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CHARACTERIZATIONS OF THE EXPONENTIAL DISTRIBUTION BASED ON CERTAIN PROPERTIES OF ITS CHARACTERISTIC FUNCTION[1]

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Two characterizations of the exponential distribution among distributions with support the nonnegative real axis are presented. The characterizations are based on certain properties of the characteristic function of the exponential random variable. Counterexamples concerning more general possible versions of the characterizations are given.

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1. INTRODUCTION

A random variable X is said to follow the exponential distribution with parameter $\vartheta > 0$, if its cumulative distribution function is

$$\begin{aligned} F(x) &= 1 - e^{-x/\vartheta}, \quad x \geq 0 \\ &= 0, \qquad\qquad x < 0, \end{aligned}$$

and the corresponding probability density function is

$$\begin{aligned} f(x) &= \vartheta^{-1} e^{-x/\vartheta}, \quad x \geq 0 \\ &= 0, \qquad\qquad x < 0. \end{aligned} \tag{1}$$

Next to the normal distribution, the exponential distribution is possibly the most widely referenced continuous probability law. It appears as a textbook or an in-class example in introductory probability and statistics courses, and constitutes the A and B of reliability and life testing. There exist numerous characterizations of the exponential distribution, most of them based on the "lack of memory" and the "constant hazard rate" properties. The lack of memory property states that the exponential is the only law satisfying, $\bar{F}(x+y) = \bar{F}(x)\bar{F}(y)$, for all x, $y \geq 0$, where

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$\bar{F}(\cdot) = 1 - F(\cdot)$. The constant hazard rate property states that the exponential is the only law for which the hazard rate $f(x)/\bar{F}(x)$ is constant (independent of x). There are many other characterizations of the exponential model depending on order statistics, regression etc. The reader is referred to the monographs of Galambos and Kotz [4] and Azlarov and Volodin [1] for a complete list of characterizations, and to Johnson et al [7], chapter 19, for some more recent work.

The characteristic function (*cf*) $\varphi(t)$, is defined as

$$\varphi(t) = \mathbb{E}\left(e^{itX}\right) = \int_{-\infty}^{\infty} e^{itx} f(x)\,\mathrm{d}x = C(t) + iS(t),\ i = \sqrt{-1}, \tag{2}$$

for real t. In (2), $C(t) = \mathbb{E}(\cos tX)$ denotes the real part and $S(t) = \mathbb{E}(\sin tX)$ denotes the imaginary part of $\varphi(t)$. In Section 2 we present two characterizations that involve the *cf*. The first depends on the ratio $S(t)/C(t)$ and the second relates the squared modulus of the *cf* to $C(t)$.

2. THE CHARACTERIZATIONS

For the exponential density in (1), we easily calculate the *cf* from (2) as

$$\varphi(t) = C(t) + iS(t) = \frac{1}{1+\vartheta^2 t^2} + i\frac{\vartheta t}{1+\vartheta^2 t^2}. \tag{3}$$

Hence we have for some $\vartheta > 0$,

$$S(t) = \vartheta t C(t), \text{ for all } t. \tag{4}$$

That is to say the ratio $S(t)/C(t)$ (we assume that $C(t) \neq 0$), is a straight line (in t) through the origin with positive slope ϑ. We will prove that under some conditions, (4) implies exponentiality.

Csörgő and Heathcote [2] proved that for symmetric distributions and for some $\delta \in \mathbb{R}$,

$$S(t) = \tan(\delta t) C(t), \text{ for all } t. \tag{5}$$

By comparing (4) and (5) one concludes that (4) can not be true for any symmetric distribution. Consequently one maybe tempted to prove that (4) implies exponentiality in the entire class of distributions over the whole real line. However this is not true as the following counterexample shows.

For $0 < p < 1$ consider the density

$$f(x) = (1-p)\,\vartheta^{-1} e^{-x/\vartheta} I_{[0,\infty)}(x) + p\,(2\vartheta)^{-1} e^{-|x|/\vartheta},$$

that is a mixture of an exponential with a zero-mean Laplace density, both with common scale $\vartheta > 0$. Then one can easily calculate the *cf* of $f(\cdot)$ as

$$\varphi(t) = \frac{1}{1+\vartheta^2 t^2} + i\frac{(1-p)\vartheta t}{1+\vartheta^2 t^2}.$$

Hence this density has the property (4) with ϑ replaced by $(1-p)\vartheta$.

Our next step is to assume that $\mathbb{P}(X \geq 0) = 1$. Then we have the following.

Theorem. Among all continuous non–negative random variables which possess smooth densities with finite limit as $x \to 0^+$ and absolutely integrable derivatives, the exponential random variable is the only one for which (4) holds.

Proof. Division by ϑ in (4) yields,

$$\vartheta^{-1} \int_0^\infty \sin(tx) f(x)\,\mathrm{d}x = t \int_0^\infty \cos(tx) f(x)\,\mathrm{d}x.$$

Then apply integration by parts to the right hand side of the last equality to get (all derivatives are with respect to x),

$$\begin{aligned} t\int_0^\infty \cos(tx) f(x)\,\mathrm{d}x &= \int_0^\infty (\sin(tx))' f(x)\,\mathrm{d}x \\ &= \sin(tx) f(x)\Big|_0^\infty - \int_0^\infty \sin(tx) f'(x)\,\mathrm{d}x \\ &= -\int_0^\infty \sin(tx) f'(x)\,\mathrm{d}x. \end{aligned}$$

Hence for all t,

$$\int_0^\infty [\vartheta^{-1} f(x) + f'(x)] \sin(tx)\,\mathrm{d}x = 0.$$

The left hand side of the last equality is a "scaled" Fourier Sine Transform of the function $\vartheta^{-1} f(x) + f'(x)$. From a well known inversion formula (Fedoryuk [3]) we conclude that for every x, $f(x) + \vartheta f'(x) = 0$. The last differential equation has as unique solution the exponential density in (1), and we are done. □

An Additional Characterization. From (3) one can also see that

$$|\varphi(t)|^2 = C^2(t) + S^2(t) = C(t), \text{ for all } t. \tag{6}$$

That is, the squared modulus and the real part of the *cf* coincide in the exponential distribution case. If the distribution is degenerate at zero we have $\varphi(t) \equiv C(t) \equiv 1$ and hence (6) trivially holds. Also if $X = 0$ or δ ($\delta \in \mathbb{R}$), with equal probability 1/2, then again the *cf* of X satisfies (6). Let us rephrase (6) in a distributional language. Let X and Y be iid with *cf* $\varphi(t)$. Then $|\varphi(t)|^2$ is the *cf* of the random variable $Z = X - Y$. Also let $W = X$ or $-X$ with equal probability 1/2. Then $C(t)$ is the *cf* of W. But, as it was suggested by a referee, the distributional equality $Z \stackrel{\mathrm{d}}{=} W$ implies (under the condition of X being non–negative and non–lattice) that X is exponentially distributed (refer to Rossberg [8], and Johnson et al. [7], p. 545).

Statistical applications of the characterizations will very naturally be based on the empirical *cf* which, given the random sample $X_1, X_2, \ldots, X_n$, is defined as $\varphi_n(t) = n^{-1} \sum_{j=1}^n \exp(itX_j)$. The reader is referred to Henze and Meintanis [5] (resp. Henze and Meintanis [6]), for goodness–of–fit tests to the exponential distribution based on (4) (resp. (6)) and $\varphi_n(\cdot)$.

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