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ON CONTINUOUS CONVERGENCE AND EPI-CONVERGENCE OF RANDOM FUNCTIONS
Part II: Sufficient Conditions and Applications

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Part II of the paper aims at providing conditions which may serve as a bridge between existing stability assertions and asymptotic results in probability theory and statistics. Special emphasis is put on functions that are expectations with respect to random probability measures. Discontinuous integrands are also taken into account. The results are illustrated applying them to functions that represent probabilities.

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1. INTRODUCTION

When considering general stability statements, one observes that there are two kinds of assumptions: conditions on the true (or limit) problem and conditions which assume continuous convergence or epi-convergence a.s. or in the deterministic sense for the objective functions and/or continuous convergence a.s. or in the deterministic sense for the constraint functions. The functions are in many problems of the same form, e. g. functions that are integrals with respect to random or deterministic probability measures. Hence, sufficient conditions for continuous convergence and epi-convergence a.s. of such functions would be useful tools for the derivation of stability statements for diverse applications.

Convergence in probability comes into play, if one has weakly consistent estimates only or dependent samples for which conditions that imply a.s. convergence cannot be verified. Stability statements in the "in probability" setting, which are similar to those in the a.s. case, are available, see Part I of our paper, i. e. [15], Chap. 4,

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or [27] for details. In short, the assumptions on the true problem are the same as in the a.s. case, and convergence of the approximating functions is now required in probability. Sufficient conditions for this case are investigated only in a few papers, mostly in form of large deviations results (cf. [10, 12, 26]).

The present paper aims at providing general approaches for the verification of continuous convergence and epi-convergence a.s. and in probability. We prove results which pave the way for the direct exploitation of asymptotic results in probability theory and statistics. Hence new consistency results for estimates or new laws of large numbers for dependent samples, for instance, may immediately be utilized for stability statements without further inspection of complicated proofs.

Firstly, we investigate the so-called case of estimated parameters. We consider random functions having the form $f_n(x,\omega) = \hat{f}(x, \Lambda_n(\omega))$ and give convergence statements which require convergence a.s. or in probability of the estimates Λ_n and semicontinuity properties of $\hat{f}$. The main part of the paper deals with approximations of a deterministic function which is the expectation of a random function. Such functions occur above all in stochastic programming and in Markovian decision processes, when the distribution function is approximated. Several problems of statistical decision theory fit into this framework, too (cf. [6]). We allow for integrands that are discontinuous, thus functions representing probabilities may also be treated within this setting.

We investigate two general approaches for this case. The first one, called "direct approach", is in the line of "Portmanteau-Theorem like" results (cf. [1, 6, 9, 18, 30, 31]), which are mostly formulated for sequences of deterministic functions, but can be extended to the a.s. case. We prove a statement which replaces usually imposed equitightness conditions by a weaker lower equiintegrability assumption and allows for a relaxation of semicontinuity properties of the integrands.

The second approach, which seems to offer a broader range of applications, is the so-called "pointwise approach". It reduces convergence considerations for random functions to convergence investigations for random variables with values in R^n and is applicable to convergence almost surely and convergence in probability as well. Furthermore, large deviations results may be derived in this way.

The paper is organized as follows. Definitions, notation and theory are placed in Part I, i. e. [15], and here, in Part II, we present sufficient conditions and applications. Section 2 deals with the case of estimated parameters. Section 3 contains statements on the approximation of a deterministic function which is the expectation of a random function and presents the direct and the pointwise approach with applications. Section 4 demonstrates how the results of Section 3 can be applied to functions that represent probabilities and, eventually, probabilistic constraints.

For convenience, let us repeat general setting of the problem considered in the paper. We assume a complete probability space $[\Omega, \mathcal{A}, P]$ be given and suppose that a random optimization problem

$$(I\!P_0) \qquad \min_{x \in \Gamma_0(\omega)} f_0(x, \omega)$$

is approximated by a sequence of surrogate problems

$$(I\!P_n) \qquad \min_{x \in \Gamma_n(\omega)} f_n(x,\omega), \quad n \in \mathbb{N},\ \omega \in \Omega,$$

where $\Gamma_n \,|\, \Omega \to 2^{\mathbb{R}^p}$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, denotes a multifunction with measurable graph, i.e. Graph $\Gamma_n \in \mathcal{A} \otimes \Sigma^p$, and the function $f_n \,|\, \mathbb{R}^p \times \Omega \to \overline{\mathbb{R}}$, $n \in \mathbb{N}_0$, is supposed to be $(\Sigma^p \otimes \mathcal{A},\ \overline{\Sigma})$-measurable. Here Σ denotes the σ-field of Borel sets of $\mathbb{R}$ and $\overline{\Sigma}$ is the σ-field of Borel sets of $\overline{\mathbb{R}}$, i.e. generated by Σ and $\{+\infty\}, \{-\infty\}$. Consequently, Σ^p denotes the σ-field of Borel sets of $\mathbb{R}^p$.

In the sequel, we present conditions for appropriate "convergence" of objective functions f_n to f_0 and constraint sets Γ_n to Γ_0 provided particular cases. Sections 2, 3 consider objective functions and Section 4 deals with probabilistic constraints.

2. FUNCTIONS DEPENDING ON A RANDOM VARIABLE

In this section we shall investigate random functions f_n of a particular shape $f_n(x,\omega) = \hat{f}(x, \Lambda_n(\omega))$, $n \in \mathbb{N}_0$, where $\hat{f} \,|\, \mathbb{R}^p \times \mathbb{R}^u \to \overline{\mathbb{R}}$ is $(\Sigma^p \otimes \Sigma^u, \overline{\Sigma})$-measurable and $\Lambda_n \mid \Omega \to \mathbb{R}^u$, $n \in \mathbb{N}_0$, is a random variable, i.e. $(\mathcal{A}, \Sigma^u)$-measurable.

The special case of $\Lambda_0(\omega) = \lambda_0\ \forall \omega \in \Omega$ often occurs in practical problems. The value λ_0 may be interpreted as an unknown parameter which is approximated by estimates Λ_n, therefore the case considered in this section is usually called "case of estimated parameters". Of course the following propositions also apply to the case that one has a sequence of random processes which is determined by a sequence of random vectors $(\Lambda_n)_{n \in \mathbb{N}}$.

Let us recall definitions we shall use in the text.

Definition 3.1. Let $X \subset \mathbb{R}^p$ and $Y \subset \mathbb{R}^u$. A function $h \,|\, \mathbb{R}^p \times \mathbb{R}^u \to \overline{\mathbb{R}}$ is said to be

i) *lower semicontinuous* (l.s.c) at $X \times Y$ if for each $x_0 \in X$, $\lambda_0 \in Y$

$$\liminf_{\substack{x \to x_0 \\ \lambda \to \lambda_0}} h(x,\lambda) \ \geq\ h(x_0, \lambda_0).$$

ii) *epi-upper semicontinuous* (epi-u.s.c) at $X \times Y$ if for each $x_0 \in X$, $\lambda_0 \in Y$

$$\sup_{V \in \mathcal{N}(x_0)} \ \inf_{W \in \mathcal{N}(\lambda_0)} \ \sup_{\lambda \in W} \inf_{x \in V} h(x,\lambda) \leq h(x_0, \lambda_0).$$

Our definition of the epi-upper semicontinuity of a function coincides with the definition in [21].

Proposition 2.1. Let $\hat{f}$ be l.s.c. at $X \times Y$ and $\Lambda_0 \in Y$ a.s. Then

i) $\left(\Lambda_n \xrightarrow{\text{a.s.}} \Lambda_0\right) \Rightarrow \left(f_n \xrightarrow[X]{\text{l-a.s.}} f_0\right)$,

ii) $\left(\Lambda_n \xrightarrow{\text{prob}} \Lambda_0\right) \Rightarrow \left(f_n \xrightarrow[X]{\text{l-prob}} f_0\right)$.

Proof. i) Let ω be such that $\Lambda_n(\omega) \to \Lambda_0(\omega) \in Y$. Furthermore, let $x_0 \in X$, $(x_n)_{n\in\mathbb{N}}$ with $x_n \to x_0$ be fixed. Since $\hat{f}$ is l.s.c. at $\{x_0\} \times \Lambda_0(\omega)$, to each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\hat{f}(x,\lambda) \geq \hat{f}(x_0, \Lambda_0(\omega)) - \varepsilon \quad \forall x \in U_\delta\{x_0\}, \quad \forall \lambda \in U_\delta\{\Lambda_0(\omega)\}.$$

Consequently there is an $n_0(\varepsilon,\omega)$ with

$$\hat{f}(x_n, \Lambda_n(\omega)) \geq \hat{f}(x_0, \Lambda_0(\omega)) - \varepsilon \quad \forall n \geq n_0(\varepsilon,\omega),$$

which implies

$$\liminf_{n\to+\infty} \hat{f}(x_n, \Lambda_n(\omega)) \geq \hat{f}(x_0, \Lambda_0(\omega))$$

and hence $f_n \xrightarrow[X]{\text{l-a.s.}} f_0$.

i) Let $(f_n)_{n\in\tilde{N}}$ be an arbitrary subsequence of $(f_n)_{n\in\mathbb{N}}$.
As $\Lambda_n \xrightarrow{\text{prob}} \Lambda_0$, the set $\tilde{N}$ contains a subset $\{n_k, k \in \mathbb{N}\}$ with $\Lambda_{n_k} \xrightarrow{\text{a.s.}} \Lambda_0$.
Using (i), $f_{n_k} \xrightarrow[X]{\text{l-a.s.}} f_0$, and by Lemma 4.1 in [15] we obtain $f_n \xrightarrow[X]{\text{l-prob}} f_0$. □

Proposition 2.2. Let $\hat{f}$ be epi-u.s.c. at $X \times Y$ and $\Lambda_0 \in Y$ a.s. Then

i) $\left(\Lambda_n \xrightarrow{\text{a.s.}} \Lambda_0\right) \Rightarrow \left(f_n \xrightarrow[X]{\text{epi-u-a.s.}} f_0\right)$,

ii) $\left(\Lambda_n \xrightarrow{\text{prob}} \Lambda_0\right) \Rightarrow \left(f_n \xrightarrow[X]{\text{epi-u-prob}} f_0\right)$.

Proof. i) We consider an $\omega \in \Omega$ with $\Lambda_n(\omega) \to \Lambda_0(\omega) \in Y$. Then, by the epi-upper semicontinuity, for each neighborhood $V \in \mathcal{N}(x_0)$ we have

$$\limsup_{n\to+\infty} \inf_{x\in V} \hat{f}(x, \Lambda_n(\omega)) \leq \hat{f}(x_0, \Lambda_0(\omega)).$$

Hence $f_n \xrightarrow[X]{\text{epi-u-a.s.}} f_0$.

ii) The second assertion can be proved via Lemma 4.3 in [15] using a similar idea as in the proof of Proposition 2.1. □

If the estimates $(\Lambda_n)_{n\in\mathbb{N}}$ converge to λ_0 in probability with a given convergence rate, similar assertions may be proved, which show that the convergence rate carries over to the convergence of the random functions and further – under additional assumptions – to the optimal values and solution sets of the surrogate problem.

3. EXPECTATIONS WITH RESPECT TO DETERMINISTIC OR RANDOM DISTRIBUTION FUNCTIONS

We are going to investigate functions f_0 which can be regarded as the expectation of a function φ_0 depending on a random variable $Z|[\Omega, \mathcal{A}, P] \to [\mathbb{R}^m, \Sigma^m]$. Such functions play an important role in stochastic programming, cf. [30], and in Markovian decision processes, cf. [16], where mostly the expected reward or the expected costs are to be optimized.

Firstly, we consider the case that the probability distribution P_0 of Z is approximated by a sequence $(P_n)_{n\in\mathbb{N}}$ of (deterministic) probability measures on $[\mathbb{R}^m, \Sigma^m]$. This case occurs for instance if a well-known but complicated distribution is approximated by simpler ones, especially for numerical reasons. Often, we also allow that the integrand φ_0 is approximated by a sequence of φ_n. Then we have to deal with the deterministic functions

$$f_{n,D}(x) := \int_{\mathbb{R}^m} \varphi_n(x,z)\,\mathrm{d}P_n(z), \quad n \in \mathbb{N}_0, \tag{3.1}$$

where $\varphi_n|\mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}$, $n \in \mathbb{N}_0$, are supposed to be integrable with respect to the second variable.

Of course statements for this case can be used to derive assertions for the almost-surely-setting, too.

Results on the continuous or epi- convergence of $(f_{n,D})_{n\in\mathbb{N}}$ have been obtained by [1, 6, 9, 11, 18], and [30]. The results by [16], which are formulated in a somewhat different manner, should be mentioned too. The authors employ modifications of the Portmanteau Theorem. With exception of [1] (semi)continuity of $\varphi_0(x_0,\cdot)$ is supposed. There are, however, several applications, where this assumption is a serious obstacle. This restriction is overcome in [1], allowing for functions which are discontinuous on a set of measure zero. We will contribute to this approach by weakening the conditions imposed in [1].

Another approach for the verification of $f_{n,D} \xrightarrow[\{x_0\}]{1} f_{0,D}$, which will be suggested here, makes use of the convergence of $\int_{\mathbb{R}^m} \inf_{x\in U_\varepsilon\{x_0\}} \varphi_n(x,z)\,\mathrm{d}P_n(z)$ if n tends to infinity. This approach seems to be useful especially in the random setting, when an unknown probability measure P_0 is estimated, e. g. by the empirical measure. We shall give an assertion which may serve as a bridge to the vast literature on asymptotic results in probability theory and statistics, such as laws of large numbers, statements on the asymptotic behavior of density estimators and so on. Using this approach, it is possible to see how weaker conditions, for instance new dependence assumptions in laws of large numbers, immediately imply corresponding stability results. Note that also large deviations results may be employed in this way, thus yielding assertions on the convergence rate for convergence in probability, see [26].

We start with the "Portmanteau-Theorem-like" approach and quote the result obtained by [1]. In [1] random variables with values in a complete separable metric space are considered. We confine ourselves – according to our framework – to $\mathbb{R}^n$.

Definition 3.1. By $\operatorname{lc} h$ we denote the lower closure of a function $h|\mathbb{R}^m \to \mathbb{R}$, i. e. $\operatorname{lc} h(z_0) := \liminf_{z \to z_0} h(z)$ for each $z_0 \in \mathbb{R}^m$.

Theorem 3.5 in [1] says: Let the following assumptions be satisfied at $x_0 \in \mathbb{R}^p$:

(AW1) $(P_n)_{n\in\mathbb{N}}$ converges weakly to P_0.

(AW2) $\varphi_0(x_0,\cdot)$ is l.s.c. P_0-almost everywhere.

(AW3) $\forall z \in \mathbb{R}^m \quad \forall (x_n)_{n\in\mathbb{N}}$ with $x_n \to x_0$ $\forall (z_n)_{n\in\mathbb{N}}$ with $z_n \to z$

$\liminf_{n\to+\infty} \varphi_n(x_n, z_n) \;\geq\; \operatorname{lc}(\varphi_0(x_0,\cdot)(z)$ (episublimit condition).

(AW4) The family $W = \{(\varphi_n(x_n,\cdot), P_n), n \in \mathbb{N}_0\}$ is equitight for all sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \to x_0$, i. e. $\forall \varepsilon > 0\ \exists K_\varepsilon \in C^m\ \exists b_\varepsilon > 0\ \forall (\varphi_n(x_n,\cdot), P_n) \in W$:

(AW4a) $P_n(\mathbb{R}^m \backslash K_\varepsilon) < \varepsilon$,

(AW4b) $|\varphi_n(x_n, z)| \leq b_\varepsilon$ for P_n-almost all $z \in K_\varepsilon$,

(AW4c) $\int_{\mathbb{R}^m\backslash K_\varepsilon} |\varphi_n(x_n,z)|\, \mathrm{d}P_n(z) < \varepsilon$ (equitightness).

Then $f_{n,D} \xrightarrow[\{x_0\}]{1} f_{0,D}$.

This result may be weakened replacing the equitightness condition by a weaker assumption.

Definition 3.2. The family $\{(f_n, P_n), n \in \mathbb{N}\}$ is said to be lower equiintegrable if

$$\lim_{\Delta\to+\infty} \inf_{n\in\mathbb{N}} \int_{\mathbb{R}^m} f_n(z)\chi_{\{f_n(z)<-\Delta\}}\ \mathrm{d}P_n(z) \;=\; 0.$$

Theorem 3.1. Let for each $n \in \mathbb{N}$ the function φ_n be written in the form $\varphi_n(x,z) = \psi_n(x,z) + \xi_n(x,z)$, where the following conditions are fulfilled at $x_0 \in \mathbb{R}^p$ for each sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to x_0$:

(V1) $(P_n)_{n\in\mathbb{N}}$ converges weakly to P_0.

(V2) $\liminf_{n\to+\infty} \psi_n(x_n, z_n) \;\geq\; \varphi_0(x_0, z)$ for P_0-almost all z and all sequences $(z_n)_{n\in\mathbb{N}}$ with $z_n \to z$.

(V3) The functions $\varphi_n(x_n,\cdot)$ are P_n-integrable for each $n \in \mathbb{N}_0$, and the family $\{(\psi_n(x_n,\cdot), P_n), n \in \mathbb{N}\}$ is lower equiintegrable,

(V4) $\inf_{n\in\mathbb{N}} \int_{\mathbb{R}^m} \xi_n(x_n,z)\,\mathrm{d}P_n(z) \;>\; -\infty$ and $\liminf_{n\to+\infty} \int_{\mathbb{R}^m} \xi_n(x_n,z)\,\mathrm{d}P_n(z) \;\geq\; 0$.

Then $f_{n,D} \xrightarrow[\{x_0\}]{1} f_{0,D}$.

P r o o f. If $\int_{\mathbb{R}^m} \varphi_0(x_0,z)\,\mathrm{d}P_0(z) = -\infty$, we have nothing to prove. Assume, therefore, this integral greater than $-\infty$ and consider

$$\int_{\mathbb{R}^m} \varphi_n(x_n,z)\,\mathrm{d}P_n(z) \;=\; \int_{\mathbb{R}^m} \psi_n(x_n,z)\,\mathrm{d}P_n(z) \;+\; \int_{\mathbb{R}^m} \xi_n(x_n,z)\,\mathrm{d}P_n(z).$$

The second summand is non-negative if $n \to +\infty$ and needs no further investigation.

In order to estimate the first integral we take an arbitrary but fixed $\epsilon > 0$. Since $\{(\psi_n(x_n,\cdot),P_n), n \in \mathbb{N}\}$ is lower equiintegrable, there is $\Delta > 0$ such that

$$\int_{\mathbb{R}^m} (\psi_n(x_n,z)+\Delta)\chi_{\{\psi_n(x_n,z)<-\Delta\}}\;\mathrm{d}P_n(z) > -\epsilon$$

for each $n \in \mathbb{N}$.

Hence, we can estimate the integral

$$\begin{aligned}
&\int_{\mathbb{R}^m} \psi_n(x_n,z)\,\mathrm{d}P_n(z)\\
=\;& \int_{\mathbb{R}^m} \max\{-\Delta,\psi_n(x_n,z)\}\,\mathrm{d}P_n(z) + \int_{\mathbb{R}^m} (\psi_n(x_n,z)+\Delta)\chi_{\{\psi_n(x_n,z)<-\Delta\}}\;\mathrm{d}P_n(z)\\
\geq\;& \int_{\mathbb{R}^m} \max\{-\Delta,\psi_n(x_n,z)\}\,\mathrm{d}P_n(z) - \epsilon \;\geq\; \int_{\mathbb{R}^m} \max\{-\Delta,\inf_{j\geq k}\psi_j(x_j,z)\}\,\mathrm{d}P_n(z) - \epsilon\\
\geq\;& \int_{\mathbb{R}^m} \max\left\{-\Delta,\,\mathrm{cl}\inf_{j\geq k}\psi_j(x_j,\cdot)(z)\right\}\,\mathrm{d}P_n(z) - \epsilon \quad \text{for each } n \geq k,\; n,k \in \mathbb{N}.
\end{aligned}$$

The function $\max\left\{-\Delta,\,\mathrm{cl}\inf_{j\geq k}\psi_j(x_j,\cdot)\right\}$ is l.s.c. and bounded from below. That together with (V1) is giving

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^m} \psi_n(x_n,z)\,\mathrm{d}P_n(z) \;\geq\; \int_{\mathbb{R}^m} \max\left\{-\Delta,\,\mathrm{cl}\inf_{j\geq k}\psi_j(x_j,\cdot)(z)\right\}\,\mathrm{d}P_0(z) - \epsilon \quad \text{for each } k \in \mathbb{N}.$$

The assumption $\liminf_{n\to+\infty}\psi_n(x_n,z_n) \;\geq\; \varphi_0(x_0,z)$ whenever $z_n \to z$ for P_0-almost all $z \in \mathbb{R}^m$ is giving $\lim_{k\to+\infty}\;\mathrm{cl}\inf_{j\geq k}\psi_j(x_j,\cdot)(z) \;\geq\; \varphi_0(x_0,z)$ for P_0-almost all $z \in \mathbb{R}^m$.

Hence, letting k tend to infinity and using monotone convergence lemma, we obtain

$$\begin{aligned}
&\liminf_{n\to+\infty}\int_{\mathbb{R}^m} \varphi_n(x_n,z)\,\mathrm{d}P_n(z) \geq \int_{\mathbb{R}^m} \max\{-\Delta,\varphi_0(x_0,z)\}\,\mathrm{d}P_0(z) - \epsilon\\
\geq\;& \int_{\mathbb{R}^m} \varphi_0(x_0,z)\}\,\mathrm{d}P_0(z) - \epsilon.
\end{aligned}$$

That concludes our proof since the inequality is fulfilled for each $\epsilon > 0$. □

If the assumptions (AW2) and (AW3) are satisfied, the condition (V2) is fulfilled for $(\varphi_n)_{n\in\mathbb{N}}$, hence one can choose $\xi_n(x,z) = 0$ for all $x \in \mathbb{R}^p$ and all $z \in \mathbb{R}^m$. (AW4) implies (V3). Thus Theorem 3.1 generalizes the result derived by [1]. The introduction of the functions ξ_n allows for a relaxation of the "lower semicontinuous" behavior of $(\varphi_n)_{n\in\mathbb{N}}$,

Example 3.1. Let $p = m = 1$, $\Omega = [0,1]$, $\mathcal{A} = \Sigma_{[0,1]}$ the σ-field of Borel subsets of $[0,1]$ and $P_0(A) = P_n(A) = \lambda A \cap [0,1]$ where $A \in \mathcal{A}$ and λ denotes the Lebesgue measure. Furthermore, suppose that $\varphi_0(x,z) = 0\ \forall x \in \mathbb{R}\ \forall z \in [0,1]$,
and for $n = 2^l + k$, $l \in \mathbb{N}_0$, $k \in \{0,1,\dots,2^l-1\}$,

$$\varphi_n(x,z) := \begin{cases} 2^{\frac{l}{2}} & \text{if } z \in [\frac{k}{2^l}, \frac{k+1}{2^l}] \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_{n,D}(x) = 2^{-\frac{l}{2}}$ and $f_{0,D}(x) = 0\ \forall x \in \mathbb{R}\ \forall n \in \mathbb{N}$. Hence $(f_{n,D})_{n\in\mathbb{N}}$ converges continuously to $f_{0,D}$ on $\mathbb{R}$. However, (AW4) is violated.

In order to show $f_{n,D} \xrightarrow[\{x_0\}]{\text{l}} f_{0,D}$, making use of Theorem 3.1, one can take $\xi_n(x,z) = 0$ and $\psi_n(x,z) = \varphi_n(x,z)\ \forall x\ \forall z\ \forall n$. The choice $\xi_n(x,z) = \varphi_n(x,z)$ and $\psi_n(x,z) = 0\ \forall x\ \forall z\ \forall n$ is also convenient. In order to show $f_{n,D} \xrightarrow[\{x_0\}]{\text{u}} f_{0,D}$, one cannot take $\xi_n(x,z) = 0$ and $\psi_n(x,z) = -\varphi_n(x,z)\ \forall x\ \forall z\ \forall n$. The choice does not fulfill (V2) because $\liminf_{n\to+\infty} \psi_n(x_n,z) = -\infty\ \forall z \in [0,1]$. Nevertheless, the choice $\xi_n(x,z) = -\varphi_n(x,z)$ and $\psi_n(x,z) = 0\ \forall x\ \forall z\ \forall n$ is convenient.

We now turn to the pointwise approach which was already considered in [28]. It may also be used to derive assertions in the deterministic sense. But as it is directly applicable to random problems too, it will be formulated in the random setting. Let $f_n(x,\omega) = \int_{\mathbb{R}^m} \varphi_n(x,z)\,\mathrm{d}P_n(z,\omega)$, $n \in \mathbb{N}$, and let $f_{0,D}$ be defined as before. Thus, dealing with a deterministic "limit function" $f_{0,D}$ we can employ the results in [27], section V, which are repeated in [15], Chap. 5.

It is well known that pointwise convergence implies the upper part of epi-convergence, hence Proposition 3.1 is obvious.

Proposition 3.1. If $(x_n)_{n\in\mathbb{N}}$ is a sequence with $x_n \to x_0$ then

(i) $$\left(f_n(x_n,\cdot) \xrightarrow{\text{u-a.s.}} f_{0,D}(x_0)\right) \Rightarrow \left(f_n \xrightarrow[\{x_0\}]{\text{epi-u-a.s.}} f_{0,D}\right),$$

(ii) $$\left(f_n(x_n,\cdot) \xrightarrow{\text{u-prob}} f_{0,D}(x_0)\right) \Rightarrow \left(f_n \xrightarrow[\{x_0\}]{\text{epi-u-prob}} f_{0,D}\right).$$

In order to prove a similar result for the lower semicontinuous approximation we introduce for each $\varepsilon > 0$ the auxiliary quantities

$$H_n^\varepsilon(x,\omega) \quad := \quad \int_{\mathbb{R}^m} \inf_{\tilde{x}\in\overline{U}_\varepsilon\{x\}} \varphi_n(\tilde{x},z)\,\mathrm{d}P_n(z,\omega),$$

$$H_{0,D}^\varepsilon(x) \quad := \quad \int_{\mathbb{R}^m} \inf_{\tilde{x}\in\overline{U}_\varepsilon\{x\}} \varphi_0(\tilde{x},z)\,\mathrm{d}P_0(z).$$

Here we shall also indicate how large deviations results may be obtained.

Theorem 3.2. Let the following assumptions be satisfied for a given $x_0 \in \mathbb{R}^p$:

(V5) The function $\varphi_0(\cdot,z)$ is l.s.c. at the point x_0 for P_0-almost all z.

(V6) There is $\tilde{\varepsilon} > 0$ such that $H_{0,D}^{\tilde{\varepsilon}}(x_0) > -\infty$ and $H_n^\varepsilon(x_0,\cdot)$, $H_{0,D}^\varepsilon(x_0)$ exist for each $0 < \varepsilon \le \tilde{\varepsilon}$ and each $n \in \mathbb{N}$.

Then

(i) $\left(H_n^\varepsilon(x_0,\cdot) \xrightarrow{\text{l-a.s.}} H_{0,D}^\varepsilon(x_0) \quad \forall \varepsilon \le \tilde{\varepsilon}\right) \Longrightarrow (f_n \xrightarrow[\{x_0\}]{p\text{-l-a.s.}} f_{0,D}),$

(ii) $\left(H_n^\varepsilon(x_0,\cdot) \xrightarrow{\text{l-prob}} H_{0,D}^\varepsilon(x_0) \quad \forall \varepsilon \le \tilde{\varepsilon}\right) \Longrightarrow (f_n \xrightarrow[\{x_0\}]{p\text{-l-prob}} f_{0,D}).$

(iii) $\left(P\left\{\omega : H_n^\varepsilon(x_0,\omega) < H_{0,D}^\varepsilon(x_0) - \varepsilon\right\} \; = \; o(\zeta_n) \quad \forall \varepsilon \le \tilde{\varepsilon}\right) \Longrightarrow$

$$\left(\forall \varepsilon > 0 \; \exists U\{x_0\} \in C^p : \; P\left\{\omega : \inf_{x\in U\{x_0\}} f_n(x,\omega) < f_{0,D}(x_0) - \varepsilon\right\} \; = \; o(\zeta_n)\right),$$

where $(\zeta_n)_{n\in\mathbb{N}}$ is a given sequence tending to zero, the so-called convergence rate.

Proof.

i) According to monotone convergence lemma we have $\sup_{\varepsilon>0} H_{0,D}^\varepsilon(x_0) = f_{0,D}(x_0)$ since $H_{0,D}^{\tilde{\varepsilon}}(x_0) > -\infty$ and (V5) are assumed.

Let $\varepsilon > 0$ be fixed and choose a δ with $0 < \delta < \tilde{\varepsilon}$ and $H_{0,D}^\delta(x_0) > f_{0,D}(x_0) - \frac{\varepsilon}{2}$. Now let ω be such that $\liminf_{n\to+\infty} \inf_{x\in\overline{U}_\delta\{x_0\}} f_n(x,\omega) < f_{0,D}(x_0) - \varepsilon$.

Because of

$$\begin{aligned}\inf_{x\in\overline{U}_\delta\{x_0\}} f_n(x,\omega) &= \inf_{x\in\overline{U}_\delta\{x_0\}} \int_{\mathbb{R}^m} \varphi_n(x,z)\,\mathrm{d}P_n(z,\omega)\\ &\ge \int_{\mathbb{R}^m} \inf_{x\in\overline{U}_\delta\{x_0\}} \varphi_n(x,z)\,\mathrm{d}P_n(z,\omega) = H_n^\delta(x_0,\omega)\end{aligned}$$

we obtain $\liminf_{n\to+\infty} H_n^\delta(x_0,\omega) < H_{0,D}^\delta(x_0) - \frac{\varepsilon}{2}$.

ii) As in part (i) we choose $\varepsilon > 0$ and to ε a $\delta > 0$. Then we have

$$P\left\{\omega : \inf_{x\in\overline{U}_\delta\{x_0\}} f_n(x,\omega) < f_{0,D}(x_0) - \varepsilon\right\} \le P\left\{\omega : H_n^\delta(x_0,\omega) < H_{0,D}^\delta(x_0) - \frac{\varepsilon}{2}\right\},$$

hence the assertion follows.

iii) The assertion immediately follows from (ii), taking into account a convergence rate and choosing $\delta \leq \frac{\varepsilon}{2}$. □

In [14] a similar approach, named scalarization, is suggested. The authors consider functions which are defined in complete separable metric spaces. However, consideration is restricted to convergence a.s.

In the rest of the paper we shall show, how Theorem 3.2 can be used in several applications. The assertions will be formulated for general functions $h_n|\mathbb{R}^m \to \mathbb{R}$, $n \in \mathbb{N}_0$. One can plug in $h_n = \varphi_n(x_n, \cdot)$ in order to derive epi-upper semicontinuous approximation applying Proposition 3.1 or $h_n = \inf_{x \in \overline{U}_\varepsilon\{x_0\}} \varphi_n(x, \cdot)$ deducing lower semicontinuous approximation from Theorem 3.2.

We distinguish three cases for the approximation of P_0, namely

a) approximation by a sequence $(P_n)_{n \in \mathbb{N}}$ of deterministic probability measures,

b) approximation by the empirical measure,

c) approximation by means of density estimators.

a) We can employ Theorem 5.5 in [3] and ideas of the proof of Theorem 5.2 in [4] to obtain Proposition 3.2.

Proposition 3.2. Let (V1) be satisfied and suppose that the following assumptions are satisfied:

(V7) $\quad P_0\{z : h_n \xrightarrow[\{z\}]{} h_0\} = 1.$

(V8) $\quad \exists C > 0 \quad \forall z \in \mathbb{R}^m \quad \forall n \in \mathbb{N}_0 : \quad |h_n(z)| \leq C.$

Then

$$\lim_{n \to +\infty} \int_{\mathbb{R}^m} h_n(z) \, \mathrm{d}P_n(z) = \int_{\mathbb{R}^m} h_0(z) \, \mathrm{d}P_0(z).$$

For the special case that only the probability measure is approximated and $\varphi_n(x, z) = \varphi(x, z) \quad \forall n \in \mathbb{N}_0$, by stability theory of parametric programming, the continuity of φ at $\overline{U}_\varepsilon\{x_0\} \times \{z_0\}$ is sufficient for the continuity of $\inf_{x \in \overline{U}_\varepsilon\{x_0\}} \varphi(x, \cdot)$ at $z_0 \in \mathbb{R}^m$ ($\varepsilon > 0$).

The condition (V8) can be weakened, if the sequence $(P_n)_{n \in \mathbb{N}}$ of probability measures fulfills strongei conditions. For instance, if $(P_n)_{n \in \mathbb{N}}$ converges in a Wasserstein metric W_p defined by

$$W_p(P, Q) := \left(\inf_{\eta \in D(P,Q)} \int_{\mathbb{R}^m \times \mathbb{R}^m} ||z - z'||^p \, \mathrm{d}\eta(z, z') \right)^{\frac{1}{p}},$$

where $p \in \mathbb{N}$, P, Q are probability measures on $[\mathbb{R}^m, \Sigma^m]$ and $D(P, Q)$ denotes the class of all probability measures on $[\mathbb{R}^{2m}, \Sigma^{2m}]$ having P and Q as marginal measures.

Proposition 3.3. Let $\lim_{n\to+\infty} W_p(P_n, P_0) = 0$ for a fixed $p \in \mathbb{N}$. Furthermore, suppose that (V7) is satisfied and $|h_n(z)| \leq ||z||^p \quad \forall n \in \mathbb{N}_0$. Then

$$\lim_{n\to+\infty} \int_{\mathbb{R}^m} h_n(z)\,\mathrm{d}P_n(z) = \int_{\mathbb{R}^m} h_0(z)\,\mathrm{d}P_0(z).$$

Proof. According to a result by [20] (see also [23]) the convergence of $(P_n)_{n\in\mathbb{N}}$ to P_0 in the metric W_p is equivalent to the weak convergence of $(P_n)_{n\in\mathbb{N}}$ to P_0 and the condition $\lim_{n\to+\infty} \int_{\mathbb{R}^m} ||z||^p\,\mathrm{d}P_n(z) = \int_{\mathbb{R}^m} ||z||^p\,\mathrm{d}P_0(z)$.
In order to show the statement, we can proceed as follows:

Let $\varepsilon > 0$ be given.

Then there is a $K \in \mathbb{R}$ such that $\int_{\mathbb{R}^m} (||z||^p - \min\{||z||^p, K\})\,\mathrm{d}P_0(z) < \frac{\varepsilon}{2}$.

As $\lim_{n\to+\infty} \int_{\mathbb{R}^m} (\min\{||z||^p, K\})\,\mathrm{d}P_n(z) = \int_{\mathbb{R}^m} (\min\{||z||^p, K\})\,\mathrm{d}P_0(z)$ by the weak convergence of $(P_n)_{n\in\mathbb{N}}$ to P_0 we find an $n_0 = n_0(\varepsilon)$ such that

$$\int_{\mathbb{R}^m} (|h_n(z)| - \min\{|h_n(z)|, K\})\,\mathrm{d}P_n(z) < \frac{\varepsilon}{2} \quad \forall n \geq n_0.$$

Hence the assertion follows from the equality

$$\begin{aligned}\int_{\mathbb{R}^m} h_n(z)\,\mathrm{d}P_n(z) &= \int_{\mathbb{R}^m} h_n(z) - \max\{\min\{h_n(z), K\}, -K\}\,\mathrm{d}P_n(z)\\ &+ \int_{\mathbb{R}^m} \max\{\min\{h_n(z), K\}, -K\} - \max\{\min\{h_0(z), K\}, -K\}\,\mathrm{d}P_n(z)\\ &+ \int_{\mathbb{R}^m} \max\{\min\{h_0(z), K\}, -K\}\,\mathrm{d}P_n(z)\end{aligned}$$

and from the fact that a function $z \mapsto \max\{\min\{h_0(z), K\}, -K\}$ is P_0-a.s. continuous because h_0 is P_0-a.s. continuous, according to (V7). □

b) Let P_n be the empirical measure, i.e.
$P_n(A,\omega) = P_n^E(A,\omega) := \frac{1}{n}\sum_{i=1}^{n} \chi_A(Z_i(\omega)), \quad A \in \Sigma^m$,
where $(Z_n)_{n\in\mathbb{N}}$ is a sequence of random variables $Z_n \mid [\Omega, \mathcal{A}, P] \to [\mathbb{R}^m, \Sigma^m]$.

We have $\int_{\mathbb{R}^m} \varphi_n(x,z)\,\mathrm{d}P_n(z) = \frac{1}{n}\sum_{i=1}^{n} \varphi_n(x, Z_i(\omega))$, thus, in the general case, laws of large numbers for triangular arrays are required. For such results see for instance [17]. An overview on results for the case $\forall n \in \mathbb{N}\ \varphi_n = \varphi_0$ is given in [7].

Epi-convergence a.s. of $(f_n)_{n\in\mathbb{N}}$ to f_0 is investigated in [13], assuming that $\forall n \in \mathbb{N}$ $\varphi_n = \varphi$ is a convex normal integrand.

Since in the case under consideration the equations

$$f_n(x,\omega) = \frac{1}{n}\sum_{i=1}^{n} \varphi_0(x, Z_i(\omega)) \quad \text{and} \quad H_n^{\varepsilon}(x,\omega) = \frac{1}{n}\sum_{i=1}^{n} \inf_{\tilde{x}\in U_\varepsilon\{x\}} \varphi_0(\tilde{x}, Z_i(\omega))$$

hold, the assumptions of Propositions 3.2 and 3.3 require laws of large numbers for sequences of random variables, namely for

$$(\varphi_0(x_0, Z_n(\omega))_{n\in\mathbb{N}} \quad \text{and} \quad \left(\inf_{x\in\overline{U}_\varepsilon\{x_0\}} \varphi_0(x, Z_n(\omega))\right)_{n\in\mathbb{N}} .$$

If the $(Z_n)_{n\in\mathbb{N}}$ are independent and identically distributed according to P_0, the condition $\int_{\mathbb{R}^m} \left| \inf_{x\in\overline{U}_{\tilde{\varepsilon}}\{x_0\}} \varphi_0(x,z)\right| \mathrm{d}P_0(z) < \infty$, $\tilde{\varepsilon} > 0$, is sufficient for the convergence assumptions by Kolmogorov's law of large numbers. This special result was proved in [25]. For more general cases we refer the reader to [8]. Note that, in general, convexity is not required. Furthermore, we can deal with dependent samples assuming ergodicity. Uniform convergence of functions obtained via empirical measures is investigated in [19]. According to Proposition 3.2 in [15] the results obtained here may also be used to ensure uniform convergence if the limit function is continuous.

c) Eventually we consider approximations of P_0 via density estimators.

Suppose that P_0 has the density $\mathfrak{p}_0$ and that $P_n(\cdot,\omega)$ is generated by a density estimator $\mathfrak{p}_n(\cdot,\omega)$. Consistency results for density estimators are given in several forms (cf [5], [17], [24]). Conditions ensuring that

$$\int_{\mathbb{R}^m} |\, \mathfrak{p}_n(z,\cdot) - \mathfrak{p}_0(z)\,|\, \mathrm{d}z \xrightarrow{\text{a.s.}} 0 \tag{3.2}$$

are investigated in [5].

The following simple proposition shows how (3.2) can be employed for our aims, especially for functions originating from chance constraints. Let h_n stand either for $\varphi_n(x_0,\cdot)$ or for $\inf\limits_{x\in\overline{U}_\varepsilon\{x_0\}} \varphi_n(x,\cdot)$, $\varepsilon > 0$.

Proposition 3.4. Let (V8) be satisfied and suppose that the following assumptions hold true.

(i) $P_0\{z : \lim\limits_{n\to+\infty} h_n(z) = h_0(z)\} = 1.$

(ii) $\int_{\mathbb{R}^m} |\, \mathfrak{p}_n(z,\cdot) - \mathfrak{p}_0(z)\,|\, \mathrm{d}z \xrightarrow{\text{a.s.}} 0.$

Then $\int_{\mathbb{R}^m} h_n(z)\mathfrak{p}_n(z,\cdot)\,\mathrm{d}z \xrightarrow{\text{a.s.}} \int_{\mathbb{R}^m} h_0(z)\; \mathfrak{p}_0(z)\,\mathrm{d}z.$

Proof. We have

$$\begin{aligned} &\left|\int_{\mathbb{R}^m} h_n(z)\mathfrak{p}_n(z,\cdot)\,\mathrm{d}z - \int_{\mathbb{R}^m} h_0(z)\mathfrak{p}_0(z)\,\mathrm{d}z\right| \\ \le\; &\left|\int_{\mathbb{R}^m} h_n(z)(\mathfrak{p}_n(z,\cdot) - \mathfrak{p}_0(z))\,\mathrm{d}z\right| + \int_{\mathbb{R}^m} |h_n(z) - h_0(z)|\; \mathfrak{p}_0(z)\,\mathrm{d}z, \end{aligned}$$

hence the first summand tends to zero because of (V8) and the L_1-convergence of the density estimator, the second because of (V8) and the Lebesgue Convergence Theorem. □

A similar assertion holds for convergence in probability.

Remark 3.1. Kernel and histogram estimators of the density fulfill (3.2).

(i) Suppose that $\mathfrak{p}_n$ is a kernel estimator, i. e.

$$\mathfrak{p}_n(z,\omega) = \frac{1}{n(h_n)^m} \sum_{i=1}^{n} \mathfrak{k}\left(\frac{z - Z_i(\omega)}{h_n}\right)$$

where $\mathfrak{k}$ is a Borel function with $\mathfrak{k}(z) \geq 0$, $\int_{\mathbb{R}^m} \mathfrak{k}(z)\,dz = 1$; $h_n > 0$, $n \in \mathbb{N}$; $\lim_{n\to+\infty} h_n = 0$, and $Z_n \,|[\Omega, \mathcal{A}, P] \to [\mathbb{R}^m, \Sigma^m]$.

Theorem 3.1 in [5] showed that (3.2) holds almost surely if the Z_i are i.i.d. and

$$\lim_{n\to+\infty} n(h_n)^m = \infty. \tag{3.3}$$

If $(Z_n)_{n\in\mathbb{N}}$ is a (strictly) stationary sequence the results in [8] may be used. For instance, each of the following two conditions is shown to be sufficient for (3.2):

(A1) $(Z_i)_{i\in\mathbb{N}}$ is φ-mixing and (3.3) holds.

(A2) $(Z_i)_{i\in\mathbb{N}}$ is α-mixing,
for some $\delta_1 > 0$ the inequality $\sum_{i=1}^{\infty} i^{\delta_1}\alpha_i < \infty$ holds and
for some $\delta_2 > \frac{2(1+\delta_1)}{\delta_1}$ the condition $\lim_{n\to+\infty} n\cdot(h_n)^{\delta_2\cdot m} = \infty$ is satisfied.
(Theorem 4.2.1 in [8]).

(ii) If $\mathfrak{p}_n$ is a so-called recursive kernel estimator, i. e.
$\mathfrak{p}_n(z,\omega) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{(h_i)^m}\mathfrak{k}\left(\frac{z-Z_i(\omega)}{h_i}\right)$, where $\mathfrak{k}$, h_i, Z_i are as in (i), similar assertions are available. For instance the condition $\lim_{n\to+\infty} \frac{n(h_n)^m}{\log\log n} = \infty$ implies (3.2) provided Z_i to be i.i.d., cf. [8].

(iii) Finally, we will have a look at histogram estimators.
Let $\mathfrak{R}_n = \{A_{n,1}, A_{n,2}, \ldots\}$ be a sequence of partitions such that $0 < \mathcal{L}(A_{n,j})$ for $n = 1,2\ldots$ and $j = 1,2\ldots$, where $\mathcal{L}$ denotes the Lebesgue-measure. The histogram estimator is defined by
$\mathfrak{p}_n(z,\omega) := \frac{P_n^E(A_{n,i},\omega)}{\mathcal{L}(A_{n,i})}$ if $z \in A_{n,i}$ denoting by P_n^E the empirical probability.

Then the following two conditions (together) imply (3.2) in the i.i.d. case (see [8]):

(A3) For each set A of positive and finite Lebesgue-measure and for each $\varepsilon > 0$ there is an n_0 such that for all $n \geq n_0$ there is

$A_n \in \sigma(\mathfrak{R}_n)$ with $\mathcal{L}(A \triangle A_n) < \varepsilon$

($\sigma(\mathfrak{R}_n)$ denotes the σ-field generated by $\mathfrak{R}_n$ and $\triangle$ the symmetric difference).

(A4) For all $M > 0$ and for all spheres S

$$\limsup_{n \to +\infty} \mathcal{L}\left(\bigcup_{j: \mathcal{L}(A_{n,j} \cap S) \leq \frac{M}{n}} A_{n,j} \cap S \right) = 0.$$

For the dependent case see also [8].

Further assertions that replace the uniform boundedness condition by integrability conditions may be derived using results obtained by [17].

4. FUNCTIONS REPRESENTING PROBABILITIES

We shall now show how probabilistic functions can be investigated within the provided framework. The derivation will be based on the second approach, i. e. Theorem 3.2, because – in our opinion – the conditions are slightly easier to check. However, for the deterministic case, the first approach may be employed in a similar way, making use of Lemma 4.1 and Lemma 4.2 below.

We shall investigate functions $h_{n,D}$ with

$$h_{n,D}(x) = P_n\{z \in \mathbb{R}^m : \gamma_l(x,\ z) \leq 0,\ l = 1, \ldots, q\}, \quad n \in \mathbb{N}_0, \tag{4.1}$$

where the functions $\gamma_l \mid \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}$ are supposed to be measurable with respect to the second variable.

In stochastic programming problems functions of the form $h_{n,D}$ can occur among the objective functions. More often, however, one has to deal with probabilistic constraints, that means, the constraint set $\Gamma_{n,\,D}$ is specified by

$$\Gamma_{n,D} = \{x \in \mathbb{R}^p \mid g^j_{n,D}(x) \leq 0,\ j \in J\}, \tag{4.2}$$

where $g^j_{n,D}(x) = \eta^j - P_n\{z \in \mathbb{R}^m : \gamma^j_l(x,\ z) \leq 0,\ l = 1, \ldots, q^j\}$ and $\eta^j,\ j \in J,$ are given probability levels.

The functions $h_{n,D}$ (and in the same way $g^j_{n,D}$) will be rewritten as

$$\begin{aligned} h_{n,D}(x) &= \int_{\mathbb{R}^m} \chi_{M(x)}(z)\,\mathrm{d}P_n(z) \quad \text{with} \\ M(x) &:= \{z \in \mathbb{R}^m : \gamma_l(x,\ z) \leq 0,\ l = 1, \ldots, q\} \quad \text{and} \\ \chi_A(z) &:= \begin{cases} 1 & \text{if } z \in A \\ 0 & \text{otherwise} \end{cases} \qquad (A \in \Sigma^m). \end{aligned}$$

Consequently, we shall investigate functions $\varphi_1(x,z) := \chi_{M(x)}(z)$ or $\varphi_2^j(x,z) := \eta^j - \chi_{M^j(x)}(z)$, where M^j is defined analogously to M. (The superscript j at φ_2 will be omitted if no misunderstandings are possible.)

Lemma 4.1 and Lemma 4.2 yield conditions that ensure condition (V5) of Theorem 3.2 and allow for conclusions on the continuity of $\inf\limits_{x\in\overline{U}_\varepsilon\{x_0\}} \varphi_n(x,\cdot)$, which are useful for the verification of (V7) in the deterministic case.

Note that the denotation concerning upper semicontinuity of multifunctions is not unique in the literature. The definition used in sequel corresponds to that of closedness in [2].

Lemma 4.1. Let the multifunction $\Psi \mid \mathbb{R}^p \to 2^{\mathbb{R}^m}$ be upper semicontinuous at x_0 (i. e. for all sequences $(x_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ with $x_n \to x_0$, $z_n \in \Psi(x_n)$ and $z_n \to z_0$ it follows that $z_0 \in \Psi(x_0)$). Then χ_Ψ is u.s.c. at $\{x_0\} \times \mathbb{R}^m$.

Proof. Let $z_0 \in \mathbb{R}^m$ be given.

If $\chi_{\Psi(x_0)}(z_0) = 1$ we have nothing to show.

Therefore, assume $\chi_{\Psi(x_0)}(z_0) = 0$.

To z_0 there is a compact neighborhood $\hat{K} \subset \mathbb{R}^m$. We consider the multifunction $\Psi_{\hat{K}}$ with $\Psi_{\hat{K}}(x) := \Psi(x) \cap \hat{K}$. $\Psi_{\hat{K}}$ is upper semicontinuous in the sense of Hausdorff at x_0, i. e. for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\Psi_{\hat{K}}(x) \subset U_\varepsilon(\Psi_{\hat{K}}(x_0)) \quad \forall x \in U_\delta\{x_0\}$.

This also implies closedness of $\Psi_{\hat{K}}(x_0)$. Since z_0 does not belong to the set $\Psi_{\hat{K}}(x_0)$, there is an $\varepsilon > 0$ with $U_\varepsilon\{z_0\} \cap U_\varepsilon(\Psi_{\hat{K}}(x_0)) = \emptyset$ and $U_\varepsilon\{z_0\} \subset \hat{K}$, hence we find a $\delta > 0$ such that $U_\varepsilon\{z_0\} \cap \Psi_{\hat{K}}(x) = \emptyset \quad \forall x \in U_\delta\{x_0\}$.

Thus, no $z \in U_\varepsilon\{z_0\}$ belongs to $\Psi(x)$, consequently $\chi_{\Psi(x)}(z) = 0 \ \forall (x,z) \in U_\delta\{x_0\} \times U_\varepsilon\{z_0\}$. □

Lemma 4.2. Let $\Psi \mid \mathbb{R}^p \to 2^{\mathbb{R}^m}$ be a multifunction. Suppose that there is a multifunction $\Psi^o \mid \mathbb{R}^p \to 2^{\mathbb{R}^m}$ with $\Psi^o(x) \subset \Psi(x) \subset \operatorname{clo}\Psi^o(x) \ \forall x \in U\{x_0\}$ which is strongly l.s.c. at x_0 (i. e. $\forall z \in \Psi^o(x_0)\ \exists\, \varepsilon > 0\ \exists \delta > 0\ \forall x \in U_\delta\{x_0\} : U_\varepsilon\{z\} \subset \Psi^o(x)$).

Then $\partial\Psi(x_0) = \partial\Psi^o(x_0)$ and χ_Ψ is l.s.c. at $\{x_0\} \times (\mathbb{R}^m \backslash \partial\Psi(x_0))$, where ∂A denotes the boundary of $A \subset \mathbb{R}^m$

Proof. The strong lower semicontinuity of Ψ^o implies that $\Psi^o(x_0)$ is an open set.

Hence $\operatorname{clo}\Psi^o(x_0) = \operatorname{clo}\Psi(x_0)$, $\operatorname{int}\Psi^o(x_0) = \operatorname{int}\Psi(x_0)$ and $\partial\Psi^o(x_0) = \partial\Psi(x_0)$

Let $z_0 \in \mathbb{R}^m$ be given. If $\chi_{\Psi(x_0)}(z_0) = 0$ there is nothing to show.

Otherwise we distinguish the cases $z_0 \in \partial\Psi^o(x_0)$ and $z_0 \in \Psi^o(x_0)$.

In the first case we cannot expect that χ_Ψ is l.s.c. at (x_0, z_0).

However, if $z_0 \in \Psi^o(x_0)$ we find an $\varepsilon > 0$ and a $\delta > 0$ such that $U_\varepsilon(z_0) \subset \Psi^o(x) \subset \Psi(x)\ \forall x \in U_\delta\{x_0\} \cap U\{x_0\}$.

This entails $\chi_{\Psi(x)}(z) = 1\ \forall z \in U_\varepsilon\{z_0\}\ \forall x \in U_\delta\{x_0\} \cap U\{x_0\}$. □

The strong lower semicontinuity of Ψ^o cannot be replaced by the lower semicontinuity of Ψ^o in the sense of Berge, consider the following example.

Example 4.1. Let $p = m = 1$. Furthermore put

$$\Psi(x) = \begin{cases} \mathbb{Q} \cap [0,1] & \text{if } x \neq 0, \\ [0,1] & \text{if } x = 0, \end{cases} \qquad \text{and } x_0 = 0.$$

Then for each open set $\mathfrak{M}$ with $\mathfrak{M} \cap \Psi(0) \neq \emptyset$ there exists a $\delta > 0$ such that $\Psi(x) \cap \mathfrak{M} \neq \emptyset \quad \forall x \in U_\delta\{x_0\}$, hence Ψ is l.s.c. in the sense of Berge at x_0.

However, for $z_0 \in (0,\ 1)$, $x_n = \frac{1}{n}$, $z_n \in [0,\ 1]\backslash\mathbb{Q}$ we obtain $\chi_{\Psi(x_n)}(z_n) = 0 < 1 = \chi_{\Psi(0)}(z_0)$.

Remark 4.1. The following sufficient conditions for the continuity assumptions on the multifunction M are well known from parametric programming, see [2]:

(i) The multifunction M is upper semicontinuous at x_0 if the functions γ_l, $l = 1, \ldots, q$, are l.s.c. at $\{x_0\} \times \mathbb{R}^m$.

(ii) Let the functions γ_l, $l = 1, \ldots, q$, be u.s.c. at $\{x_0\} \times M(x_0)$.
Furthermore, suppose that $M(x) \subset \text{clo}\,\{z \in \mathbb{R}^m : \gamma_l(x,\ z) < 0,\ l = 1 \ldots q$ $\forall x \in U\{x_0\}$.
Then the multifunction M^o with $M^o(x) = \{z \in \mathbb{R}^m : \gamma_l(x,\ z) < 0,\ l = 1, \ldots, q\}$ is strongly l.s.c. at x_0 and we have $M^o(x) \subset M(x) \subset \text{clo}\, M^o(x)\ \forall x \in U\{x_0\}$.

Combining Theorem 4.2, Proposition 4.2, Theorem 9 from [27], and the above considerations, we obtain immediately the following assertions.

Proposition 4.1. Let the functions γ_l, $l = 1, \ldots, q$, be l.s.c. on $\{x_0\} \times \mathbb{R}^m$ and suppose that (V1) is satisfied. Then $g_{n,\,D} \xrightarrow[\{x_0\}]{\mathrm{l}} g_{0,\,D}$ and $h_{n,\,D} \xrightarrow[\{x_0\}]{\mathrm{u}} h_{0,\,D}$.

Proposition 4.2. Let, additionally to (V1), the following assumptions be satisfied:

(V9) The functions γ_l, $l = 1, \ldots, q$, are continuous at $\{x_0\} \times \mathbb{R}^m$.

(V10) $M(x) \subset \text{clo}\,\{z \in \mathbb{R}^m : \gamma_l(x,z) < 0,\ l = 1, \ldots, q\} \quad \forall x \in U\{x_0\}$.

(V11) $P_0\,(\partial M(x_0)) = 0$.

Then $g_{n,D} \xrightarrow[\{x_0\}]{} g_{0,\,D}$ and $h_{n,D} \xrightarrow[\{x_0\}]{} h_{0,\,D}$.

Under the assumptions of Proposition 4.2, continuity of $g_{n,D}$ and $h_{n,D}$ is ensured by Proposition 5 in [26]. Assertions on the a.s. and "in probability" sense can be derived in a similar way.

Since $g^j_{n,D} \xrightarrow[\{x_0\}]{\mathrm{l}} g^j_{0,\,D} \quad \forall x_0 \in \mathbb{R}^m \quad \forall j \in J$ implies $\limsup_{n \to +\infty} \Gamma_{n,D} \subset \Gamma_{0,D}$ (cf. Theorem 3.1 in [27]), using the above considerations one can weaken the assumptions of the corresponding results given by [29].

When considering nonrandom objective functions and their modifications

$$\tilde{f}_{n,D}(x) := \begin{cases} f_{n,D}(x) & \text{if } x \in \Gamma_{n,D}, \\ +\infty & \text{otherwise,} \end{cases} \tag{4.3}$$

we need assertions ensuring that $\tilde{f}_{n,D} \xrightarrow[\{x_0\}]{\text{epi-u}} \tilde{f}_{0,\,D}$ only, instead of $\tilde{f}_{n,D} \xrightarrow[\{x_0\}]{\text{u}} \tilde{f}_{0,\,D}$.

For instance, according to Theorem 4.2 in [27], $\tilde{f}_{n,D} \xrightarrow[\{x_0\}]{\text{epi-u}} \tilde{f}_{0,D}\ \forall x_0 \in \mathbb{R}^m$ ensures that the optimal values of the programming problems $\min\limits_{x\in\mathbb{R}^p} \tilde{f}_{n,D}(x)$ behave "upper semiconvergent".

Let us denote an indicator of the constraint set by

$$\vartheta_{\Gamma_{n,D}}(x) := \begin{cases} 0 & \text{if } x \in \Gamma_{n,D}, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.4}$$

As in the closely related parametric case, considered by [22] and [9], $f_{n,D} \xrightarrow[\{x_0\}]{\text{u}} f_{0,\,D}$ and $\vartheta_{\Gamma_{n,D}} \xrightarrow{\text{epi-u}} \vartheta_{\Gamma_{0,\,D}}$ together imply $\tilde{f}_{n,D} \xrightarrow[\{x_0\}]{\text{epi-u}} \tilde{f}_{0,\,D}$. Further, $\Gamma_{0,D} \subset \liminf\limits_{n\to+\infty} \Gamma_{n,D}$ entails $\vartheta_{\Gamma_{n,D}} \xrightarrow{\text{epi-u}} \vartheta_{\Gamma_{0,\,D}}$. Note that these relations hold for arbitrary constraint sets $\Gamma_{n,D}$ which are not necessarily governed by probabilistic constraints.

Concerning the "lower semicontinuous" behavior of $(\Gamma_{n,D})_{n\in\mathbb{N}}$, Theorem 3.1.5. in [2] yields Proposition 4.3 below, which is also closely related to assertions obtained by [29].

Proposition 4.3. Let the assumptions of Proposition 4.2 be satisfied at all $x_0 \in \Gamma_{0,D}$ and for all functions γ_l^j. Furthermore, assume that J is a finite set and that

(V12) $\Gamma_{0,D} \subset \operatorname{clo}\{x \in \mathbb{R}^p : g^j_{0,\,D}(x) < 0 \quad \forall j \in J\}$ is fulfilled.

Then $\Gamma_{0,D} \subset \liminf\limits_{n\to+\infty} \Gamma_{n,D}$.

Sufficient conditions for assumption (V12) are considered in [26] and (in a somewhat different form) in [29].

Finally, we consider another approach, which was already used in [30]. Proposition 4.4 is inspired by Proposition 6.3 in [30].

Proposition 4.4. Suppose that, additionally to (V1), the following assumption is satisfied:

(V13) There exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x_0$,
$g^j_{0,\,D}(x_n) < 0\ \forall n \in \mathbb{N} \quad \forall j \in \{1, \dots, j_0\}$ and
$P_0\left(\partial M^j(x_n)\right) = 0$ for all $n \in \mathbb{N}_0,\ j \in \{1, \dots, j_0\}$.

Then $\vartheta_{\Gamma_{n,D}} \xrightarrow[\{x_0\}]{\text{epi-u}} \vartheta_{\Gamma_{0,D}}$.

Proof. We consider a sequence with the properties required in (V13). Because of $g^j_{0,\,D}(x_n) < 0$ we have $P_0(M^j(x_n)) > \eta^j \quad \forall j \in \{1,\dots,j_0\}$.
$M^j(x_n)$ being a P_0-continuity set, to each $n \in \mathbb{N}_0$ we find a k_n such that $P_k(M^j(x_n)) > \eta^j \quad \forall k \geq k_n \quad \forall j \in \{1,\dots,j_0\}$.

Hence $g^j_{k,\,D}(x_n) < 0 \quad \forall k \geq k_n \quad \forall j \in \{1,\dots,j_0\}$.

Now, let $\tilde{k}_1 := 1$ and $\tilde{k}_n$ is defined recursively by $\tilde{k}_n := \max\{\tilde{k}_{n-1}+1, k_n\}$. Furthermore, $\tilde{x}_k := x_n$ for all k with $\tilde{k}_n \leq k < \tilde{k}_{n+1}$.

The sequence $(\tilde{x}_k)_{k\in\mathbb{N}}$ constructed in this way has the properties $\tilde{x}_k \to x_0$ and $\vartheta_{\Gamma_{k,D}}(\tilde{x}_k) = 0 \quad \forall k \in \mathbb{N}$, hence $\limsup_{k\to+\infty} \vartheta_{\Gamma_{k,D}}(\tilde{x}_k) = 0 \leq \vartheta_{\Gamma_{0,D}}(x_0)$. □

Sufficient conditions for (V13) in the case of probabilistic constraints are given in [27].

In the random setting similar relations may be proved.

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