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SOME IDEAS FOR COMPARISON OF BELLMAN CHAINS

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In this paper we are exploiting some similarities between Markov and Bellman processes and we introduce the main concepts of the paper: comparison of performance measures, and monotonicity of Bellman chains. These concepts are used to establish the main result of this paper dealing with comparison of Bellman chains.

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1. INTRODUCTION

The main motivation of this work is that it has recently emerged the fact that the Maslov's idempotent measure theory allows an optimization theory to be derived at the same level of generality as probability and stochastic process theory. Applying Bellman optimality principle to optimization process leads to the idempotent version of the classical Markov causality principle (see e. g. [3] and references therein).

This paper is a first attempt to compare Bellman chains using some well-known arguments and results on stochastic comparisons of Markov chains (see [5] and references therein). We present basic results in very simple case where state space $S = \{1, \dots, s\}$, $s \geq 1$, is discrete and finite. The starting point of the paper is that the dynamics of Markov and Bellman chains are linear in some specific sense. This fact was already noticed in e. g. [1] (see also [6] and references therein).

Here we only develop the algebraical approach of the comparison result. Measure interpretation is a further work. The aim of this paper is only to show similarities between Markov and Bellman chains comparisons.

More formally, let us consider the semiring:

$$\mathbb{M} = (\mathbb{R}_+, +, \cdot, 0, 1\,; \leq)$$

where $\leq$ is the classical order on $\mathbb{R}$, set of real numbers. And the idempotent semiring

$$\mathbb{D} = (\mathbb{R}_- \cup \{-\infty\}, \max, +, -\infty, 0; \prec)$$

which will be denoted:

$$\mathbb{D} \stackrel{\text{def}}{=} (\mathbb{R}_- \cup \{-\infty\}, \oplus, \otimes, \epsilon, e; \prec)$$

where $\prec$ is defined by:

$$\forall a, b \ ((a \oplus b = b) \Longleftrightarrow (a \prec b)).$$

Note that in this case $\leq$ and $\prec$ are equivalent.

We denote $\mathcal{M}_{n,p}(\mathbb{D})$ (resp. $\mathcal{M}_{n,p}(\mathbb{M})$) the semimodule of $n \times p$ matrices with entries in $\mathbb{D}$ (resp. $\mathbb{M}$). When $n = p$ we write $\mathcal{M}_n(\mathbb{D})$ (resp. $\mathcal{M}_n(\mathbb{M})$).

A probability density measure on S is a row vector $x = (x_1, \ldots, x_s)$ such that $\forall 1 \leq i \leq s$, $0 \leq x_i$ and $x \ \vec{1}^T = 1$ where $\vec{1}$ is the row vector which all components are 1 and $(\cdot)^T$ denotes transpose operator.

A cost density measure on S is a row vector $x = (x_1, \ldots, x_s)$ such that $\forall 1 \leq i \leq s$, $\epsilon \prec x_i$ and $x \otimes \vec{e}^T = e$ where $\vec{e}$ is the row vector which all components are e and (if $y = (y_1, \ldots, y_s)$)

$$x \otimes y^T = \oplus_{i=1}^{s} (x_i \otimes y_i) \stackrel{\text{def}}{=} \max_{i=1,\ldots,s} (x_i + y_i).$$

Evolution of state probabilities of an S-valued Markov chain $(X_n)_{n \geq 0} = (x^0, A)$ can be represented by the classical $(+, \cdot)$-linear system

$$\begin{cases} x^0 \\ x^{n+1} = x^n \ A, \ \forall n \in \mathbb{N}, \end{cases}$$

or equivalently:

$$x^n = x^0 \ A^n, \ \forall n \in \mathbb{N}. \tag{1}$$

Where $\mathbb{N}$ denotes set of integers, $A^n = \underbrace{A \cdots A}_{n \text{ times}}$. The product of two matrices $C \in \mathcal{M}_{n,p}(\mathbb{M})$ and $D \in \mathcal{M}_{p,l}(\mathbb{M})$ which is an element of $\mathcal{M}_{n,l}(\mathbb{M})$ denoted by $C \cdot D$ (or simply $C \ D$) and defined by:

$$C \cdot D = \left[\sum_{k=1}^{p} C(i,k) \cdot D(k,j) \right]_{i=1,\ldots n \ ; j=1,\ldots,l}$$

and $A = [A(i,j)]_{i,j \in S}$ is a Markov matrix, i.e. a non-negative matrix ($\forall i, j$, $0 \leq A(i,j)$) such that

$$A \vec{1}^T = \vec{1}^T.$$

Evolution of state cost measures of an S-valued Bellman chain $(X_n)_{n \geq 0} = (x^0, A)$ can be represented by a $(\oplus, \otimes)$-linear system:

$$\begin{cases} x^0 \\ x^{n+1} \quad = x^n \otimes A, \ \forall n \in \mathbb{N}, \end{cases}$$

or equivalently:

$$x^n = x^0 \otimes A^{\otimes n}, \forall n \in \mathbb{N}. \tag{2}$$

Where $A^{\otimes n} = \underbrace{A \otimes \ldots \otimes A}_{n \text{ times}}$, with the $\otimes$-product of two matrices $C \in \mathcal{M}_{n,p}(\mathbb{D})$ and $D \in \mathcal{M}_{p,l}(\mathbb{D})$ which is an element of $\mathcal{M}_{n,l}(\mathbb{D})$ denoted by $C \otimes D$ and defined by:

$$C \otimes D = [\oplus_{k=1}^{p} C(i,k) \otimes D(k,j)]_{i=1,\ldots n\ ;j=1,\ldots,l}$$

and the matrix $A = [A(i,j)]_{i,j \in S}$ is a Bellman matrix, i.e. a non-negative matrix ($\forall i, j,\ \epsilon \prec A(i,j)$) such that

$$A \otimes \vec{e}^T = \vec{e}^T.$$

Noticing similarities between evolution equations of Markov chains (1) and Bellman chains (2) we develop results on comparison of Bellman chains based on Keilson and Kester's work on Markov chains comparison [4] where the matrix

$$K = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{3}$$

plays a fundamental role for comparing Markov chains. Our work and results are based on the fundamental matrix

$$\mathbb{K} = \begin{pmatrix} e & \epsilon & \epsilon & \cdots & \epsilon \\ e & e & \epsilon & \cdots & \epsilon \\ e & e & e & \epsilon \cdots & \epsilon \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ e & e & e & e & e \end{pmatrix}. \tag{4}$$

The paper is organized as follows. Section 2 presents key-ideas to compare Markov chains. Section 3 contains all new results to compare Bellman chains. In Section 4 we present a numerical example to illustrate main concepts of the paper. Section 5 offers a conclusion.

2. STOCHASTIC COMPARISON OF MARKOV CHAINS

We recall main concepts and results of Keilson and Kester [4] which provide our main results, Section 3.

Let X (resp. Y) be an S-valued random variable with probability distribution $x = (x_1, \ldots, x_s)$ (resp. $y = (y_1, \ldots, y_s)$).

Definition 1. (K-comparison) We say that X is *K-smaller to Y* iff

$$x\,K \leq y\,K \text{ (component-wise)}, \tag{5}$$

recalling that K is an $s \times s$ matrix defined by

$$K = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

If the previous condition (5) is fulfilled then we write $X \leq_K Y$ or also $x \leq_K y$.

Let us consider A a stochastic matrix.

Definition 2. (K-monotonicity) Let A be an $s \times s$ stochastic matrix. A is *K-monotone* iff

$$\forall x, y, \ ((x \leq_K y) \Longrightarrow (x\,A \leq_K y\,A)).$$

From this definition it is interesting to mention the necessary and sufficient condition (NSC) for K-monotonicity in the following result.

Result 1. A is an $s \times s$ K-monotone matrix iff

$$\forall i = 1, \ldots, s-1, \ A(i, \cdot) \leq_K A(i+1, \cdot). \tag{6}$$

Where $A(i, \cdot)$ denotes the ith row of matrix A.

Let us now recall the main result on stochastic majorization.

Result 2. (K-comparison of Markov chains) Let $(X_n)_{n \geq 0} = (x^0, A)$ (resp. $(Y_n)_{n \geq 0} = (y^0, B)$) be an S-valued Markov chain.

If

(i) $x^0 \leq_K y^0$,

(ii) $A\,K \leq B\,K$ (coefficient-wise),

(iii) A or B is K-monotone.

Then $\forall n \geq 0$, $X_n \leq_K Y_n$, which is equivalent to:

$$\forall n \geq 0, \ x^0 A^n \leq_K y^0 B^n.$$

3. COMPARISON OF BELLMAN CHAINS

This is the main part of this paper. We aim to present the main results dealing with comparison of Bellman chains.

Let X (resp. Y) be an S-valued decision variable with cost density $x = (x_1, \dots, x_s)$ (resp. $y = (y_1, \dots, y_s)$). We propose to define $\mathbb{K}$-comparison of decision variables based on the K-comparison of random variables, Definition 1.

Definition 3. ($\mathbb{K}$-comparison) We say that X is *$\mathbb{K}$-smaller to Y* iff

$$x \otimes \mathbb{K} \prec y \otimes \mathbb{K} \text{ (component-wise)}, \tag{7}$$

recalling that $\mathbb{K}$ is an $s \times s$ matrix defined by

$$\mathbb{K} = \begin{pmatrix} e & \epsilon & \epsilon & \cdots & \epsilon \\ e & e & \epsilon & \cdots & \epsilon \\ e & e & e & \epsilon \cdots & \epsilon \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ e & e & e & e & e \end{pmatrix}.$$

If the previous condition (7) is fulfilled then we write $X \prec_{\mathbb{K}} Y$ or also $x \prec_{\mathbb{K}} y$.

Let us give some basic properties of the relation $\prec_{\mathbb{K}}$ which proofs are obvious.

Property 1. The relation $\prec_{\mathbb{K}}$ is

(i) reflexive,

(ii) and transitive.

Monotonicity can be defined in the same way as for matrices with entries in $\mathbb{M}$, Definition 2.

Definition 4. ($\mathbb{K}$-monotone Bellman matrix) Let A be an $s \times s$ Bellman matrix. A is said to be *$\mathbb{K}$-monotone* iff

$$\forall x, y,\ ((x \prec_{\mathbb{K}} y) \Longrightarrow (x \otimes A \prec_{\mathbb{K}} y \otimes A)).$$

The next theorem is a logically equivalent characterization for $\mathbb{K}$-monotonicity. It is inspired by the NSC for K-monotonicity, Result 1.

Theorem 1. (NSC for $\mathbb{K}$-monotonicity) Let A be an $s \times s$ Bellman matrix. A is $\mathbb{K}$-monotone iff

$$\forall i = 1, \dots, s-1,\ A(i, \cdot) \prec_{\mathbb{K}} A(i+1, \cdot), \tag{8}$$

where $A(i, \cdot)$ denotes the ith row of matrix A.

Proof. The *(Only if)* part of the proof is due to a remark of S. Gaubert during a discussion.

(Only if). Let us note that $e_i \prec_{I\!K} e_{i+1}$, $i = 1, \dots s-1$, where e_i denotes vector where the ith component is e and the others are ϵ. Thus, because A is $I\!K$-monotone, $e_i \otimes A = A(i, \cdot) \leq_{I\!K} e_{i+1} \otimes A = A(i+1, \cdot)$.

(If). Let us consider x, y such that $x \prec_{I\!K} y$. We write:

$$y \otimes A \otimes I\!K = \oplus_{i=1}^{s} y_i \otimes A(i, \cdot) \otimes I\!K. \tag{9}$$

Because of (8) and by transitivity of $\prec$ we have:

$$A(1, \cdot) \otimes I\!K \prec A(2, \cdot) \otimes I\!K \prec \dots \prec A(s, \cdot) \otimes I\!K.$$

This could be rewritten using idempotency of $\oplus$:

$$\begin{aligned} A(2, \cdot) \otimes I\!K &= A(1, \cdot) \otimes I\!K \oplus A(2, \cdot) \otimes I\!K, \\ A(3, \cdot) \otimes I\!K &= A(1, \cdot) \otimes I\!K \oplus A(2, \cdot) \otimes I\!K \oplus A(3, \cdot) \otimes I\!K, \\ \dots & \quad \dots \\ A(s, \cdot) \otimes I\!K &= A(1, \cdot) \otimes I\!K \oplus \dots \oplus A(s, \cdot) \otimes I\!K. \end{aligned} \tag{10}$$

Where the $\oplus$-sum of two matrices $C \in \mathcal{M}_{n,p}(I\!D)$ and $D \in \mathcal{M}_{n,p}(I\!D)$ which is an element of $\mathcal{M}_{n,p}(I\!D)$ denoted by $C \oplus D$ and defined by:

$$C \oplus D = [C(i,j) \oplus D(i,j)]_{i=1,\dots n; j=1,\dots,p}.$$

Now using the fact y is a cost density vector (i. e., $\oplus_{i=1}^{s} y_i = e$), result (10) and associativity of $\oplus$ and distributivity of $\oplus$ over $\otimes$, we have:

$$\begin{aligned} y \otimes A \otimes I\!K = A(1, \cdot) \otimes I\!K \oplus (\oplus_{i=2}^{s} y_i) \otimes A(2, \cdot) \otimes I\!K \oplus \dots \\ \dots \oplus \dots \oplus y_s \otimes A(s, \cdot) \otimes I\!K. \end{aligned}$$

Because $x \prec_{I\!K} y$, that is $\forall j$, $(\oplus_{i=j}^{s} x_i) \oplus (\oplus_{i=j}^{s} y_i) = \oplus_{i=j}^{s} y_i$, we obtain:

$$y \otimes A \otimes I\!K = x \otimes A \otimes I\!K \oplus y \otimes A \otimes I\!K \text{ (component-wise)},$$

which is equivalent to $x \otimes A \prec_{I\!K} y \otimes A$ and ends the proof. □

The next theorem is the main result of this paper. Once again let us note that it is inspired by K-comparison of Markov chains, Result 2.

Theorem 2. ($I\!K$-comparison of Bellman chains) Let $(X_n)_{n \geq 0} = (x^0, A)$ (resp. $(Y_n)_{n \geq 0} = (y^0, B)$) be an S-valued Bellman chain.

If

(i) $x^0 \prec_{I\!K} y^0$,

(ii) $A \otimes I\!K \prec B \otimes I\!K$ (coefficient-wise),

(iii) A or B is $\mathbb{K}$-monotone.

Then $\forall n \geq 0$, $X_n \prec_{\mathbb{K}} Y_n$, which is equivalent to:

$$\forall n \geq 0,\ x^0 \otimes A^{\otimes n} \prec_{\mathbb{K}} y^0 \otimes B^{\otimes n}.$$

Proof. Assume that A is $\mathbb{K}$-monotone. Because of (ii), we have

$$y^0 \otimes A \otimes \mathbb{K} \prec y^0 \otimes B \otimes \mathbb{K} \text{ (component-wise)}.$$

Because of (i) and Definition 3 (apply to $x = x^0$, $y = y^0$ and matrix A), we thus have

$$x^0 \otimes A \otimes \mathbb{K} \prec y^0 \otimes A \otimes \mathbb{K}.$$

By transitivity of $\prec$ we obtain:

$$x^0 \otimes A \otimes \mathbb{K} \prec y^0 \otimes B \otimes \mathbb{K}.$$

Thus we proved that $x^0 \prec_{\mathbb{K}} y^0 \Longrightarrow x^1 = x^0 \otimes A \prec_{\mathbb{K}} y^0 \otimes B = y^1$. Now, the proof is easily achieved by induction on n. □

4. ILLUSTRATIVE EXAMPLE

Let us recall that the main motivation to study Bellman chains is its link with dynamic programming (see e.g. [2]). However, in this section we only develop a small example to illustrate the main concepts of the paper.

We consider the state space $S = \{1, 2, 3\}$. Then the matrix $\mathbb{K}$ defined by (4) is

$$\mathbb{K} = \begin{pmatrix} e & \epsilon & \epsilon \\ e & e & \epsilon \\ e & e & e \end{pmatrix}.$$

First we present $\mathbb{K}$-comparison of performance measure. Then we illustrate necessary and sufficient condition for $\mathbb{K}$-monotonicity of a Bellman matrix (see Theorem 1). Finally, we illustrate the main result of the paper dealing with comparison of Bellman chains (see Theorem 2).

4.1. $\mathbb{K}$-comparison of cost functions

Let $x^0 = (0, -2, -3)$ and $y^0 = (-20, 0, -1)$ be two cost density measures.

By computing $x^0 \otimes \mathbb{K} = (0, -2, -3)$ and $y^0 \otimes \mathbb{K} = (0, 0, -1)$ we conclude that:

$$x^0 \otimes \mathbb{K} \prec y^0 \otimes \mathbb{K} \text{ (component-wise)}.$$

Thus x^0 is $\mathbb{K}$-smaller than y^0.

4.2. $\mathbb{K}$-monotone matrix

Let us consider the following 3×3 Bellman matrix

$$B = \begin{pmatrix} 0 & -4 & -7 \\ 0 & -3 & -6 \\ -\infty & 0 & -2 \end{pmatrix}.$$

The aim is to show that B is $\mathbb{K}$-monotone because it satisfies (8), Theorem 1. Let us compute $B \otimes \mathbb{K}$:

$$B \otimes \mathbb{K} = \begin{pmatrix} 0 & -4 & -7 \\ 0 & -3 & -6 \\ 0 & 0 & -2 \end{pmatrix}.$$

We check that: $(0,-4,-7) \prec (0,-3,-6) \prec (0,0,-2)$ (component-wise). This is the NSC for $\mathbb{K}$-monotonicity, Theorem 1. The dynamical aspects of a $\mathbb{K}$-monotone Bellman chain are illustrated in Table 1.

Table 1. Monotone Dynamics.

n	$x^0 \otimes B^{\otimes n}$	$y^0 \otimes B^{\otimes n}$
0	$(0,-2,-3)$	$(-20,0,-1)$
1	$(0,-3,-5)$	$(0,-1,-3)$
2	$(0,-4,-7)$	$(0,-3,-5)$
3	$(0,-4,-7)$	$(0,-4,-7)$

4.3. $\mathbb{K}$-comparison of Bellman Chains

In Table 2 we illustrate the result of Theorem 2 on comparison of Bellman chains (x^0, A) and (y^0, B) where:

$$x^0 = (0,-2,-3), \; y^0 = (-20,0,-1)$$

and

$$B = \begin{pmatrix} 0 & -4 & -7 \\ 0 & -3 & -6 \\ -\infty & 0 & -2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -5 & -10 \\ 0 & -7 & -11 \\ 0 & -1 & -5 \end{pmatrix}.$$

Note that the following conditions of Theorem 2 are fulfilled, i.e.:

(i) $x^0 \prec_{\mathbb{K}} y^0$,

(ii) $A \otimes \mathbb{K} \prec B \otimes \mathbb{K}$ (coefficient-wise),

(iii) B is $\mathbb{K}$-monotone.

And we can check in Table 2 that $\forall n, x^0 \otimes A^{\otimes n} \prec_{I\!K} y^0 \otimes B^{\otimes n}$.

Table 2. Comparison results.

n	$x^0 \otimes A^{\otimes n}$	$y^0 \otimes B^{\otimes n}$
0	$(0,-2,-3)$	$(-20,0,-1)$
1	$(0,-4,-8)$	$(0,-1,-3)$
2	$(0,-5,-10)$	$(0,-3,-5)$

5. CONCLUSION

In this paper we presented fundamental tools for comparing Bellman chains. It is inspired by Keilson and Kester's work on monotone Markov chains and the fact that Bellman chains are analogue of Markov chains up to a semimodule.

As further work we aim to develop theoretical aspects on comparison of cost measures. We also aim to develop algebraic approaches for bounding Bellman chains with large number of states by Bellman chains with reduced state space.

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