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COMPLEX CALCULUS OF VARIATIONS

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In this article, we present a detailed study of the complex calculus of variations introduced in [4]. This calculus is analogous to the conventional calculus of variations, but is applied here to $\mathbf{C}^n$ functions in $\mathbf{C}$. It is based on new concepts involving the minimum and convexity of a complex function. Such an approach allows us to propose explicit solutions to complex Hamilton–Jacobi equations, in particular by generalizing the Hopf–Lax formula.

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1. INTRODUCTION

The objective of this article is to present a detailed study of the complex calculus of variations introduced in [4]. While the complex calculus of variations studied in [4] is similar to the conventional calculus of variations (Euler's equation and Hamilton–Jacobi's equation), we apply it here to value functions as defined in $\mathbf{C}^n$. The present study is based on two new concepts that we develop in Section 2: The minimum of a complex value function as defined on $\mathbf{C}^n$ and the definition of convexity for such functions. These concepts then lead us to defined a Fenchel transform whose properties are analysed in Section 3. Finally, in Section 5, we propose explicit solutions to Hamilton–Jacobi equations for complex value functions defined on $\Re^n$ or $\mathbf{C}^n$, in particular by generalizing the Hopf–Lax formula. This new approach should make it possible to take into account certain extensions of the calculus of variations that are required by modern physics, particulary in quantum mechanics. In this way, complex Hamilton–Jacobi equations have already been introduced in quantum mechanics by many authors such as Balian and Bloch [1] and Voros [6].

These authors show that complex Hamilton–Jacobi equations are necessary to carry out certain approximations more completely, such as the BKW approximation.

2. MINIMUM OF A COMPLEX FUNCTION

Let $f(z) = f(x + iy)$ be a complex function of an open set Ω of $\mathbf{C}^n$ in $\mathbf{C}$, expressed in the form $f(z) = P(x, y) + iQ(x, y)$ with $P(x, y)$ continuous in x and y.

Definition 2.1.

1. $z_0 = x_0 + iy_0$ is a local minimum of f in Ω if a neighbourhood $v(z_0) \subset \Omega$ exists such that: (x_0, y_0) is a saddle point of $P(x,y)$ on $v(z_0)$:
$$P(x_0, y) \leq P(x_0, y_0) \leq P(x, y_0) \;\; \forall x : x + iy_0 \in v(z_0); \forall y : x_0 + iy \in v(z_0).$$
2. z_0 is a global minimum of f in Ω if (x_0, y_0) is a saddle point of $P(x,y)$ in the whole of Ω:
$$P(x_0, y) \leq P(x_0, y_0) \leq P(x, y_0) \;\; \forall x : x + iy_0 \in \Omega; \forall y : x_0 + iy \in \Omega.$$
3. Ω is convex for all values of $z_1, z_2 \in \Omega$ if the segment $[z_1, z_2] := \{\lambda z_1 + (1-\lambda) z_2 : \lambda \in [0,1]\}$ is contained within Ω.
4. $f(z)$ is (strictly) convex in Ω if $P(x,y)$ is (strictly) convex for x in Ω and (strictly) concave for y in Ω.

Proposition 2.1. If $f(z)$ is strictly convex in Ω, then it will assum a unique global minimum value in Ω.

Proof. Let $z_0 = x_0 + iy_0$ and $z_1 = x_1 + iy_1$ denote two global minima of f in Ω, with $z_0 \neq z_1$. Then, according to the definition of the global minimum, we obtain:
$$P(x_0, y) \leq P(x_0, y_0) \leq P(x, y_0) \;\; \forall x : x + iy_0 \in \Omega \;\text{ et }\; \forall y : x_0 + iy \in \Omega$$
and
$$P(x_1, y) \leq P(x_1, y_1) \leq P(x, y_1) \;\; \forall x : x + iy_1 \in \Omega \;\text{ et }\; \forall y : x_1 + iy \in \Omega.$$
But $z_0 \neq z_1$ implies $x_0 \neq x_1$ or $y_0 \neq y_1$.

If $y_0 \neq y_1$, we obtain $P(x_0, y_0) \leq P(x_1, y_0) < P(x_1, y_1) \leq P(x_0, y_1) < P(x_0, y_0)$ which gives rise to a contradiction. We find the same contradiction for $x_0 \neq x_1$. As a result, $z_0 = z_1$. □

Proposition 2.2. If $f(z)$ is convex in a convex Ω, then any local minimum is global.

Proof. Let us take $z_0 = x_0 + iy_0$ as a local minimum of f. Then, since $\forall z = x + iy \in v(z_0)$, we can write:
$$P(x_0, y) \leq P(x_0, y_0) \leq P(x, y_0).$$
Take for instance $z_1 = x_1 + iy_1$ as any point of Ω. Assuming $z = \lambda z_0 + (1 - \lambda) z_1$ with $0 \leq \lambda \leq 1$, we show that:
$$P(x_0, y_1) \leq P(x_0, y_0) \leq P(x_1, y_0).$$
We can write $P(x, y_0) = P(\lambda x_0 + (1-\lambda) x_1, y_0)$. But P is convex in x, so $P(x, y_0) \leq \lambda P(x_0, y_0) + (1 - \lambda) P(x_1, y_0)$. If we assume that $P(x_1, y_0) < P(x_0, y_0)$, this leads to: $P(x, y_0) < \lambda P(x_0, y_0) + (1 - \lambda) P(x_0, y_0) = P(x_0, y_0)$. This gives rise to a contradiction, so as a result, $P(x_0, y_0) \leq P(x_1, y_0)$.

In the same way, we can show that $P(x_0, y_1) \leq P(x_0, y_0)$ using the fact that P is concave in y. □

Proposition 2.3. If $f(z)$ is holomorphic, then $P(x,y)$ is (strictly) convex for x in Ω, and is equivalent to $P(x,y)$ (strictly) concave for y in Ω. In this case, the condition of optimality is simply written as:

$$P(x_0,y_0) \leq P(x,y_0) \quad \forall x : x+iy_0 \in \Omega.$$

Proof. Since f is holomorphic, then the Cauchy conditions are satisfied for all values of i and j: $\frac{\partial P}{\partial x_i} = \frac{\partial Q}{\partial y_i}$ et $\frac{\partial P}{\partial y_j} = -\frac{\partial Q}{\partial x_j}$. These conditions imply $\frac{\partial^2 P}{\partial x_i \partial x_j} = -\frac{\partial^2 P}{\partial y_i \partial y_j}$.

P is (strictly) convex in $x = (x_1, \ldots, x_n)$ if and only if the proper values of the Hessian matrix $\left(\frac{\partial^2 P}{\partial x_i \partial x_j}\right)_{i=1,n;j=1,n}$ are $(>0) \geq 0$ in Ω. From this, we can deduce that P is (strictly) convex in x if and only if the proper values of the matrix $\left(\frac{\partial^2 P}{\partial y_i \partial y_j}\right)_{i=1,n;j=1,n}$ are $(<0) \leq 0$ in Ω, and, therefore, if and only if P is (strictly) concave in $y = (y_1, \ldots, y_n)$. □

Let us give some examples of strictly convex complex functions:

1. The function $f(z) = \frac{1}{2}z^2$ is strictly convex on $\mathbf{C}$. More generally, $g(z) = \frac{1}{2}z^2$, in which $z^2 = z_1^2 + z_2^2 + z_3^2$, is strictly convex on $\mathbf{C}^3$.

2. $f(z) = \frac{1}{2}z^t Q z$, where Q is defined as positive, is strictly convex over the whole of $\mathbf{C}^n$.

3. $f(z) = \frac{1}{3}z^3$ is strictly convex on $\mathbf{C}_+^* := \{x+iy,\, x>0\}$. In fact, $f''(z) = 2z$, so $\mathrm{Re}(f''(z)) = 2x > 0$. From this, we derive the strict convexity of f on $\mathbf{C}_+^*$.

4. $f(z) = \frac{1}{\alpha}z^\alpha$ with α being an integer $\alpha > 2$, is convex in $\Omega_\alpha = \{x+iy; |y| \leq x \tan\frac{\pi}{2(\alpha-2)}\}$ on $\mathbf{C}_+^*$. In fact, $f''(z) = (\alpha-1)z^{\alpha-2}$.

   For $z = re^{i\theta}$, we obtain $\mathrm{Re}(f''(z)) = (\alpha-1)r^{(\alpha-2)}\cos(\alpha-2)\theta$.
   $\mathrm{Re}(f''(z)) \geq 0$ for $-\frac{\pi}{2(\alpha-2)} \leq \theta \leq \frac{\pi}{2(\alpha-2)}$ and therefore $|y| \leq x\tan\frac{\pi}{2(\alpha-2)}x$.

5. $f(z) = \sum_{n=1}^{\infty} a_n z^n$ with $z \in \mathbf{C}_+^*$, $a_2 > 0$ and $a_n \geq 0$. The function is convex in $\Omega = \{x+iy$ such that $\exists\epsilon_x > 0$, $|y| \leq \epsilon_x x\}$. In fact, $f''(z) = 2a_2 + \sum_{n=3}^{\infty} n(n-1)a_n z^{n-2}$. $\frac{\partial^2 P}{\partial x^2}(x,0) = 2a_2 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} > 0$ so $f(x)$ is therefore strictly convex on $\Re$. Since $\mathrm{Re}(f''(0)) = 2a_2 > 0$, then, by continuity, there is a value $\epsilon_x > 0$ such that we obtain $\mathrm{Re}(f''(z)) > 0$ in the open set defined by $|y| \leq \epsilon_x x$.

6. $f(z) = c^2\left(1 - \sqrt{1-\frac{z^2}{c^2}}\right)$ with $c > 0$ and $|x| \leq c$, where $\sqrt{Z}$ is the square root of Z having a real positive part. This function is convex in the open set $\Omega = \{x+iy \;;\; |x| \leq c, |y| \leq \epsilon_x|x|\}$. We can write $f''(z) = \left(1-\frac{z^2}{c^2}\right)^{-\frac{3}{2}}$.
   But $f''(x) = \left(1-\frac{x^2}{c^2}\right)^{-\frac{3}{2}} > 0$ is strictly convex for$|x| < c$ and, as deduced from Example 5, for all values of x there is a $\epsilon_x > 0$ for which we obtain $\mathrm{Re}(f''(z)) \geq 0$ in the open set defined by $|y| \leq \epsilon_x x$.

7. $f(z) = \frac{2}{5}z^{\frac{5}{2}}$ is convex on $\mathbf{C}_+^*$, with $z \in \mathbf{C}_+^*$, and where $\sqrt{Z}$ is the square root of Z having a real positive part. In fact, since $f''(z) = \frac{3}{2}z^{\frac{1}{2}}$ then $\mathrm{Re} f''(z) \geq 0$.

The following proposition provides a general framework for the above examples.

Proposition 2.4. If $f(x)$ is a real analytical function strictly convex on an open set Λ of $\Re^n$, then its analytical prolongation $f(z)$ is strictly convex on a neighbourhood Ω of Λ in $\mathbf{C}^n$ having the form $\Omega = \bigcup_{x\in\Lambda} v_x$, where

$$v_x = \{z' \in \mathbf{C}\ ,\ \exists \epsilon_x > 0 \text{ with } |z' - x| \leq \epsilon_x\}.$$

Futhermore, $f''(z)$ is reversible in a neighbourhood Ω' ($\Lambda \subset \Omega' \subset \Omega$) having the form: $\Omega' = \bigcup_{x\in\Lambda} v'_x$ where

$$v'_x = \{z' \in \mathbf{C}\ ,\ \exists \delta_x > 0 \text{ with } |z' - x| \leq \delta_x \leq \epsilon_x\}.$$

Proof. $f''(z) = f''(x+iy) = u(x,y) + iv(x,y)$. For $y = 0$, $f''(x) = u(x,0) + iv(x,0)$. Since $f(x)$ is strictly convex on Λ, there is a value $\alpha_x \in \Re$ such that $f''(x) \geq \alpha_x > 0$. This inequality leads to $u(x,0) \geq \alpha_x$ and by continuity we obtain: $\forall x\ , \exists \beta_x\ \ : |y| \leq \beta_x$ and $u(x,y) \geq \alpha_x > 0$. From this, we derive the strict convexity of f.

We show that $f''(z)$ is reversible in Ω' ($\Lambda \subset \Omega' \subset \Omega$).
In dimension 1, the fact that $f(z)$ is strictly convex in Ω, while $f''(z) = u(x,y) + iv(x,y)$, gives us $(f''(z))^{-1} = \frac{u-iv}{u^2+v^2}$ or $u(x,y) > 0$, thus $(f''(z))^{-1}$ is well defined on $\Omega' = \Omega$.

In dimension n, with $f(z) = f''(z) = U(x,y) + iV(x,y)$, we need to find two matrices X and Y such that $(U+iV)(X+iY) = I$. The strict convexity of $f(z)$ leads to the fact that $U(x,y)$ is a reversible matrix. This strict convexity of f(x) is expressed by:

$$\forall x \in \Lambda\ \ \exists \delta_x :\ \ |y| \leq \delta_x\ \ U(x,y) \geq \alpha I > 0$$

$$\forall x \in \Lambda\ \ \exists \gamma_x :\ \ |y| \leq \gamma_x\ \ V(x,y) \leq \frac{\alpha}{10} I$$

and therefore $U^{-1}V < I$, which leads to $U^{-1} < 2X$ and, as a result, $X > \frac{Y}{2}$ in $v'_x = \{z' \in \mathbf{C}\ ,\ |z' - x| \leq \min(\delta_x, \gamma_x, \epsilon_x)\}$. □

Proposition 2.5. If $f(z)$ is a holomorphic function on a convex open set Ω, then a necessary condition for $z_0 \in \Omega$ to be a local minimum of $f(z)$ in Ω is that $f'(z_0) = 0$. It is sufficient if, in addition, f is convex in the neighbourhood of z_0.

Proof. $z_0 = x_0 + iy_0$ is a local minimum of $f(z)$ in Ω, hence:

$$P(x_0, y_0) = \min_{x;\ x+iy_0 \in v(z_0)} P(x, y_0) = \max_{y;\ x_0+iy \in v(z_0)} P(x_0, y).$$

Thus, $\frac{\partial P}{\partial x}(x_0, y_0) = \frac{\partial P}{\partial y}(x_0, y_0) = 0$. Since f is holomorphic in Ω, The Cauchy conditions imply $f'(z_0) = \frac{1}{2}(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}) - \frac{i}{2}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) = \frac{\partial P}{\partial x}(x_0, y_0) - i\frac{\partial P}{\partial y}(x_0, y_0) = 0$.

If $f(z)$ is convex in the neighbourhood of z_0, then $P(x, y)$ is convex for x in the open set $\{x;\ x + iy \in v(z_0)\}$.Thus, we obtain $\forall x :\ x + iy_0 \in v(z_0),\ \ P(x_0, y_0) \leq P(x, y_0) + \frac{\partial P}{\partial x}(x_0, y_0)(x_0 - x)$.

$f'(z_0) = 0$ implies that $\frac{\partial P}{\partial x}(x_0, y_0) = 0$, so therefore $P(x_0, y_0) \leq P(x, y_0)\ \ \forall x :\ x + iy_0 \in v(z_0)$. □

Observation. If Ω is convex, then a necessary condition for z_0 to be a global minimum of $f(z)$ in Ω is that $f'(z_0) = 0$.

3. FENCHEL COMPLEX TRANSFORM

Definition 3.1. Each complex function $f(z) : z \in \Omega \subset \mathbf{C}^n \mapsto \mathbf{C}$ is associated with its Fenchel complex transform $\widehat{f}_{\Omega,\Sigma}(p) : p \in \Sigma \subset \mathbf{C}^n \mapsto \mathbf{C}$, which, if it exists, is defined by:

$$\forall p \in \Sigma\ ,\ \ \widehat{f}_{\Omega,\Sigma}(p) = \max_{z\in\Omega}(p \cdot z - f(z)).$$

Examples.

1. For the real values m and σ ($\sigma \neq 0$), let us assume that $f_{m,\sigma}(z) = \frac{1}{2}\left(\frac{z-m}{\sigma}\right)^2$ and calculate $\widehat{f}_{m,\sigma}(p) = \max_{z\in\mathbf{C}}\left(p \cdot z - \frac{1}{2}\left(\frac{z-m}{\sigma}\right)^2\right)$. It is easy to check that the function $g(z)$ defined by: $g(z) = p \cdot z - \frac{1}{2}\left(\frac{z-m}{\sigma}\right)^2$ can be derived out of $\mathbf{C}$ an its derivative $g'(z)$ is cancelled out at a unique point $z = p\sigma^2 + m$. The maximum is thus attained (cf. Proposition 2.5), for $z = p\sigma^2 + m$ and $\widehat{f}(p) = \frac{1}{2}p^2\sigma^2 + mp$.

2. For $f(z) = \frac{1}{2}z^tQz$, where Q is symmetric and defined as positive, we obtain: $\widehat{f}(p) = \max_{z\in\mathbf{C}^n}\left(p^t \cdot z - \frac{1}{2}z^tQz\right)$. The function $g(z) = p^t \cdot z - \frac{1}{2}z^tQz$ can be derived out of $\mathbf{C}^n$ an its derivative is cancelled out for $z = Q^{-1}p$. The maximum is therefore attained at this point (necessary condition of optimality), and $\widehat{f}(p) = \frac{1}{2}p^tQ^{-1}p$.

3. For $f(z) = \frac{1}{3}z^3$ with $z \in \mathbf{C}^*_+$, we define $\widehat{f}(p)$ for $p \in \mathbf{C}^*_+$ as follows: $\widehat{f}(p) = \max_{z\in\mathbf{C}^*_+}(p \cdot z - \frac{1}{3}z^3)$. The necessary condition of optimality leads to $f'(z_p) = 0$, that is to say $p = z_p^2$. This equation allows a single root in $\mathbf{C}^*_+$, hence a single local maximum $z_p = \sqrt{p}\ -\frac{\pi}{4} \leq \arg(z_p) \leq \frac{\pi}{4}$. Since $f(z)$ is convex in $\mathbf{C}^*_+$, then z_p is the global maximum of $pz - f(z)$ in $\mathbf{C}^*_+$ (cf. Proposition 2.2). In this case, $\widehat{f}(p)$ exists and $\widehat{f}(p) = \frac{2}{3}p^{\frac{3}{2}}$.

4. Let $f(z) = c^2(1 - \sqrt{1 - \frac{z^2}{c^2}})$ with $c > 0$ and $z \in \Omega = \{z, |\mathrm{Re}(z)| \leq c\}$, where $\sqrt{Z}$ is the root of Z having a real positive part. The function $g(z) = p \cdot z - c^2\left(1 - \sqrt{1 - \frac{z^2}{c^2}}\right)$ can be derived out of $\mathbf{C}$ and its derivative is cancelled

out for $z_p = p/\sqrt{1+\frac{p^2}{c^2}}$. The Fenchel transform of $f(z)$ therefore only exists for p in a sub-set Σ of $\mathbf{C}$ such that:

$$\Sigma = \left\{ p \in \mathbf{C} \; : \; \left| \mathrm{Re}\left(\frac{p}{\sqrt{1+\frac{p^2}{c^2}}} \right) \right| \leq c \right\}.$$

Then, for all values of $p \in \Sigma$, we can define the complex Fenchel transform by $\widehat{f}_{\Omega,\Sigma}(p) = \max_{z\in\Omega}\left(p\cdot z - c^2\left(1-\sqrt{1-\frac{z^2}{c^2}}\right)\right)$ and we can write $\widehat{f}_{\Omega,\Sigma}(p) = c^2\left(\sqrt{1+\frac{p^2}{c^2}}-1\right)$.

For all real values of p, we obtain $p \in \Sigma$. By continuity, we observe that Σ is a closed set containing $\Re$.

5. For $f(z) = \frac{1}{4}z^4$ with $z \in \mathbf{C}^*_+$, we can define $\widehat{f}(p)$ for $p \in \mathbf{C}^*_+$, as follows: $\widehat{f}(p) = \max_{z\in\mathbf{C}^*_+}(p\cdot z - \frac{1}{4}z^4)$. The necessary condition of optimality leads to $f'(z_p) = 0$, that is to say $p = z_p^3$. If we assume that $Z^{\frac{1}{3}}$ is the cube root of Z having a real positive part, then the equation $p = z_p^3$ allows a single root in $\mathbf{C}^*_+$, so there is only a single local maximum $z_p = p^{\frac{1}{3}}$ with $-\frac{\pi}{6} \leq \arg(z_p) \leq \frac{\pi}{6}$ (or $\sqrt{3}|y_p| \leq x_p$). As $f(z)$ is convex in the cone $\Omega := \{x+iy$ such that $|y| \leq x\}$ of $\mathbf{C}$, then z_p is a global maximum in Ω. However, it is not a global maximum in $\mathbf{C}^*_+$. To demonstrate this, taking $P(x,y) = \mathrm{Re}(p\cdot z - \frac{1}{4}z^4)$, it suffices to find a point $(x_p, \tilde{y_p})$ such that $P(x_p,y_p) \geq P(x_p,\tilde{y_p})$.

   For $p = \alpha + i\beta$ and $z = x+iy$ we obtain $P(x,y) = \alpha x - \beta y - \frac{1}{4}(x^2-y^2)^2 + x^2y^2$. Let $\tilde{y_p} = y_p + 6x_p$, then it follows $P(x_p,y_p) - P(x_p,\tilde{y_p}) = 36x_p^2(6x_py_p + \frac{3}{2}y_p^2 + \frac{15}{2}x_p^2) \geq 0$ if and only if $(6x_py_p + \frac{3}{2}y_p^2 + \frac{15}{2}x_p^2) \geq 0$. As $y_p \geq -x_p$, we obtain $6x_py_p + \frac{3}{2}y_p^2 + \frac{15}{2}x_p^2 \geq -6x_p^2 - \frac{3}{2}x_p^2 + \frac{15}{2}x_p^2 \geq 0$, which leads to the conclusion that (x_p,y_p) is not a saddle point of $P(x,y)$ on $\mathbf{C}^*_+$. Hence, we can only define the Fenchel transform on the con Ω: $\widehat{f}_{\Omega,\Sigma}(p) = \max_{z\in\Omega}(pz - \frac{1}{4}z^4) = \frac{3}{4}p^{\frac{4}{3}}$.

Theorem 3.1. Let us assume that:

— Ω and Σ are two sets of $\mathbf{C}^n$,

— $f(z)$ is a function of Ω in $\mathbf{C}$,

— $\widehat{f}_{\Omega,\Sigma}(p)$ of Σ in $\mathbf{C}$ is the complex Fenchel transform of $f(z)$,

For a convex Σ and a holomorphic and strictly convex f, $\widehat{f}$ (if it exists) is also convex in Σ.

Proof. Let us note $p = \alpha + i\beta$, $z = x+iy$ and $f(z) = P(x,y) + iQ(x,y)$. The strict convexity of $f(z)$ in Ω means that $\mathrm{Re}(p\cdot z - f(z)) = \alpha x - \beta y - P(x,y)$ is strictly concave in x and strictly convex in y, thus allowing, if it exists, a unique saddle point

(x_p, y_p) of $\alpha x - \beta y - P(x,y)$ with $z_p = x_p + iy_p \in \Omega$. Otherwise, $\widehat{f_\Omega}(p) = p \cdot z_p - f(z_p)$ leads to $\widehat{P}(\alpha,\beta) = \alpha x_p - \beta y_p - P(x_p, y_p)$ and $\widehat{Q}(\alpha,\beta) = \alpha y_p + \beta x_p - Q(x_p, y_p)$. Moreover, we can readily verify the convexity of $\widehat{P}(\alpha,\beta)$ in α and the concavity in β. In fact, since Σ is convex, for $\alpha = \theta\alpha_1 + (1-\theta)\alpha_2$ with $\theta \in [0,1]$, we can write:

$$\widehat{P}(\alpha,\beta) = \max_x \min_y ((\theta\alpha_1 + (1-\theta)\alpha_2)x - \beta y - p(x,y))$$

$$\begin{aligned}\widehat{P}(\alpha,\beta) \leq\ & \theta \max_x \left\{ \alpha_1 x + \min_y(-\beta y - p(x,y)) \right\} \\ & +(1-\theta)\max_x \left\{ \alpha_2 x + \min_y(-\beta y - p(x,y)) \right\}\end{aligned}$$

$$\widehat{P}(\alpha,\beta) \leq \theta \max_x \min_y \{\alpha_1 x - \beta y - p(x,y)\} + (1-\theta)\max_x \min_y \{\alpha_2 x - \beta y - p(x,y)\}.$$

So $\widehat{P}(\alpha,\beta) \leq \theta\widehat{P}(\alpha_1,\beta) + (1-\theta)\widehat{P}(\alpha_2,\beta)$ and consequently, $\widehat{P}(\alpha,\beta)$ is convex in α. In the same way, we can demonstrate that $\widehat{P}(\alpha,\beta)$ is concave in β. From this, we deduce that if $\widehat{f_\Omega}$ exists, z_p is the unique solution in Ω such that $f'(z_p) = p$. □

Observation. The equation $f'(z_p) = p$ indicates the existence of a Fenchel transform if $\Sigma = f'(\Omega)$.

Theorem 3.2. Take a convex set of $\mathbf{C}^n$ as well as a function $f(z)$ of Ω in $\mathbf{C}$, which is holomorphic and strictly convex in Ω. Let Ω' be the open set of Ω where $f''(z)$ is reversible. Then, for all $p \in \Sigma' = f'(\Omega')$, $\widehat{f}$ is involutive:

$$\forall z \in \Omega' \quad \widehat{\widehat{f}}_{\Omega',\Sigma'}(z) = \max_{p\in\Sigma'}(p \cdot z - \widehat{f}_{\Omega',\Sigma'}(p)) = f(z).$$

P r o o f. The strict convexity of f in Ω means that $f''(z)$ is reversible in Ω'. Since $\Sigma' = f'(\Omega')$, z_p exists and the equation $f'(z_p) = p$ leads us to the fact that $\frac{\partial z_p}{\partial p}$ exists and is equal $(f''(z_p))^{-1}$. Furthermore, we obtain the $\max_{p\in\Sigma'}\left(p \cdot z - \widehat{f}_{\Omega',\Sigma'}(p)\right)$ for p by verifying $z - z_p - p \cdot \frac{\partial z_p}{\partial p} + f'(z_p)\frac{\partial z_p}{\partial p} = 0$, That is $z = z_p$. From this, we obtain $\widehat{\widehat{f}}_{\Omega',\Sigma'}(z) = \max_{p\in\Sigma'}(p \cdot z - p \cdot z_p + f(z)) = f(z)$. □

Observation. In dimension 1, we obtain $\Omega = \Omega'$ and $\Sigma = \Sigma'$. In this case, $\widehat{\widehat{f}}_{\Omega,\Sigma}(z) = f(z)\ \ \forall z \in \Omega$ (cf. Theorem 2.4).

4. COMPLEX CALCULUS OF VARIATIONS

Let $L : \mathbf{C}^n \times \Re^n \times \Re^+ \longrightarrow \mathbf{C}$, where $L(z,q,t)$ is a holomorphic function in z and q which can be derived in t. This function is referred to as a complex Lagrange function. In addition, we may assume that $L(z,q,t)$ is strictly convex in q.

Definition 4.1. Let us establish z_0 and $z_f \in \mathbf{C}^n$ and $t \geq 0$. We define the functional complex action J by:

$$J[w(\cdot)] = \int_0^t L\left(w(s), \frac{\mathrm{d}w(s)}{\mathrm{d}s}, s\right) \mathrm{d}s \tag{1}$$

and the class of allowable functions:

$$A = \{w(\cdot) : [0,t] \to \mathbf{C}^n \text{ holomorphic}/w(0) = z_0, w(t) = z_f\}. \tag{2}$$

The problem of the complex calculus of variations is then to define a curve $w_0(\cdot) \in A$ such that:

$$J[w_0(\cdot)] = \min_A J[w(\cdot)] \tag{3}$$

where min is the global minimum taken in the sense of the complex min in definition 2.1: noting that $w_0(t) = u_0(t) + iv_0(t)$ and $J(w_0(t)) = P(u_0(t), v_0(t)) + iQ(u_0(t), v_0(t))$, while for all $w(t) = u(t) + iv(t) \in A$:

$$P(u_0(t), v(t)) \leq P(u_0(t), v_0(t)) \leq P(u(t), v_0(t)).$$

Theorem 4.1. (Complex Euler equation) If function $z(\cdot)$ is a holomorphic solution of (1), (2), (3), then $z(\cdot)$ satisfies the complex Euler equation:

$$-\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial q}(z(s), \frac{\mathrm{d}z}{\mathrm{d}s}, s) + \frac{\partial L}{\partial z}(z(s), \frac{\mathrm{d}z}{\mathrm{d}s}, s\right) = 0 \ \ \forall 0 \leq s \leq t. \tag{4}$$

Proof. Let us consider the function $G : [0,t] \to \mathbf{C}^n$ such that $G(0) = G(t) = 0$. We define the function $w(\cdot) = z(\cdot) + \tau G(\cdot)$ for $\tau \in \mathbf{C}$. Let $B_\tau = \{w(\cdot) + \tau G(\cdot)/G(\cdot) \in \mathbf{C}^2([0,t]; \mathbf{C}^n)$, with $G(0) = G(t) = 0\}$. It is evident that $B_\tau \subset A$ and that $z(\cdot) \in B_\tau$ ($\tau = \alpha + i\beta$ where $\alpha = \beta = 0$). According to the Lemma 4.1 given below, the function g defined by: $g(\tau) = J[z(\cdot) + \tau G(\cdot)]$ has a complex minimum in $\tau = 0$. As a result, $g'(0) = 0$.

$$\begin{aligned} g(\tau) &= \int_0^t L(z(s) + \tau G(s), z'(s) + \tau G'(s), s)\, \mathrm{d}s \\ g'(\tau) &= \int_0^t \frac{\partial L}{\partial z}(z + \tau G, z' + \tau G', s)\, G\, \mathrm{d}s + \int_0^t \frac{\partial L}{\partial q}(z + \tau G, z' + \tau G', s)\, G' \mathrm{d}s \\ g'(0) &= \int_0^t \frac{\partial L}{\partial z}(z(s), z'(s), s)\, G(s)\, \mathrm{d}s + \int_0^t \frac{\partial L}{\partial q}(z(s), z'(s), s)\, G'(s)\, \mathrm{d}s = 0. \end{aligned}$$

After integrating each part, we find:

$$\int_0^t \frac{\partial L}{\partial z}(z(s), z'(s), s)\, G(s)\, \mathrm{d}s - \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial q}(z(s), z'(s), s)\right) G(s)\, \mathrm{d}s = 0.$$

For all G satisfying the boundary conditions. Thus, $\forall 0 \leq s \leq t$

$$-\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial q}(z(s), z'(s), s)\right) + \frac{\partial L}{\partial z}(z(s), z'(s), s) = 0.$$

□

Lemma 4.1. Let us consider the variational problem:

$$J[z(\cdot)] = \min_{w(\cdot)\in A} J[w(\cdot)].$$

Let us assume B is a sub-set of A such that $z(\cdot) \in B$. Then:

$$J[z(\cdot)] = \min_{w(\cdot)\in B} J[w(\cdot)].$$

Proof. Let us note $z(\cdot) = x(\cdot) + iy(\cdot)$, $w(\cdot) = u(\cdot) + iv(\cdot)$ and $J[w(\cdot)] = P(u,v) + iQ(u,v)$. According to Definition 2.1, $(x(\cdot), y(\cdot))$ is a saddle point of $P(u,v)$: $P(x(\cdot), v(\cdot)) \leq P(x(\cdot), y(\cdot)) \leq P(u(\cdot), y(\cdot))$ $\forall u : u(\cdot) + iy(\cdot) \in A$ $\forall v : x(\cdot) + iv(\cdot) \in A$. But $B \subset A$. Thus, the inequalities given above are valid $\forall u : u(\cdot) + iy(\cdot) \in B$ $\forall v : x(\cdot) + iv(\cdot) \in B$, so therefore $J[z(\cdot)] = \min_{w(\cdot)\in B} J[w(\cdot)]$. □

Definition 4.2. We define the complex action $S(z,t)$ as the complex minimum of the integral of the complex Lagrange function:

$$S(z(t),t) = \min_{\nu(s)\,,\,0\leq s\leq t} \left\{ S_0(z') + \int_0^t L(\nu(s))\,\mathrm{d}s \right\} \tag{5}$$

where the complex minimum is taken for all the tests $\nu(s)$, $s \in [0,t]$, while the change of state $z(s)$ is given by the system evolution equations:

$$\frac{\mathrm{d}z(s)}{\mathrm{d}s} = \nu(s) \quad \text{et} \quad z(0) = z'$$

where S_0 and z' are given, and where $z(t)$ is a holomorphic function.

Theorem 4.2. Let us assume that $L(\cdot)$ is convex. Then, for $z \in \mathbf{C}^n$ and $t > 0$, the function $K(z,t,z') = tL\left(\frac{z-z'}{t}\right)$ is the complex minimum of $\int_0^t L(w'(s))\,\mathrm{d}s$, where $w(\cdot) \in A$, that is to say $w(\cdot)$ is holomorphic, while $w(0) = z'$ and $w(t) = z$.

The demonstration of this theorem is based on an inequality that is equivalent to the one used by Jensen on complex functions:

Lemma 4.2. Let f be a holomorphic and convex function of $\mathbf{C}^n$ in $\mathbf{C}$, which can be expressed in the form $f(z) = P(x,y) + iQ(x,y)$. Let $w = u + iv$ be a holomorphic function on an open bounded set Ω of $\mathbf{C}^n$. Then, we can write:

$$\oint P\left(\oint u, v\right) \leq P\left(\oint u, \oint v\right) \leq \oint P\left(u, \oint v\right) \tag{6}$$

where $\oint u(x)\,\mathrm{d}x := \frac{1}{t}\int_0^t u(s)\,\mathrm{d}s$.

Proof. f is convex and holomorphic, while for all $(x_0, y_0) \in \Re^n \times \Re^n$, there is r_1, $r_2 \in \mathbf{C}^n$ such that

$$P(x, y_0) \geq P(x_0, y_0) + r_1(x - x_0) \text{ and } P(x_0, y) \leq P(x_0, y_0) + r_2(y - y_0)$$

for all $x \in \Re^n$, and for all $y \in \Re^n$. Let us assume that

$$X_0 = \oint u(x)\ ,\ Y_0 = \oint v(x)\ ,\ X = u(x) \quad \text{and} \quad Y = v(x).$$

Accordingly

$$P\left(u(x), \oint v(x)\right) \geq P\left(\oint u(x), \oint v(x)\right) + r_1\left(u(x) - \oint u(x)\right)$$

and

$$P\left(\oint u(x), v(x)\right) \leq P\left(\oint u(x), \oint v(x)\right) + r_2\left(v(x) - \oint v(x)\right).$$

Thus

$$\oint P\left(\oint u(x), v(x)\right) \leq P\left(\oint u(x), \oint v(x)\right) \leq \oint P\left(u(x), \oint v(x)\right).$$

□

Proof of Theorem 4.2. Let us assume $z = x + iy$, $z' = x' + iy'$, $w' = u' + iv'$, $L(w'(s)) = P(u'(s), v'(s)) + iQ(u'(s), v'(s))$. By noting $w_0(s) = z' + \frac{s}{t}(z - z')$ $(0 \leq s \leq t)$, we show that $(u_0'(s)\,, v_0'(s)) = (\frac{x-x'}{t}, \frac{y-y'}{t})$ is a saddle point of $\mathrm{Re}\left(\int_0^t L(w'(s))\,\mathrm{d}s\right)$. We obtain $\mathrm{Re}\left(\int_0^t L(w'(s))\,\mathrm{d}s\right) = \int_0^t P(u'(s), v'(s))\,\mathrm{d}s$. Since $\oint w'(s)\,\mathrm{d}s = \frac{1}{t}(z - z')$, then $\oint u' = \frac{1}{t}(x - x')$ and $\oint v' = \frac{1}{t}(y - y')$. Thus $\int_0^t P(u_0'(s)\,, v_0'(s))\,\mathrm{d}s = \int_0^t P\left(\frac{x-x'}{t}, \frac{y-y'}{t}\right) = \int_0^t P\left(\oint u', \oint v'\right)$. We then apply Lemma 4.2, taking into account that $v_0'(s) = \oint v'$ and $u_0'(s) = \oint u'$,

$$\int_0^t P\left(\oint u', \oint v'\right) \leq \int_0^t \oint P(u', v_0')\,\mathrm{d}s = \int_0^t P(u', v_0')\,\mathrm{d}s\ \forall u'.$$

In the same way,

$$\int_0^t P\left(\oint u', \oint v'\right) \geq \int_0^t \oint P(u_0', v')\,\mathrm{d}s = \int_0^t P(u_0', v')\,\mathrm{d}s\ \forall v'.$$

Thus, $(\frac{x-x'}{t}, \frac{y-y'}{t})$ is a saddle point of $\int_0^t P(u'(s), v'(s))\,\mathrm{d}s$ and therefore

$$\inf_{w'(s)}\left(\int_0^t L(w'(s))\,\mathrm{d}s\right) = \int_0^t L(w_0'(s))\,\mathrm{d}s = tL\left(\frac{z - z'}{t}\right).$$

□

Corollary 4.1. Let us assume that $L(-p)$ is also convex, and that $S_0(z)$ is holomorphic as well as strictly convex. Hence, for $z \in \mathbf{C}^n$ and $t > 0$, the function

$$S(z,t) = \inf_{z'} \left(tL\left(\frac{z - z'}{t}\right) + S_0(z') \right) \tag{7}$$

is the solution of

$$\inf_{z'} \inf_{w'} \left\{ \int_0^t L(w'(s))\, \mathrm{d}s + S_0(z') \right\} \tag{8}$$

where inf is taken on w', with $w(\cdot)$ being holomorphic and $w(t) = z$, and on $z' = w(0)$.

Proof. For a given value of z', Theorem 4.2 implies: $tL\left(\frac{z-z'}{t}\right) + S_0(z') = \inf_{w'} \left\{ \int_0^t L(w'(s))\, \mathrm{d}s + S_0(z') \right\}$. Since $L(-p)$ is convex, then $L\left(\frac{z-z'}{t}\right)$ is convex in z' and $tL\left(\frac{z-z'}{t}\right) + S_0(z')$ is a holomorphic and convex function in z'. Thus, $\inf_{z'} \left(tL\left(\frac{z-z'}{t}\right) + S_0(z')\right)$ is well defined and is equal to $\inf_{z'} \inf_{w'} \left\{ \int_0^t L(w'(s))\, \mathrm{d}s + S_0(z') \right\}$. □

5. SOLUTIONS OF THE COMPLEX HAMILTON–JACOBI EQUATION

Let us consider the following system of partial differential equations, which we use to look for the functions $a(x,t)$ and $b(x,t) \in C^2(\Re^n \times \Re^+; \Re)$:

$$\frac{\partial a}{\partial t} + \frac{1}{2}(\nabla a)^2 - \frac{1}{2}(\nabla b)^2 = 0 \quad \forall (x,t) \in \Re^n \times \Re^+ \tag{9}$$

$$\frac{\partial b}{\partial t} + \nabla a \cdot \nabla b = 0 \quad \forall (x,t) \in \Re^n \times \Re^+ \tag{10}$$

$$a(x,0) = a_0(x) \quad b(x,0) = b_0(x) \quad \forall x \in \Re^n \tag{11}$$

where $a_0(x)$ and $b_0(x)$ are analytical functions of $\Re^n$ in $\Re$, $a_0(x)$ is strictly convex in an open set O of $\Re^n$, while $b_0(x)$ is affine. By assuming $S(x,t) = a(x,t) + ib(x,t)$, the previous system is equivalent to the complex Hamilton–Jacobi equation:

$$\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2 = 0 \quad \forall (x,t) \in \Re^n \times \Re^+ \tag{12}$$

$$S(x,0) = S_0(x) \quad \forall x \in \Re^n \tag{13}$$

where $S_0(x) = a_0(x) + ib_0(x)$. Let $S_0(z)$ be the analytical extension of $S_0(x)$ having the form $a_0(z) + ib_0(z)$. According to Proposition 2.4, $a_0(z)$ (and thus $S_0(z)$ as well, since $b_0(z)$ is affine) is strictly convex in a neighbourhood Ω of $\Re^n$ in $\mathbf{C}^n$.

Theorem 5.1. The function $S(x,t)$ defined by:

$$S(x,t) = \min_{z \in \Omega} \left(S_0(z) + \frac{(z-x)^2}{2t} \right) \tag{14}$$

is a solution for small values of t in the system (12),(13).

Proof. $S_0(z) + \frac{(z-x)^2}{2t}$ is a holomorphic and convex function in Ω. The necessary condition of optimality follows only if $z_{x,t}$ is the solution of:

$$\nabla S_0(z) + \frac{(z-x)}{t} = 0. \tag{15}$$

In other words: $z_{x,t} = x - t\nabla S_0(z_{x,t})$ is an element of Ω, which corresponds to the optimal solution. The equality (15) is continuous in z. It is satisfied in x for $t = 0$. As Ω contains a ball with centre x in $\mathbf{C}^n$, the equality (15) allows a solution $z_{x,t}$, within this ball for sufficiently small values of t. In this case, $S(x,t) = S_0(z_{x,t}) + \frac{(z_{x,t}-x)^2}{2t}$ and $\nabla S(x,t) = \left(\nabla S_0(z_x,t) + \frac{(z_{x,t}-x)}{t}\right)\nabla z_{x,t} - \frac{z_{x,t}-x}{t} = \frac{x-z_{x,t}}{t}$. Since $\frac{\partial S}{\partial t}(x,t) = -\frac{(x-z_{x,t})^2}{2t^2}$, then $\frac{\partial S}{\partial t}(x,t) = -\frac{1}{2}(\nabla S)^2$ $\forall x \in \Re^n$ and at small values of t. □

Corollary 5.1. When $S_0(z)$ is quadratic, then $S_0(z)$ is convex in $\mathbf{C}^n$ and the function

$$S(x,t) = \min_{z \in \mathbf{C}^n} \left(S_0(z) + \frac{(z-x)^2}{2t} \right)$$

is a solution of (12),(13) for all values of t.

Observation. We may note that, to obtain the real solution to the problem (9), (10), (11), it was necessary to make use of complex variables. This appears to be a general principle and proves advantageous in calculating the complex variations introduced above. Moreover, the various formulae for resolving the Hamilton–Jacobi equations, cf. [9], can be generalized for complex Hamilton–Jacobi equations. In particular, we obtain the following generalization of the Hopf–Lax formula.

Theorem 5.2. Take the complex Hamilton–Jacobi equation:

$$\frac{\partial S}{\partial t} + H(\nabla S) = 0 \quad \forall (z,t) \in \Omega \times \Re^+ \tag{16}$$

$$S(z,0) = S_0(z) \quad \forall z \in \Omega \tag{17}$$

where H and S_0 are two holomorphic functions, and Ω is a convex set.

Let us assume that S_0 is strictly convex in $\Omega = H'(\Sigma)$, and that H is strictly convex in $\Sigma = S_0'(\Omega)$. If the complex Fenchel transform $\widehat{H}$ of H is holomorphic in Σ, then, for small values of t, the function $S(z,t)$ defined by:

$$S(z,t) = \min_{z' \in \Omega} \left(S_0(z') + t\widehat{H}\left(\frac{z-z'}{t}\right) \right) \tag{18}$$

is a solution of the system (16),(17).

P r o o f. H is holomorphic and strictly convex in Σ. $\widehat{H}$ is holomorphic and convex in Ω. Consequently, $S_0(z')+t\widehat{H}(\frac{z-z'}{t})$ is convex and holomorphic in Ω. The necessary condition of optimality means that, if z_0, the solution of:

$$\nabla S_0(z') - \nabla \widehat{H}\left(\frac{z-z'}{t}\right) = 0 \tag{19}$$

is in Ω, and it will correspond to the optimal solution. Since H is holomorphic and strictly convex in Σ, then the necessary condition of optimality only applies if p, the solution of:

$$\nabla H(p) = \frac{z-z'}{t} \tag{20}$$

is in Σ, thus corresponding to the optimal solution.

For $t=0$, $z'=z \in \Omega$. For small values of t, $z' \in v_x \subset \Omega$ where $x = \mathrm{Re}(z)$. Thus, $\frac{z-z'}{t} \in \Omega$. Since $\Omega = H'(\Sigma)$, then there exists $p_{z'} \in \Sigma$ such that

$$\widehat{H}\left(\frac{z-z'}{t}\right) = \frac{z-z'}{t}\cdot p_{z'} - H(p_{z'}) \quad \forall z' \in \Omega.$$

$$\nabla S_0(z') - \nabla \widehat{H}\left(\frac{z-z'}{t}\right) = \nabla S_0(z') + (z-z')\nabla p_{z'} - p_{z'} - t\nabla H(p_{z'})\nabla p_{z'} = 0.$$

The equality (20) leads to $\nabla S_0(z') - p_{z'} = 0$. Since $S_0'(\Omega) = \Sigma$, then there exists $z_0 \in \Omega$ such that $S(z,t) = S_0(z_0) + t\widehat{H}\left(\frac{z-z_0}{t}\right)$. $\nabla S(z,t) = \nabla S_0(z_0)\nabla z_0 + \nabla \widehat{H}\left(\frac{z-z_0}{t}\right)(\mathrm{I}-\nabla z_0)$. The equality (19) leads to $\nabla S(z,t) = \nabla \widehat{H}\left(\frac{z-z_0}{t}\right)$.

$$\nabla \widehat{H}\left(\frac{z-z_0}{t}\right)\left(\frac{I-\nabla z_0}{t}\right) = \left(\frac{I-\nabla z_0}{t}\right)\cdot p_0 + \frac{z-z_0}{t}\cdot \nabla p_0 - \nabla H(p_0)\cdot \nabla p_0$$

$$\nabla \widehat{H}\left(\frac{z-z_0}{t}\right) = p_0 + \frac{z-z_0-t\nabla H(p)}{I-\nabla z_0}\cdot \nabla p_0 = p_0 = \nabla S$$

$$\frac{\partial S}{\partial t}(z,t) = \widehat{H}\left(\frac{z-z_0}{t}\right) - \left(\frac{z-z_0}{t}\right)\frac{\partial \widehat{H}}{\partial t}\left(\frac{z-z_0}{t}\right).$$

But,

$$\frac{\partial}{\partial t}\widehat{H}\left(\frac{z-z_0}{t}\right)\left(\frac{z_0-z}{t}\right) = \frac{z_0-z}{t}\cdot p_0$$

then

$$\frac{\partial S}{\partial t}(z,t) = \widehat{H}\left(\frac{z-z_0}{t}\right) + \frac{z_0-z}{t}\cdot p_0 = -H(p_0).$$

Thus,

$$\frac{\partial S}{\partial t} + H(\nabla S) = 0.$$

□

Examples.

1) For small values of t, the function $S(z,t) = \min_{z'\in\Omega}\left\{S_0(z') + \frac{(z-z')^3}{3t^2}\right\}$ is a solution of:

$$\begin{aligned} \frac{\partial S}{\partial t} + \frac{2}{3}(\nabla S)^{\frac{3}{2}} &= 0 \quad \forall (z,t) \in \Omega \times \Re^+ \\ S(z,0) &= S_0(z) \quad \forall z \in \Omega. \end{aligned}$$

In fact, for $\Omega = \Sigma = \mathbf{C}_+^*$, the function $H(p) = \frac{2}{3}p^{\frac{3}{2}}$ is convex in $\mathbf{C}_+^*$ and $\forall p \in \mathbf{C}_+^*$, $H'(p) = p^{\frac{1}{2}} \in \mathbf{C}_+^*$. Otherwise, we choose S_0 to be holomorphic and convex in $\mathbf{C}_+^* = \Omega$, and such that $S_0'(\Omega) = \mathbf{C}_+^* = \Sigma$ (for example, $S_0(z) = \frac{1}{2}z^2$). The conditions of Theorem 5.2 are thus well satisfied.

2) For small values of t, the function $S(z,t) = \min_{z'\in\Omega}\left\{S_0(z') + \frac{(z-z')^4}{4t^3}\right\}$ is a solution of:

$$\begin{aligned} \frac{\partial S}{\partial t} + \frac{3}{4}(\nabla S)^{\frac{4}{3}} &= 0 \quad \forall (z,t) \in \Omega \times \Re^+ \\ S(z,0) &= S_0(z) \quad \forall z \in \Omega \end{aligned}$$

In fact, for $\Omega = \{x+iy/|y| \leq |x|\} \bigcap \mathbf{C}_+^*$ and $\Sigma = \mathbf{C}_+^*$ the function $H(p) = \frac{3}{4}p^{\frac{4}{3}}$ is convex in Σ. We choose S_0 to be holomorphic and convex in Ω, such that $S_0'(\Omega) = \mathbf{C}_+^* = \Sigma$ (for example, $S_0(z) = \frac{1}{2}z^2$ or $S_0(z) = \frac{1}{3}z^3$). The conditions of Theorem 5.2 are thus well satisfied.

Observation. In the case where H and S_0 are quadratic functions, they are strictly convex in $\mathbf{C}^n$, and

$$S(z,t) = \min_{z'\in\mathbf{C}^n}\left(S_0(z') + t\widehat{H}\left(\frac{z-z'}{t}\right)\right)$$

is a solution of the complex Hamilton–Jacobi equation (16), (17) for all $(z,t) \in \mathbf{C}^n \times \Re^+$.

Corollary 5.2. Let Ω be an open set of $\mathbf{C}^n$. The complex action $S(z,t)$ defined by (5) satisfies the complex Hamilton–Jacobi equation:

$$\frac{\partial S}{\partial t} + H(\nabla S) = 0 \quad \forall (z,t) \in \Omega \times \Re^+ \tag{21}$$

$$S(z,0) = S_0(z) \quad \forall z \in \Omega \tag{22}$$

where $H(p)$ is the complex Fenchel transform of $L(q)$.

Proof. This corollary can be directly deduced from Theorems 4.2 and 5.2. □

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