equations (8) and (9). Computational complexity which is given in the algorithm may be an interesting topic for future research.

APPENDIX

The rth moment of the Erlang distribution about the origin is given by

$$\mu_r = \int_0^\infty x^r f(x)\,\mathrm{d}x = \frac{\Gamma(r+m)}{\lambda^r\,\Gamma(m)}, \tag{A1}$$

in particular,

$$\mathsf{E}(X) = \frac{m}{\lambda} \quad \text{and} \quad \mathsf{Var}(X) = \frac{m}{\lambda^2}.$$

by letting $\lambda_i = \frac{m}{\mu_i}$, some special cases arose from equations (7) and (8).

For $k = 1$.

Case 1. $n = 2$ and $m = 2$.

$$\mathsf{E}\left(\max(X_1, X_2)\right) = \mu_{2:2} = \mu_1 + \mu_2 - \left(\frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_2}} + \frac{1}{\mu_1\mu_2(\frac{1}{\mu_1} + \frac{1}{\mu_2})^3}\right). \tag{A2}$$

Case 2. $n = 3$ and $m = 2$.

$$\mathsf{E}(\max(X_1, X_2, X_3)) = \mu_{3:3} = \mu_1 + \mu_2 + \mu_3 - \left(\frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_2}} + \frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_3}} + \frac{1}{\frac{1}{\mu_2} + \frac{1}{\mu_3}}\right.$$
$$\left. + \frac{1}{\mu_1\mu_2(\frac{1}{\mu_1} + \frac{1}{\mu_2})^3} \frac{1}{\mu_1\mu_3(\frac{1}{\mu_1} + \frac{1}{\mu_3})^3} + \frac{1}{\mu_2\mu_3(\frac{1}{\mu_2} + \frac{1}{\mu_3})^3}\right) + \frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}}$$
$$+ \frac{\mu_1 + \mu_2 + \mu_3}{\mu_1\mu_2\mu_3(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3})^3} + \frac{3}{\mu_1\mu_2\mu_3(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3})^4}. \tag{A3}$$

Case 3. $n = 2$ and $m = 3$.

$$\mu_{2:2} = \mu_1 + \mu_2 - \left(\frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_2}} + \frac{1}{\mu_1\mu_2(\frac{1}{\mu_1} + \frac{1}{\mu_2})^3} + \frac{2}{\mu_1^2\mu_2^2(\frac{1}{\mu_1} + \frac{1}{\mu_2})^5}\right). \tag{A4}$$

Case 4. $n = 3$ and $m = 3$.

$$\mu_{3:3} = \mu_1 + \mu_2 + \mu_3 - \left(\frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_2}} + \frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_3}} + \frac{1}{\frac{1}{\mu_2} + \frac{1}{\mu_3}} + \frac{1}{\mu_1\mu_2(\frac{1}{\mu_1} + \frac{1}{\mu_2})^3}\right.$$
$$+ \frac{1}{\mu_1\mu_3(\frac{1}{\mu_1} + \frac{1}{\mu_3})^3} + \frac{1}{\mu_2\mu_3(\frac{1}{\mu_2} + \frac{1}{\mu_3})^3} + \frac{2}{\mu_1^2\mu_2^2(\frac{1}{\mu_1} + \frac{1}{\mu_2})^5} + \frac{2}{\mu_1^2\mu_3^2(\frac{1}{\mu_1} + \frac{1}{\mu_3})^5}$$
$$\left. + \frac{2}{\mu_2^2\mu_3^2(\frac{1}{\mu_2} + \frac{1}{\mu_3})^5}\right)$$
$$+ \frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}} + \frac{\frac{1}{\mu_1}(\mu_2 + \mu_3) + \frac{1}{\mu_2}(\mu_1 + \mu_3) + \frac{1}{\mu_3}(\mu_1 + \mu_2) + 6}{\mu_1\mu_2\mu_3(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3})^4}$$
$$2\frac{\mu_1^2 + \mu_2^2 + \mu_3^2}{\mu_1^2\mu_2^2\mu_3^2(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3})^5} + 10\frac{\mu_1 + \mu_2 + \mu_3}{\mu_1^2\mu_2^2\mu_3^2(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3})^6} + 30\frac{1}{\mu_1^2\mu_2^2\mu_3^2(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3})^7}. \tag{A5}$$

For $k = 2$.

Case 1. $n = 2$ and $m = 2$.

$$\mu_{2:2}^{(2)} = \frac{3}{2}\left[\mu_1^2 + \mu_2^2 - \left(\frac{1}{(\frac{1}{\mu_1} + \frac{1}{\mu_2})^2} + \frac{2}{\mu_1\mu_2(\frac{1}{\mu_1} + \frac{1}{\mu_2})^4}\right)\right]. \tag{A6}$$

Case 2. $n = 2$ and $m = 3$.

$$\mu_{2:2}^{(2)} = \frac{4}{3}\left[\mu_1^2 + \mu_2^2 - \left(\frac{1}{(\frac{1}{\mu_1} + \frac{1}{\mu_2})^2} + \frac{2}{\mu_1\mu_2(\frac{1}{\mu_1} + \frac{1}{\mu_2})^4} + \frac{5}{\mu_1^2\mu_2^2(\frac{1}{\mu_1} + \frac{1}{\mu_2})^6}\right)\right]. \quad \text{(A7)}$$

Case 3. $n = 3$ and $m = 2$.

$$\mu_{3:3}^{(2)} = \frac{3}{2}\left[\mu_1^2 + \mu_2^2 + \mu_3^2 - \left(\frac{1}{(\frac{1}{\mu_1} + \frac{1}{\mu_2})^2} + \frac{1}{(\frac{1}{\mu_1} + \frac{1}{\mu_3})^2} + \frac{1}{(\frac{1}{\mu_2} + \frac{1}{\mu_3})^2}\right.\right.$$

$$\left. + \frac{2}{\mu_1\mu_2(\frac{1}{\mu_1} + \frac{1}{\mu_2})^4} + \frac{2}{\mu_1\mu_3(\frac{1}{\mu_1} + \frac{1}{\mu_3})^4} + \frac{2}{\mu_2\mu_3(\frac{1}{\mu_2} + \frac{1}{\mu_3})^4}\right) + \frac{1}{(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3})^2}$$

$$\left. + \frac{2(\mu_1 + \mu_2 + \mu_3)}{\mu_1\mu_2\mu_3(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3})^4} + \frac{8}{\mu_1\mu_2\mu_3(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3})^5}\right]. \quad \text{(A8)}$$

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