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DERIVATION OF EFFECTIVE TRANSFER FUNCTION MODELS BY INPUT, OUTPUT VARIABLES SELECTION[1]

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Transfer function models used for early stages of design are large dimension models containing all possible physical inputs, outputs. Such models may be badly conditioned and possibly degenerate. The problem considered here is the selection of maximal cardinality subsets of the physical input, output sets, such as the resulting model is nondegenerate and satisfies additional properties such as controllability and observability and avoids the existence of high order infinite zeros. This problem is part of the early design task of selecting well-conditioned progenitor models on which successive design has to be carried out. The conditions for different type of degeneracy are investigated and this leads to necessary and sufficient conditions required to guarantee nondegeneracy. The sufficient conditions for nondegeneracy also lead to models with no infinite zeros. Furthermore, additional conditions are derived which guarantee controllability and observability of the resulting model. The results are then used to develop a selection procedure for natural subsets of inputs and outputs, which guarantee transfer function and input, output nondegeneracy, as well as controllability and observability of the resulting system. A parameterisation of solutions that satisfy the above requirements is given.

1. INTRODUCTION

The derivation of models that can be used for early design stages studies of processes requires the use of the process flowsheet (system interconnection graph), the availability of simple models describing the fundamental dynamics of subprocesses and the selection of control (input) and measurement (output) variables. Before we embark on the investigation of the properties of the resulting model it is useful to include all possible inputs and outputs; at a later stage we can then determine the effective subsets of inputs, outputs using different "controllability", "operability" criteria. Such models corresponding to all possible inputs and all possible outputs are referred to as *progenitor models* [7]. Progenitor models are derived on the basis that possible inputs, outputs are selected using heuristics, physical arguments and thus the resulting transfer function may be of large dimensions and possibly not well behaved. The essential feature of such models is that the input, output variables are

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physical variables, on which specifications may be imposed, and that this transfer function contains as parts all possible (smaller dimension) transfer functions that may be used in an actual design. Transfer functions corresponding to subsets of the potential input and output sets are referred to as *effective models* and are submatrices of the progenitor transfer function. Different families of effective models may be defined by fixing the cardinality of the input, output effective sets, or by requiring that the input, output sets contain certain fixed physical variable sets. Characterising such families of models, in terms of a range of important properties, is an important part of the "process controllability" studies [17].

This paper deals with a specific problem within the general area of selecting effective models, when we use as criteria the nondegeneracy of the effective transfer function, the nonredundancy of the instrumentation schemes (independence of selected sensors and actuators) and the controllability and observability of the resulting system. Nondegeneracy is a fundamental property for the effective model, since it is linked to the output function controllability [18], and thus to the solvability of a number of control problems. Conditions for the characterisation of system degeneracy and redundancy of the input, output structure of the system are derived in terms of the state space parameters; these conditions also indicate the criteria required to guarantee nondegeneracy and input, output scheme nonredundancy. For the cases of proper and strictly proper progenitor models simple and quite broad sufficient conditions of the rank type are given, which guarantee nondegeneracy and nonredundancy. The characterisation of the controllability and observability properties is performed here using the McMillan degree and the associated properties of Hankel matrices [1]. Such approach is faster and more suitable for selecting effective models. The selection of maximal dimension effective models, which have all of the previous properties, is then tackled by deploying a procedure that defines the "most orthogonal basis" [14] for a given set of vectors, without transforming the data of the set. The approach suggested here leads to a parameterisation of all maximal dimension effective models, which are nondegenerate and input, output nonredundant. The elements of this set may then be used for the selection of models having additional desirable properties, such as avoiding high order infinite zeros. Amongst the additional properties that may be considered are those of avoiding nonminimum phase properties of the resulting models, as well as more general criteria expressing overall control for control design and known as "process controllability" [15]. The work here is considered as a first stage in the process of selection of "good" early stage design models.

The paper is structured as follows: In Section 2 we introduce the problem as part of the early systems design and describe the objectives of the work. In Section 3 we deal with the problem of Input, Output Redundancy and establish their links to system degeneracy. In Section 4 we examine the type of degeneracy, which is not linked to input, output redundancy, but it is a property of the internal model structure of the system. The sufficient conditions for avoiding this type of degeneracy also guarantee the absence of infinite zeros for the resulting model. In Section 5 we deal with the characterisation of the family of controllable and observable effective models based on the characterisation of McMillan degree of Hankel matrices. In

Section 6 we use the results of the previous sections to parameterise and select maximal cardinality well-conditioned models (as far as degeneracy, input, output redundancy, controllability and observability). Finally, in Section 7 we illustrate the results in terms of examples. The proof of the results is given in the Appendix.

2. STATEMENT OF THE PROBLEM

The development of models, which may be used for evaluation of alternatives is an integral part of the Early Process Design of process plants [17]. Such models are usually developed for the entire plant, are based on the selected process flowsheet (interconnection graph and involve the use of simple models of the subprocesses). As such, they are large dimension models and their final structure is determined when the control structure is decided.

The selection of control structures is a topic that has attracted a lot of interest within the process control area ([4, 5, 8, 16, 17, 19] and references there in). This problem involves a number of key subproblems [8], which are: (i) The classification of process variables into potential inputs, outputs and referred to as *Model Orientation Problem* (MOP). (ii) Specification of effective sets of inputs, outputs on an oriented model and referred to as *Model Projection Problem* (MPP). (iii) Deciding on the way we couple effective inputs and outputs for control design purposes and referred to as *Input – Output Coupling Problem* (I-O.C.P.). Most of the attention so far has been focused on I-O.C.P., when heuristics and diagnostic indicators have been used. For the first two problems, less attention has been given, especially from the Control Theory viewpoint, with the exception of the work in [4, 8, 9, 16] on some specific problems. In this paper we are concerned with the selection of the effective sets of inputs, outputs on a system, in order to satisfy certain criteria for the resulting transfer function, such as the system nondegeneracy, the nonredundancy of the input, output scheme and controllability, observability of the resulting model. Such problems belong to the MPP family.

We assume that we are given a linearised model of a system, for which the classification of system variables (implicit variables) into systems and outputs has been already decided. At the early stages of design it is desirable to include as inputs, all possible variables that can be used as variables to be controlled and measured; these inputs, outputs are referred to as *potential* sets. The model that corresponds to the potential inputs, outputs provides the basis for deriving all subsequent models based on *effective* input, output sets and it is thus referred to as the *progenitor model.* The characteristic of the progenitor model is that all inputs and outputs are physical variables that can be acted upon and measured. Given that the classification of internal variables into inputs, outputs has been done mainly with physical, process based criteria, a progenitor model may not be well behaving. That is the transfer function may be degenerate and there is redundancy in the input, output schemes and a number of other fundamental properties may not have good values (i. e. condition numbers etc.). Note that a progenitor model represents all our knowledge about the system at a given stage of early design and the McMillan degree of the progenitor transfer function represents the *natural order* n of the system.

System models, which are degenerate, are not good for subsequent design since they do not satisfy the basic condition of the output function controllability. It is thus desirable to select subsets of the potential inputs and outputs (by elimination of some elements of the potential sets), such that the resulting transfer function is "well-conditioned" in some sense. Amongst the basic criteria we can use are the properties of nondegeneracy, controllability and observability of the system model and nonredundancy of the input and output scheme. Any submodel that satisfies the above three properties and has maximal cardinality for the input and output set will be called a *normal progenitor model*; clearly, a system may have more than one such models. The problem we consider also here is the parameterisation and systematic construction (by avoiding listing and testing of all possible submodels) of the family of normal progenitor models.

We will assume that the progenitor model is described by the minimal state space equations:

$$\underline{\dot{x}} = A\underline{x} + B\underline{u}, \quad A \in \Re^{n\times n}, \quad B \in \Re^{n\times r} \tag{2.1a}$$

$$\underline{y} = C\underline{x} + D\underline{u}, \quad C \in \Re^{q\times n}, \quad D \in \Re^{q\times r} \tag{2.1b}$$

with a corresponding transfer function $H(s) = C(sI - A)^{-1}B + D \in \Re^{q\times r}(s)$ and let $\rho = \text{rank}_{\Re(s)}\{H(s)\}$ be the normal rank of $H(s)$. Clearly $\rho \leq \min(q, r)$ and whenever strict inequality holds, then the system is called *degenerate*; when equality holds the system is called *nondegenerate*. The significance of ρ is described below [18].

Remark 2.1. ρ defines the maximal number of output variables that may be controlled independently (output function controllability criterion). Furthermore, ρ defines the minimal number of independent inputs required to control ρ outputs.

Definition 2.1. For the system $S(A, B, C, D)$ for which r, $q \leq n$, we define the numbers:

$$\tau_r \triangleq \text{rank}\left\{\begin{bmatrix} B \\ D \end{bmatrix}\right\} \leq r \quad \text{and} \quad \tau_\ell = \text{rank}\{[C, D]\} \leq q. \tag{2.2}$$

If $\tau_r < r$ $(\tau_\ell < q)$, the system will be said to have *input* (*output*) redundancy; otherwise, i.e. if $\tau_r = r$ $(\tau_\ell = q)$, then it will be said to be *regular*.

Regularity of the model is clearly equivalent to nonredundancy of both sensor and actuator schemes and it is a desirable property, which however may not hold on a progenitor model. The problem we consider here is described below:

PROBLEM. Given the progenitor model described by $H(s)$, or with $S(A, B, C, D)$ define:

(i) A maximal cardinality subset of the potential input and output sets such as that the resulting transfer function is nondegenerate, has the maximal possible normal rank and it is also regular.

(ii) Amongst the solutions of (i), determine whether there exist solutions, which have McMillan degree equal to that of $H(s)$.

(iii) Parameterise all solutions with the properties described above.

The solution of problem (i) will be referred to as *well-conditioning of Progenitor models* and part (ii) describes the property that the resulting model is both controllable and observable. Note that controllability and observability are notions defined on $S'(A, B', C', D')$ where A corresponds to the minimal realisation of $H(s)$. The latter problem will be referred to as *normal-conditioning of Progenitor Models.* The existence of such solutions, as well as the parameterisation of them (when such solutions exist) will be examined here. Within the same classes of problems we may also consider more relaxed cases such as stabilisability and detectability [6] and more detailed model properties such as absence of nonminimum phase properties, avoidance of high order infinite zeros etc. More general properties referred to as "process controllability" [15] may be used for subsequent evaluations. The overall problem under consideration is the study of properties of the submatrices of the rational transfer function matrix $H(s)$ obtained by elimination of certain sets of columns, rows. Of special interest is the definition of those submatrices $H'(s)$, which preserve certain properties of $H(s)$, but avoid certain undesirable properties. The study of well-conditioning is considered first.

3. INPUT, OUTPUT REDUNDANCY AND SYSTEM DEGENERACY

The notion of redundancy of the input, output map of the progenitor model is linked to some type of redundancy of the resulting model and it is the topic of this section. This form of degeneracy will be referred to simply as simple, to distinguish it from an alternative form of degeneracy characterised by properties of the internal mechanism and referred to as strong. The latter is examined in the following section.

The unifying thing between redundancy and degeneracy is that they both relate to properties of kernels of transfer function, or matrix pencil models. The state space description $S(A, B, C, D)$ may be represented in the s-domain as

$$\underbrace{\begin{bmatrix} sI_n - A & -B \\ -C & -C \end{bmatrix}}_{\triangleq P(s)} \begin{bmatrix} \underline{x}(s) \\ \underline{y}(s) \end{bmatrix} = \begin{bmatrix} \underline{x}_0 \\ -\underline{y}(s) \end{bmatrix} \tag{3.1}$$

where $P(s) \in \Re^{(n+q)\times(n+r)}[s]$ is the Rosenbrock System Matrix pencil [18]. We may define:

Definition 3.1. For the system described by $S(A, B, C, D)$ we shall denote by $Z_r \triangleq N_r\{P(s)\}$, $Z_\ell \triangleq N_\ell\{P(s)\}$ the right, left null spaces of $P(s)$. Then,

(i) A pair of polynomial vectors $\underline{x}(s) \in \Re^n[s]$, $\underline{u}(s) \in \Re^r[s]$ will be said to be a *right pair* and the composite vector $\underline{\zeta}(s) = [\underline{x}(s)^t, \underline{u}(s)^t]^t$ a *right vector*, if

$$P(s)\underline{\zeta}(s) = \underline{0}. \tag{3.2}$$

(ii) A pair of polynomial vectors $\underline{y}(s) \in \Re^n[s]$, $\underline{v}(s) \in \Re^q[s]$ will be said to be a *left pair* and the composite vector $\underline{\xi}(s)^t = [\underline{y}(s)^t, \underline{v}(s)^t]$ a *left vector*, if

$$\underline{\xi}(s)^t P(s) = \underline{0}. \tag{3.3}$$

For a right (left) pair $\underline{\zeta}(s)$ we define by $\partial[\underline{\zeta}(s)]$ its degree. An interesting property of the degree is described below [21]:

Remark 3.1. For any right pair $(\underline{x}(s), \underline{u}(s))$, left pair $(\underline{y}(s), \underline{v}(s))$ we have that

$$\partial[\underline{u}(s)] = \partial[\underline{x}(s)] + 1, \quad \partial[\underline{v}(s)] = \partial[\underline{y}(s)] + 1. \tag{3.4}$$

Furthermore, all right pairs $(\underline{x}(s), \underline{u}(s))$ with $\partial[\underline{\zeta}(s)] = 0$, we have $\underline{x}(s) = \underline{0}$ and $\underline{u}(s) = \underline{u} \in \Re^r$. Similarly, for all left pairs $(\underline{y}(s), \underline{v}(s))$ with $\partial[\underline{\xi}(s)] = 0$, we have $\underline{y}(s) = \underline{0}$ and $\underline{v}(s) = \underline{v} \in \Re^q$.

The above leads to the following interpretation of the significance of right, left constant vectors [3]:

Proposition 3.1. For the system $S(A, B, C, D)$ the following holds true:

(a) There exists a right constant vector $\underline{\zeta} = [\underline{0}^t, \underline{u}^t]^t \neq \underline{0}$ if and only if

$$\begin{bmatrix} B \\ D \end{bmatrix} \underline{u} = \underline{0}, \quad \underline{u} \neq \underline{0} \iff \operatorname{rank}\left\{ \begin{bmatrix} B \\ D \end{bmatrix} \right\} < r. \tag{3.5a}$$

(b) There exists a left constant vector $\underline{\xi}^t = [\underline{0}^t, \underline{v}^t]^t \neq \underline{0}^t$, if and only if

$$\underline{v}^t [C, D] = \underline{0}^t, \quad \underline{v}^t \neq \underline{0} \iff \operatorname{rank}\{[C, D]\} < q. \tag{3.5b}$$

The above readily follows from the definition and clearly establishes the presence of input, or output redundancy as equivalent to the existence of constant, right, or left vectors correspondingly. In the following we shall denote by:

$$\eta = \dim N_r\{P(s)\}, \quad \theta = \dim N_\ell\{P(s)\}. \tag{3.6}$$

The following result establishes some interesting properties of η, θ numbers.

Proposition 3.2. For the system $S(A, B, C, D)$, let $\tau = \operatorname{rank}_{\Re(s)}\{P(s)\}$ and $\rho = \operatorname{rank}_{\Re(s)}\{H(s)\}$. Then the following properties hold true:

(i) $\tau = n + \rho$, where n is the number of states.

(ii)
$$\eta \triangleq \dim N_r\{P(s)\} = \dim N_r\{H(s)\} = r - \rho \tag{3.7a}$$

$$\theta \triangleq \dim N_\ell\{P(s)\} = \dim N_\ell\{H(s)\} = q - \rho. \tag{3.7b}$$

A direct consequence of the above lemma is:

Remark 3.2. The system is degenerate, if and only if $\tau = \operatorname{rank}_{\Re(s)}\{P(s)\} < \min(n+r, n+q)$. That is we can use either $P(s)$ or $H(s)$ for characterisation of the property. Furthermore, degeneracy implies that both null spaces of $P(s)$ or $H(s)$ are nontrivial and degeneracy is equivalent to that possibly only one of the two null spaces is nontrivial ($\neq \{0\}$).

Remark 3.3. The property of degeneracy is linked to the loss of output (input) function controllability [1, 18], since the existence of a right inverse of $H(s)$ is a necessary and sufficient condition for output function controllability. Thus, $N_r\{P(s)\} = 0$, or $N_r\{H(s)\} = 0$ are conditions for output function controllability of the corresponding model.

Some relationships between degeneracy and input, output loss of regularity are described below:

Proposition 3.3. For the system $S(A, B, C, D)$ the following properties hold true:

(i) If $q \geq r$ ($q \leq r$) and the system is not input (output) regular, then it is degenerate.

(ii) If a system is not input and not output regular, then it is degenerate.

(iii) Let $\tau_\ell = \operatorname{rank}[C, D]$, $\tau_r = \operatorname{rank}[B^t, D^t]$. Then,

(a) If $q \geq r$ and $\tau_\ell < r$, the system is degenerate.

(b) If $q \leq r$ and $\tau_r < q$, then the system is degenerate.

For the pencil $P(s)$, the right, left null spaces $N_r\{P(s)\}$, $N_\ell\{P(s)\}$ are characterised by a set of column, row minimal indices (cmi, rmi) [3], which also here may be referred to as *right*, *left indices* of $P(s)$ [2]. Such sets are denoted by $I_P^c = \{\varepsilon_i : i = 1, \ldots, \eta = n - \rho\}$, $I_P^r = \{\mu_j : j = 1, \ldots, \theta = q - \rho\}$ and may have t_r zero cmi and t_ℓ zero rmi; in fact,

$$t_r = r - \operatorname{rank}\left\{ \left[B^t, D^t\right]^t \right\} = r - \tau_r \leq r - \rho \tag{3.8a}$$

$$t_\ell = q - \operatorname{rank}\{[C, D]\} = q - \tau_\ell \leq q - \rho \tag{3.8b}$$

The numbers t_r, t_ℓ which characterise 0-cmi, 0-rmi respectively, express the order of input, output redundancy and will be referred to as *input-*, *output-redundancy index* correspondingly. The use of t_r, t_ℓ indices provides some additional insight on redundancy and leads to the following remarks.

Remark 3.3. The numbers $\tau_r = \operatorname{rank}\{[B^t, D^t]\}$ and $\tau_\ell = \operatorname{rank}\{[C, D]\}$ provide bounds for $\rho = \operatorname{rank}_{\Re(s)}\{H(s)\}$ and in particular

$$\rho \leq \min(\tau_r, \tau_\ell). \tag{3.9}$$

The case of $\rho = \min(\tau_r, \tau_\ell)$ implies:

(a) If $\tau_r = \min(\tau_r, \tau_\ell)$, then all indices in I_P^c are zero, or the set is empty; in particular, if $r > \tau_r$, then all cmi are zero and if $r = \tau_r$, then I_P^c is empty and the system is nondegenerate.

(b) If $\tau_\ell = \min(\tau_r, \tau_\ell)$, then all indices in I_P^r are zero, or the set is empty; in particular, if $q > \tau_\ell$ then all indices in I_P^r are zero and if $q = \tau_\ell$, then I_P^r is empty and the system is nondegenerate.

(c) If $\rho = \tau_r = \tau_\ell$ and at least one of r, q is equal to ρ, then clearly we have nondegeneracy and redundancy for the index that is greater than ρ. If r, $q > \rho$, then we have both degeneracy and input, output degeneracy.

The case where $t_r = r - \rho$ $(t_\ell = q - \rho)$ is referred to as *total input- (output-) irregularity*. When at least one such condition holds true, that implies that degeneracy of the transfer function may be removed by eliminating redundancy in the corresponding part of the instrumentation map. The results in this section show that there is link between input, output redundancy and system degeneracy. The type of system degeneracy inferred from the input, output redundancy will be called *simple*. Another type of degeneracy that may exist even under input and output regularity is considered next; this is linked to properties of the internal mechanism and shall be referred to as *strong degeneracy*.

4. STRONG SYSTEM DEGENERACY

In the previous section we examined issues of degeneracy and input, output redundancy, which are linked to zero values of cmi, rmi. Here we will consider the case of nonzero indices. We shall denote by $Z_r \triangleq N_r\{P(s)\}$, $Z_\ell \triangleq N_l\{P(s)\}$ and $\tau_\ell = \operatorname{rank}[C, D]$, $\tau_r = \operatorname{rank}[B^t, D^t]$. The study of strong degeneracy is an issue that is linked to nonzero minimal indices. The sets of indices I_p^c, I_p^r associated with Z_r, Z_ℓ respectively may contain nonzero indices and this is characterised by the following result.

Proposition 4.1. For any system $S(A, B, C, D)$ with r inputs, q outputs, transfer function $H(s)$ and $\rho = \operatorname{rank}_{\Re(s)}\{H(s)\}$ the following properties hold true:

(a) The numbers ρ, τ_r, τ_ℓ, r, q satisfy the conditions:

$$\rho \leq \tau_r \leq r \quad \text{and} \quad \rho \leq \tau_\ell \leq q. \tag{4.1}$$

(b) The system has $\tau_r - \rho$ nonzero cmi, if and only if

$$\rho < \tau_r \leq r \tag{4.2}$$

and all such indices are nonzero, if $\tau_r = r$.

(c) The system has $\tau_\ell - \rho$ nonzero rmi, if and only if

$$\rho < \tau_\ell \leq q, \tag{4.3}$$

and all such indices are nonzero, if $\tau_\ell = q$.

The above lead to the following result:

Proposition 4.2. The system $S(A,B,C,D)$ with $q \geq r$ and $\rho < r$ has a right index with value k at most, if and only if there exists a set of vectors $\{\underline{u}_0, \underline{u}_1, \ldots, \underline{u}_k, \underline{u}_k \neq \underline{0}\}$ such that the following conditions are satisfied:

$$\begin{bmatrix} A^kB & A^{k-1}B & A^{k-2}B & \dots & A^2B & AB & B \\ CA^{k-1}B & CA^{k-2}B & CA^{k-3}B & \dots & CAB & CB & D \\ CA^{k-2}B & CA^{k-3}B & CA^{k-4}B & \dots & CB & D & 0 \\ \vdots & \vdots & \vdots & & & & \vdots \\ CAB & CB & D & \dots & 0 & 0 & 0 \\ CB & D & 0 & \dots & 0 & 0 & 0 \\ D & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{u}_k \\ \underline{u}_{k-1} \\ \underline{u}_{k-2} \\ \vdots \\ \underline{u}_2 \\ \underline{u}_1 \\ \underline{u}_0 \end{bmatrix} = \underline{0}. \tag{4.4}$$

The above condition may now be used to derive conditions for non-degeneracy of transfer functions and thus also procedures for redesign of the system to guarantee non-degeneracy. For the given system, we define the following set of matrices:

$$M_0 = \begin{bmatrix} B \\ D \end{bmatrix}, \quad M_1 = \begin{bmatrix} AB & B \\ \cdots & \cdots \\ CB & D \\ D & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} A^2B & AB & B \\ \cdots & \cdots & \cdots \\ CAB & CB & D \\ CB & D & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \ldots,$$

$$\begin{aligned} M_k &= \begin{bmatrix} A^kB & A^{k-1}B & \dots & AB & B \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ CA^{k-1}B & CA^{k-2}B & \dots & CB & D \\ CA^{k-2}B & CA^{k-3}B & \dots & D & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ CAB & CB & \dots & 0 & 0 \\ CB & D & \dots & 0 & 0 \\ D & 0 & \dots & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A^kB \;\; \dots \;\; AB & \vdots & B \\ \cdots\cdots\cdots & & \cdots \\ & N_k & \end{bmatrix}. \end{aligned} \tag{4.5}$$

In terms of the above matrices, we may state some tests for nondegeneracy as shown below. We first note:

Lemma 4.1. If $q \geq r$, then the maximal possible value of right index of $P(s)$ is:

(i) If $D \neq 0$ and $\text{rank}(D) = \delta$, then $\varepsilon_{\max} = n - q + 2\delta - 1$.

(ii) If $D = 0$, then $\varepsilon_{\max} = n - q - 1$.

Theorem 4.1. For the system $S(A, B, C, D)$ with $q \geq r$, the following properties hold true:

(i) If D has full rank, then the system has no right indices of any value and it is thus non-degenerate.

(ii) If $D \neq 0$ and $\text{rank}(D) = \delta < r$, then the system is non-degenerate, if and only if the matrix M_σ is full rank, where $\sigma = n - q + 2\delta - 1$.

The above results for the case of strictly proper systems have the following form. First, define the matrices:

$$\widetilde{M}_0 = [B], \quad \widetilde{M}_1 = \begin{bmatrix} AB & \vdots & B \\ \cdots & \cdots & \cdots \\ CB & \vdots & 0 \end{bmatrix}, \quad \widetilde{M}_2 = \begin{bmatrix} A^2B & AB & \vdots & B \\ \cdots & \cdots & \cdots & \cdots \\ CAB & CB & \vdots & 0 \\ CB & 0 & \vdots & 0 \end{bmatrix}, \quad \ldots,$$

$$\widetilde{M}_k = \begin{bmatrix} A^kB & \ldots & A^2B & AB & \vdots & B \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ CA^{k-1}B & \ldots & CAB & CB & \vdots & 0 \\ CA^{k-2}B & \ldots & CB & 0 & \vdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ CAB & \ldots & 0 & 0 & \vdots & 0 \\ CB & \ldots & 0 & 0 & \vdots & 0 \end{bmatrix} \tag{4.6}$$

$$= \begin{bmatrix} A^kB & \ldots & A^2B & AB & \vdots & B \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & \widetilde{N}_k & & \vdots & 0 \end{bmatrix}.$$

Theorem 4.1 leads to the following corollary:

Corollary 4.1. For the system $S(A, B, C)$ with $q \geq r$, the following properties hold true:

(i) If CB is full rank, then the system has no right indices and the system is non-degenerate.

(ii) The system with CB rank deficient is non-degenerate, if and only if the matrix $\widetilde{M}_{\tau'}$ is full rank, wher, $\tau' = n - q - 1$.

The results in this section provide criteria for a type of degeneracy, and thus loss of output function controllability, which depends on the model's inner structure and will be referred to as *strong degeneracy*. The distinction between the simple and strong type is the nature of associated indices, that is zero and non-zero respectively. Note that the characterisation of this type of degeneracy is based on the right nullity properties of matrices $M_k, \widetilde{M}_k$, which have as integral parts the matrices $N_k, \widetilde{N}_k$introduced by the partitioning of $M_k, \widetilde{M}_k$as indicated by (4.5), (4.6). These matrices are of the Toeplitz type and their right nullity properties are linked to the characterisation of state space infinite zeros of the system [10]. The state space characterisation of infinite zeros of a system $S(A, B, C, D)$ (based on the notion of infinite elementary divisors of the associated system matrix $P(s)$) leads to a result that shows the links of strong degeneracy and infinite zeros.

Let us first denote the following sequence of matrices for the system $S(A, B, C, D)$

$$Q_0 = [D], Q_1 = \begin{bmatrix} D & 0 \\ CB & D \end{bmatrix}, \ldots, Q_k = \begin{bmatrix} D & 0 & 0\ldots & 0 & 0 \\ CB & D & 0 & 0 & 0 \\ CAB & CB & D & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ CA^{k-1}B & CA^{k-1}B & \ldots & CB & D \end{bmatrix} \tag{4.7}$$

and for the strictly proper case the sequence

$$\widetilde{Q}_1 = [CB], \widetilde{Q}_2 = \begin{bmatrix} CB & 0 \\ CAB & CB \end{bmatrix}, \ldots, \widetilde{Q}_k = \begin{bmatrix} CB & 0 & \ldots & 0 \\ CAB & CB & 0 & 0 \\ \vdots & & \ddots & \vdots \\ CA^{k-1}B & CA^{k-2}B & \ldots & CB \end{bmatrix}. \tag{4.8}$$

If we denote by $\gamma_i \triangleq \eta_r(Q_i), \tilde{\gamma}_i \triangleq \eta_r\left(\widetilde{Q}_i\right)$ the right nullities of the above matrices, then we have the following characterisation of infinite zeros, which summarise the results on infinite zeros.

Theorem 4.2. Assume that $S(A, B, C, D)$ be nondegenerate and let $H(s)$ be the corresponding transfer function. Then,

(i) The sequence $J_\infty = \{\gamma_0, \gamma_1, \ldots, \gamma_k, \ldots\}$ is Piecewise Arithmetic Progression [11], that satisfies the relationship

$$2\gamma_i \geq \gamma_{i-1} + \gamma_{i+1}, \quad i = 0, 1, \ldots, \ \gamma_{-1} = 0 \tag{4.9a}$$

and the singular points defined by those i for which

$$\delta_i = 2\gamma_i - \gamma_{i-1} - \gamma_{i+1} > 0, \quad i = 0, 1, 2, \ldots \tag{4.9b}$$

characterise the degrees of infinite elementary divisors (ied) of $P(s)$ and δ_i denote their corresponding multiplicity.

(ii) If $\{\hat{s}^{q_1}, i = 1, 2, \ldots, \mu\}$ is the set of ied of $P(s)$, then the orders of infinite zeros of $H(s)$ are defined by the nontrivial elements ($\neq 0$) of the set $\{\hat{q}_i : \hat{q}_i = q_i - 1, \forall i \in \underline{\mu}\}$.

Remark 4.1. The sufficient conditions for avoiding strong degeneracy, i.e. D full rank (proper systems), CB full rank (strictly proper systems) imply that the sequence J_∞ is $\{0\}$ for the proper case, or $J_\infty = \{rk, k = 1, 2, \ldots\}$, i.e. arithmetic progression for the strictly proper case. In either case the transfer function $H(s)$ has no infinite zeros (in the algebraic sense).

The above suggests that the sufficient conditions for avoiding nondegeneracy i.e. D, or CB full rank, have the additional property that they force the corresponding transfer function not to have infinite zeros. Such systems have the advantage that they can be controlled in a relatively simple way.

5. NORMAL CONDITIONING OF PROGENITOR MODELS

Given the progenitor model described by the transfer function matrix $H(s) \in \Re^{q\times r}(s)$ and with a McMillan degree n, there is always a minimal realisation $S(A, B, C, D)$. It is this model which represents our entire knowledge for the system. By deleting a subset α of inputs and a subset β of the outputs we obtain a resulting system $S(A, B_\alpha, C_\beta, D_{\alpha,\beta})$; this model may be well conditioned (nondegenerate and input, output regular), but it may not necessarily be controllable and observable. Clearly, the standard tests for controllability and observability on all possible system representations $S(A, B_\alpha, C_\beta, D_{\alpha,\beta})$ may be used, but the procedure is rather cumbersome. Here we shall use alternative tests based on the McMillan degree, which may combine with the conditions for nondegeneracy in a more natural way. We note first the following standard results from linear systems [1, 6].

Lemma 5.1. Let $H(s)$ be a transfer function and $S(A, B, C, D)$ be a realisation of $H(s)$. $S(A, B, C, D)$ is a minimal realisation of $H(s)$, if the McMillan degree of $H(s)$ is: $\delta_M(H) = \partial\{\,|sI - A|\,\}$.

Using the above result we note the following:

Proposition 5.1. Let $H(s)$ be a transfer function, $S(A, B, C, D)$ the corresponding minimal system and $H_{\alpha,\beta}(s)$ be the submatrix defined from $H(s)$ by eliminating the α set of inputs and β set of outputs. If $S(A, B_\alpha, C_\beta, D_{\alpha,\beta})$ is the resulting system, then it is minimal if and only if $\delta_M(H) = \delta_M(H_{\alpha,\beta})$.

The result follows directly from Lemma 5.1 and the construction of $H_{\alpha,\beta}(s)$, or $S(A, B_\alpha, C_\beta, D_{\alpha,\beta})$. We now consider the state space characterisation of the

McMillan degree, which is established as shown below. Let us consider the Laurent series expression of $H(s)$ [1], i.e.

$$H(s) = H_0 + \hat{H}(s) = H_0 + H_1 \cdot s^{-1} + H_2 \cdot s^{-2} + H_3 \cdot s^{-3} + \ldots \tag{5.1}$$

where $\hat{H}(s)$ is the strictly proper part and the $q \times r$ real matrices H_0, H_1, ... are the Markov parameters where

$$H_0 = D, \quad H_i = CA^{i-1}B, \quad i = 1, 2, \ldots \tag{5.2}$$

The Hankel matrix $M_H(i,j)$ of order (i,j) corresponding to the Markov parameter sequence H_1, H_2,... is defined as the $iq \times jr$ matrix given by:

$$M_H(i,j) \triangleq \begin{bmatrix} H_1 & H_2 & \ldots\ldots & H_j \\ H_2 & H_3 & \ldots\ldots & H_{j+1} \\ \vdots & \vdots & & \vdots \\ H_i & H_{i+1} & \ldots\ldots & H_{i+j-1} \end{bmatrix}. \tag{5.3}$$

Lemma 5.2. [1] The McMillan degree of the transfer function $H(s)$ is the rank of $M_H(v,v)$, where v is the degree of the least common denominator of the entries of $H(s)$.

By computing the least common multiple (lcm) of the entries of $H(s)$, say $d_H(s)$, then $v = \partial\{d_H(s)\}$. Using the Markov parameters $\{CB, CAB, \ldots)$ we may define the matrix:

$$M_H(v,v) \triangleq M_H^v = \begin{bmatrix} CB & CAB & \ldots\ldots & CA^{v-1}B \\ CAB & CA^2B & \ldots\ldots & CA^vB \\ \vdots & \vdots & & \vdots \\ CA^{v-1}B & CA^vB & \ldots\ldots & CA^{2v-1}B \end{bmatrix}. \tag{5.4}$$

Clearly, rank $\{M_H^v\} = \delta_M(H)$ and a searching procedure for the submatrices $H_{\alpha,\beta}(s)$ with the same McMillan degree with $H(s)$ can be defined as indicated below:

Definition 5.1. Let $\left\{CB, CAB, \ldots, CA^kB, \ldots\right\}$ be the Markov parameters associated with the $H(s)$ progenitor model, $\alpha = (i_1, \ldots, i_\sigma)$ be a set of indices characterising inputs of the $\{1, 2, \ldots, r\}$ set and $\beta = (j_1, \ldots, j_\tau)$ be a set of indices characterising outputs of the $\{1, 2, \ldots, q\}$ set. We shall denote by $C_\beta A^k B_\alpha$ the submatrix of CA^kB obtained by eliminating the α set of columns and β set of rows of CA^kB. We define as the $M_{H_{\alpha,\beta}}^v$ *Hankel submatrix* of M_H^v the matrix:

$$M_{H_{\alpha,\beta}}^v = \begin{bmatrix} C_\beta B_\alpha & C_\beta AB_\alpha & \ldots\ldots & C_\beta A^{v-1}B_\alpha \\ C_\beta AB_\alpha & C_\beta A^2B_\alpha & \ldots\ldots & C_\beta A^vB_\alpha \\ \vdots & \vdots & & \vdots \\ C_\beta A^{v-1}B_\alpha & C_\beta A^vB_\alpha & \ldots\ldots & C_\beta A^{2v-1}B_\alpha \end{bmatrix}. \tag{5.5}$$

Using the matrices $M_{H_{\alpha,\beta}}^v$ we may now state the following result.

Corollary 5.1. Let $S(A, B, C, D)$ be a minimal realisation of $H(s)$ and $S(A, B_\alpha, C_\beta, D_{\alpha,\beta})$ the subsystem obtained by deleting the α set of inputs and β set of outputs. The subsystem $S(A, B_\alpha, C_\beta, D_{\alpha,\beta})$ is both controllable and observable, if and only if

$$\operatorname{rank}(M_H^v) = \operatorname{rank}\left(M_{H_{\alpha,\beta}}^v\right). \tag{5.6}$$

The above result readily follows from Proposition 5.1 and Lemma 5.2. This result may be used to formulate the basis for a searching method for controllable and observable subsystems of $H(s)$.

Remark 5.2. For strictly proper transfer functions $H(s)$, a search for maximal rank $M_{H_{\alpha,\beta}}^v$ submatrices of M_H^v which is based on a full rank $C_\beta B_\beta$, guarantees nondegeneracy, no infinite zeros and minimality (controllability and observability) of the resulting subsystem.

6. WELL CONDITIONING OF TRANSFER FUNCTIONS: SELECTION PROCEDURES AND PARAMETERISATIONS

The results in the previous sections provide criteria for selecting subsystems of $H(s)$, or $P(s)$ which satisfy the input, output regularity requirements and the conditions for non-degeneracy. Although, input, output redundancy may imply degeneracy, input, output regularity does not guarantee non-degeneracy. Guaranteeing non-degeneracy may be achieved by using the sufficient conditions based on the D, CB matrices, or testing selections using the full rank tests based on $M_\tau, \widetilde{M}_{\tau'}$ matrices. Note that conditions based on $M_\tau, \widetilde{M}_{\tau'}$ are not easy to use for making initial selections, which are made using input, output regularity as a selection criterion. Two different strategies for model selection can be made:

(I) *Direct Method*: Selection based on sufficient conditions.

(II) *Indirect Method*: Selection based on input, output regularity and search for nondegeneracy.

Each one of them is described below.

6.1. Direct method for well-conditioning

We assume that $q \geq r$ and that the $S(A, B, C, D)$ model is degenerate. If the system is proper, $D \neq 0$, then degeneracy implies that D is rank deficient and if the system is strictly proper, then necessarily CB has to be rank deficient.

Remark 6.1. If the system $S(A, B, C, D)$ with $q \geq r$ is degenerate, a redesign procedure leading to $\widetilde{S}(A, \widetilde{B}, \widetilde{C}, \widetilde{D})$ with $\widetilde{D}$ full rank guarantees the creation of a system which is non-degenerate and has full rank input and output structure. Similarly, if the system $S(A, B, C)$ with $q \geq r$ is degenerate, a redesign procedure leading to $\widetilde{S}(A,\widetilde{B},\widetilde{C})$ with $\widetilde{C}\widetilde{B}$ full rank guarantees the creation of a system which is non-degenerate and has full rank input and output structure.

The meaning of redesign of D, or CB is that we aim to define a maximal subset of the columns of D, or CB that guarantee the maximal full rank property. This procedure is clearly sufficient, but not necessary and leads to a system of smaller dimensions, as far as input, output structure is concerned. Note that we would like to achieve this selection without transforming the matrices D, CB, since it is desirable to keep the physical variables involved in the original model. The redesign problem, clearly becomes trivial, if general input, output coordinate transformations are used. The problem under study here is important only when we want to retain the original set of physical variables. In the following we shall use the definition:

Definition 6.1. Let $T = [\underline{t}_1, \underline{t}_2, \ldots, \underline{t}_r] \in \Re^{q\times r}, / q \geq r$ with $\mathrm{rank}(T) = \rho < \min(q,r)$. Any ρ-subset of the set $\{\underline{t}_i, i \in \underline{r}\}$ of columns that is linearly independent is said to form a *natural basis* for the space colsp $\{T\}$. If the set $\{\underline{t}_i, i \in \underline{r}\}$ is normalised $\{\|\underline{t}_i\| = 1,\, i \in \underline{r}\}$, every natural basis has a measure of orthogonality σ and thus every natural basis $\left\{\underline{t}_{i_1}, \ldots, \underline{t}_{i_\rho}\right\}$ may be referred to as a *σ-natural basis.* The natural basis with the highest degree of orthogonality will be called a *proper basis* of colsp $\{T\}$.

The selection of a proper basis for a set of vectors has been previously addressed in algebraic computations [14] as a problem of selection of "best uncorrupted base" and an algorithm for achieving this has been introduced in [14]. In the above definition an important ingredient is the notion of orthogonality of the set. This may be introduced using the notion of the Grammian [3], or condition numbers. Here we shall adopt the Grammian notion.

Definition 6.2. [3] Let $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m$ be vectors $\in \Re^n$. The matrix defined by

$$G = \begin{bmatrix} (\underline{x}_1, \underline{x}_1) & (\underline{x}_1, \underline{x}_2) & \cdots & (\underline{x}_1, \underline{x}_m) \\ (\underline{x}_2, \underline{x}_1) & (\underline{x}_2, \underline{x}_2) & \cdots & (\underline{x}_2, \underline{x}_m) \\ \vdots & \vdots & & \vdots \\ (\underline{x}_m, \underline{x}_1) & (\underline{x}_m, \underline{x}_{21}) & \cdots & (\underline{x}_m, \underline{x}_m) \end{bmatrix} \tag{6.1}$$

where $(,)$ denotes inner product, is called the Gram matrix of the vectors $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m$ and the determinant $G_m = G(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m) = |G|$ is called their *Grammian.*

Note [3] that the vectors $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m$ are linearly independent, if and only if their Grammian is nonzero; in general we have that $|G| \geq 0$ and we have that the following property holds true (Hadamard's inequality):

$$G(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m) \leq G(\underline{x}_1) \cdot G(\underline{x}_2) \cdot \ldots \cdot G(\underline{x}_m). \tag{6.2}$$

Note that $G(\underline{x}_i) = \|\underline{x}_i\|_2$ and if the vectors are of unit length (i. e. $\|\underline{x}_i\|_2 = 1, i = 1, 2, \ldots, m$), then

$$0 \leq G(\underline{x}_1,\, \underline{x}_2, \ldots,\, \underline{x}_m) \leq 1. \tag{6.3}$$

Remark 6.2. An alternative test for closeness to normality of a normalised selected set with a basis matrix A, can be based on the condition number of the corresponding matrix. In fact, the deviation from unity of the $\|\cdot\|_2$ condition number is a measure of proximity to orthogonality. This number measures the elongation of the hyperellipsoid associated with A i. e. $\{A\underline{x}: \|\underline{x}\|_2 = 1\}$.

We will use the Grammian as the criterion for selection of natural bases with degree of orthogonality greater than a given number $\underline{\sigma}$ $(0 < \underline{\sigma} \leq 1)$. A procedure for such selection will be described later on and will be referred to as *natural basis selection*. The set of all natural bases with orthogonality $\sigma : \underline{\sigma} \leq \sigma \leq 1$ will be called the $\{\underline{\sigma}\}$-set of natural bases. We may now summarise the selection procedure as follows:

Direct method for well-conditioning

Let $T = [\underline{t}_1, \underline{t}_2, \ldots, \underline{t}_r] \in \Re^q \times r$, $q \geq r$ be a matrix that may represent D, or CB, $\rho = \text{rank}\{T\}$ and assume all its columns to be normalised (i. e. $\|\underline{t}_i\| = 1$). The selection of the well-conditioned model involves:

Step 1: Select an acceptable order of orthogonality $\underline{\sigma}$ and using the natural basis selection we define $\{\underline{\sigma}\}$-set of matrices $\left\{T_a: T_a = \left[\underline{t}_{i_1}, \underline{t}_{i_2}, \ldots, \underline{t}_{i_\rho}\right] \in \Re^{q\times\rho}\right\}$ such that the corresponding set has orthogonality degree $\sigma \geq \underline{\sigma}$.

Step 2: For every set of indices $a = (i_1, i_2, \ldots, i_\rho)$ associated with the $\{\underline{\sigma}\}$-set of matrices $\{T_a\}$, define the subsystems $\{H, \underline{\sigma}\} = \{H_a: a = (i_1, i_2, \ldots, i_\rho\}$, having as inputs those corresponding to the set $a = (i_1, i_2, \ldots, i_\rho)$ of indices defined before. This procedure leads to a set of systems $\langle\zeta, \underline{\sigma}\rangle = \{H_a(s), \underline{\sigma}\}$ for which D_a, or CB_a is a matrix with orthogonality order at least $\underline{\sigma}$.

The above procedure produces submodels, which are always non-degenerate and are input, output regular. However, it may lead to systems with unnecessarily small numbers of inputs (outputs), if rank of D, CB are small. The second approach aims at avoiding such problems.

6.2. Indirect method for well-conditioning

The second approach is based on the selection and parameterisation of all subsets of inputs and outputs for which input and output regularity is guaranteed and then testing for non-degeneracy using the tests derived before. Once we rely on the selection of natural bases for selecting the suitable input, output sets of variables. For the progenitor model $S(A, B, C, D)$ we denote by:

$$F = \begin{bmatrix} B \\ D \end{bmatrix} = \left[\underline{f}_1, \underline{f}_2, \ldots, \underline{f}_r\right] \in \Re^{(q+n)\times r}, \quad G = [C, D] = \begin{bmatrix} \underline{g}_1^t \\ \vdots \\ \underline{g}_q^t \end{bmatrix} \in \Re^{q\times(n+r)} \tag{6.4}$$

and let $\mathrm{rank}(F) = \tau_r \leq r$, $\mathrm{rank}(G) = \tau_\ell \leq q$ and $\mathrm{rank}_{\Re(s)}\{H(s)\} = \rho$. Without loss of generality we may also assume that the columns of F and the rows of G are normalised.

Definition 6.3. For the matrices F, G we shall denote by

$$\{F\} \triangleq \left\{F_\beta \colon F_\beta = \left[\underline{f}_{j_1}, \ldots, \underline{f}_{j_{\tau_r}}\right] \in \Re^{(q+n)\times\tau_r}, \beta = (j_1, \ldots, j_{\tau_r})\right\} \quad (6.5)$$
$$\{G\} \triangleq \left\{G_\gamma \colon G_\gamma = \left[\underline{g}_{l_1}, \ldots, \underline{g}_{l_{\tau_l}}\right]^t \in \Re^{\tau_l\times(n+r)}, \gamma = (l_1, \ldots, l_{\tau_\ell})\right\}$$

the set of all submatrices of F, G which correspond to the natural bases of F, G respectively. The subsets of $\{F\}, \{G\}$, which have a degree of orthogonality greater or equal to some value $\{\underline{\sigma}\}$, will be denoted by $\{F\}_{\underline{\sigma}}, \{G\}_{\underline{\sigma}}$ correspondingly. The set of sequences defined by:

$$\Omega_F \triangleq \{\forall\beta : \beta = (j_1, \ldots, j_{\tau_r})\}, \quad \Omega_G \triangleq \{\forall\gamma : \gamma = (l_1, \ldots, l_{\tau_\ell})\} \quad (6.6)$$

characterising the natural bases of F, G will be referred to as the *characteristics* of F, G respectively. For every $\beta \in \Omega_F$ and $\gamma \in \Omega_G$ we shall denote by $S_{\beta,\gamma}(A,B,C,D)$ the subsystem of $S(A,B,C,D)$ corresponding to the β set of inputs and γ set of outputs.

Remark 6.3. For proper systems $S(A,B,C,D)$, $D \neq 0$, the subsystem $S_{\beta,\gamma}(A,B,C,D)$ that corresponds to some $\beta \in \Omega_F$ and $\gamma \in \Omega_G$ is not necessarily input and output regular. This implies that the process of selecting sets $\beta \in \Omega_F$ and $\gamma \in \Omega_G$ to guarantee input and output regularity are not always independent. In fact, although we can always make the system input regular with cardinality τ_r, or output regular with cardinality τ_ℓ, achieving both may not be possible.

The above indicates that progenitor models may be classified as shown below:

Definition 6.4. Given a system $S(A,B,C,D)$ we say that:

(i) It is *input-output independent*, if *any* selection of the maximal τ_r number of independent inputs does not affect the selection of the maximal number τ_ℓ of independent outputs and vice versa; otherwise, it is called *input-output dependent.*

(ii) It is called *input-output regularisable,* if for at least a $\beta \in \Omega_F$ there is a $\gamma \in \Omega_G$ such that the subsystem $S_{\beta,\gamma}(A,B,C,D)$ is input, output regular; otherwise, it is called *input-output non-regularisable.*

The above notions are important in the construction of well conditioned systems and are examined below:

Proposition 6.1. The system $S(A, B, C, D)$ is input-output independent if the following conditions holds true:

$$\begin{aligned} \operatorname{rank}[C,\ D] &= \operatorname{rank}[C] && (6.7) \\ \operatorname{rank}[B^t,\ D^t] &= \operatorname{rank}[B^t]. && (6.8) \end{aligned}$$

Remark 6.4. A strictly proper system $S(A, B, C)$ is always an input-output independent system. Furthermore, any input-output independent system is always input-output regularisable.

An input-output dependent system, may, or may not be input-output regularisable. The selection of the maximal number of inputs, outputs in order to guarantee input and output regularity is more complicated and requires a searching method that will be described below. We first note:

Remark 6.5. For any progenitor model $S(A, B, C, D)$ the maximal number of inputs and outputs required to guarantee input and output regularity is τ_r, τ_ℓ respectively. These values can always be achieved for input-output independent systems, but not necessarily for the case of input-output dependent, where they act as upper bounds.

The problem of determining the maximal values of cardinality of inputs, outputs, as well as the parameterisation of the corresponding family of systems is considered below in an algorithmic manner. The overall family of such systems will be denoted by $\langle f \rangle$ and every subfamily, with (r', q') input, output cardinality (which is input-output regular) will be denoted by $\langle f \rangle_{r', q'}$. $\langle f \rangle$ will be referred to as the *input-output regular* family and can always be partitioned as a union of subsets with different indices (r', q').

Searching algorithm for determining the input-output regular family $\langle f \rangle$

Consider the progenitor model $S(A, B, C, D)$ and let $\tau_r = \operatorname{rank}[B^t, D^t] = \tilde{r}$, $\tau_\ell = \operatorname{rank}[C, D] = \tilde{q}$ and assume for the sake of simplicity of presentation that $\tilde{r} < \tilde{q}$. Defining $\langle f \rangle$ and the corresponding indices (r', q') involves the following:

CASE (I): Input–output independent systems

For this case the maximal cardinality is $(\tilde{r}, \tilde{q})$ and the family of $\langle f \rangle_{\tilde{r}, \tilde{q}}$ systems is constructed as:

Maximal cardinality family: Consider the sets of indices $\Omega_F = \{\beta = (j_1, \ldots, j_{\tilde{r}})\}$, $\Omega_G = \{\gamma = (l_1, \ldots, l_{\tilde{q}})\}$. If B_β, C_γ, $D_{\beta,\gamma}$ denote the submatrices corresponding to these indices then for $\forall \beta \in \Omega_F$ and $\forall \gamma \in \Omega_G$ the subsystem $S(A, B_\beta, C_\gamma, D_{\beta,\gamma})$ is a maximal cardinality $(\tilde{r}, \tilde{q})$ input-output regular subsystem.

CASE (II): Input–output dependent systems

For this case the search involves a number of steps:

Step 1: For all $\beta \in \Omega_F$ define the submatrices D_β corresponding to the set β of columns, $q_\beta = \text{rank}\,[C, D_\beta]$, and let $q_1 = \max\{q_\beta, \forall\beta \in \Omega_F\}$

(a) $q_1 = \tilde{q}$: Then the search stops and the maximal number of inputs, outputs that guarantee regularity is $(\tilde{r}, \tilde{q})$ and the system is input-output regularisable. For this case the parameterisation of the family is done as follows:

Maximal Cardinality Family. Let Ω'_F be the subset of sequences of Ω_F for which $q_\beta = \tilde{q}$. For every such $\beta \in \Omega'_F$ we shall denote by $\{\gamma(\beta)\}$ all sequences in Ω_G, which correspond to natural bases of G row space. Thus, we define the set of sequences $\Omega_{F,G} \triangleq \{(\beta,\gamma)\,\forall\beta \in \Omega'_F \text{ and } \gamma \in \gamma\{\beta\}\}$ and for all $(\beta,\gamma) \in \Omega_{F,G}$ the maximal cardinality $(\tilde{r}, \tilde{q})$ regular family is defined by $S(A, B_\beta, C_\gamma, D_{\beta,\gamma})$.

(b) $q_1 < \tilde{q}$: Then the system is not input, output regularisable and $(\tilde{r}, q_1)$ is a maximal number of inputs solution. The corresponding family of solutions with $(\tilde{r}, q_1 < \tilde{q})$ cardinality is constructed as before.

If a reduced input cardinality and increased output cardinality is desirable, then we proceed to the following step.

Step 2: For the matrix F, define all sets of $\tilde{r} - 1$ independent vectors of the columns of F (lexicographically ordered, denote this set by $\{F\}_1$ and let the corresponding set of indices be $\Omega_F^1 \triangleq \{\beta^1 \colon \beta^1 = (j_1, \ldots, j_{\tilde{r}-1})\}$.

For the set Ω_F^1 repeat Step 1 and this leads to a new solution pair $(\tilde{r} - 1, q_2)$ where $q_2 \geq q_1$. The construction of the corresponding family of subsystems follows along the lines described in Step 1.

The above algorithmic procedure defines the maximal cardinality for input, output regularity, as well as producing a parameterisation of $\langle f \rangle_{\tilde{r},\tilde{q}}$ family, as well as families with orders less than $(\tilde{r}, \tilde{q})$. We can now proceed to the description of the overall methodology for well-conditioning using the Indirect Method.

Indirect method for well-conditioning

For the system $S(A, B, C, D)$ we define the maximal cardinality pair $(\tilde{r}, \tilde{q})$ for which input, output regularity is guaranteed and let $\langle f \rangle_{\tilde{r},\tilde{q}}$ be the corresponding family of input, output regular models parameterised by pairs of sequences $(\beta,\gamma) \in \Omega_{F,G}$ with $\beta = (j_1, \ldots, j_{\tilde{r}})\,, \gamma = (l_1, \ldots, l_{\tilde{q}})$. The general element of this family is denoted by $S_{\beta,\gamma} \triangleq S\left(A, B_\beta, C_\gamma, D_{\beta,\gamma}\right)$. For each $\mathrm{S}_{\beta,\gamma}$ we proceed with testing as follows:

Step 1: If $D_{\beta,\gamma} \neq 0$ and $\text{rank}(D_{\beta,\gamma}) = \min(\tilde{r}, \tilde{q})$ or $\widetilde{D}_{\beta,\gamma} = 0$ and $\text{rank}(C_\gamma B_\beta) = \min(\tilde{r}, \tilde{q})$, then system is degenerate and search stops.

Step 2: If $D_{\beta,\gamma} \neq 0$ and $\text{rank}(D_{\beta,\gamma}) < \min(\tilde{r}, \tilde{q})$, or $\widetilde{D}_{\beta,\gamma} = 0$ and $\text{rank}(C_\gamma B_\beta) <$

$\min(\tilde{r},\tilde{q})$, then test for full rank of the Toeplitz matrix M_τ (Theorem 4.1), or respectively Toeplitz matrix $\widetilde{M}_{\tau'}$ (Corollary 4.1). If M_τ, $\widetilde{M}_{\tau'}$ are full rank, then the system is nondegenerate and the search stops. Otherwise, the system is degenerate and we proceed to the testing of another $S_{\beta,\gamma}$ subsystem.

Step 3: If all elements of $\langle f\rangle_{\tilde{r},\tilde{q}}$ have been tested for degeneracy and there is no element, which is nondegenerate, repeat the analysis of Steps 1, 2 for the smaller order family $\langle f\rangle_{\tilde{r}-1,\tilde{q}}$ etc. The overall procedure always leads to a nondegenerate system.

The system of $(\tilde{r},\tilde{q})$-maximal cardinality subsystems, which are input-output regular and nondegenerate, will be denoted by $\langle f\rangle^0_{\tilde{r},\tilde{q}}$ and $\Psi_{F,G}$ will denote the corresponding pairs of (β,γ) sequences.

6.3. Selection of natural bases

The analysis presented so far is based on selection of all possible natural bases and frequently that subset that satisfies certain orthogonality conditions. The construction of such bases is considered here. Let $T=[\underline{t}_1,\underline{t}_2,\ldots,\underline{t}_r]\in\Re^{q\times r}$, $q\geq r$ with rank (T) $=\rho$, $\rho<\min(q,r)$. The set of all natural bases from the set $\{\underline{t}_1,\underline{t}_2,\ldots,\underline{t}_r\}$ may be constructed as follows:

Construction of natural bases

Let $C_\rho(T)\in\Re^{\binom{q}{q}\times\binom{r}{\rho}}$ denote the ρth compound matrix of T [13] and let $\omega=[i_1,i_2,\ldots,i_\rho]\in Q_{\rho,r}$ be the sequences characterising the columns of $C_\rho(T)$, i.e.

$$C_\rho(T)=[\ldots,\underline{t}_\omega\wedge,\ldots],\qquad \underline{t}_\omega\wedge=\underline{t}_{i_1}\wedge\ldots\wedge\underline{t}_{i_\rho} \tag{6.9}$$

where $\underline{t}_\omega\wedge=\underline{t}_{i_1}\wedge\ldots\wedge\underline{t}_{i_\rho}$ denotes exterior product of the corresponding vectors. If $\Psi_{\rho,r}$ denotes the subset of $Q_{\rho,r}$ that corresponds to nonzero vectors $\underline{t}_\omega\wedge$, then any set $\left\{\underline{t}_{i_1},\ldots,\underline{t}_{i_\rho}:\underline{t}_\omega\wedge\neq 0\right\}$ is a natural basis. This produces a parameterisation of all such bases in terms of the sequences of $\Psi_{\rho,r}$.

Selection of natural bases with given orthogonality

The set $\Psi_{\rho,r}$ of sequences of $Q_{\rho,r}$ parameterises all proper bases. However, different bases may have different orthogonality properties. Without loss of generality we may assume that the columns of T are normalised, i.e. $\|\underline{t}_i\|=1$, $\forall i\in\underline{r}$. If we use the value of the Grammian as the measure of orthogonality, a classification of the natural bases may be achieved using the following result [14]:

Proposition 6.2. Let $T=[\underline{t}_1,\underline{t}_2,\ldots,\underline{t}_r]\in\Re^{q\times r}$, $\|\underline{t}_i\|=1$, $\forall i\in\underline{r}$, $\rho=\text{rank}\,(T)\leq\min(r,q)$, let $G=G(\underline{t}_1,\ldots,\underline{t}_r)=T^tT\in\Re^{r\times r}$ be the Gram matrix of the vectors $\{\underline{t}_1,\underline{t}_2,\ldots,\underline{t}_r\}$ and let

$$C_\rho(G)=C_\rho(T^tT)\in\Re^{\binom{r}{\rho}\times\binom{r}{\rho}}=[c_{ij}] \tag{6.10}$$

be the ρ-compound of G. The diagonal elements c_{ii} correspond to all sequences $\omega = (i_1, \dots, i_\rho) \in Q_{\rho,r}$ and represent $\|\underline{t}_\omega \wedge\|$. In particular:

a) $c_{ii} = 0$, if $\underline{t}_\omega \wedge = 0$, i.e. $\{\underline{t}_{i_1}, \dots, \underline{t}_{i_\rho}\}$ dependent.

b) $c_{ii} > 0$, if $\underline{t}_\omega \wedge \neq 0$, i.e. $\{\underline{t}_{i_1}, \dots, \underline{t}_{i_\rho}\}$ is a natural basis.

c) The element with the maximal value c^* corresponds to a sequence $\omega = (i_1, \dots, i_\rho) \in \Psi_{\rho,r}$ which characterises the most orthogonal natural basis of T.

The above result readily follows from the definition of the Grammian and the interpretation of the Binet–Cauchy Theorem [13]. Clearly by inspection of all the $\binom{r}{\rho}$ diagonal elements of $C_\rho(G)$ we can order all natural bases according to degree of orthogonality.

7. CONCLUSIONS

The problem of selecting subsystems of a progenitor model $S(A, B, C, D)$, or $H(s)$, which have maximal input and output cardinality, are input-output regular and are nondegenerate has been considered in detail. We have given criteria for the presence of input, output redundancy and system degeneracy, and suggested procedures for how we can avoid such properties. The results lead to parameterisation of all subsystems, which are input-output regular and nondegenerate and have maximal cardinality $(\tilde{r}, \tilde{q})$, and leads to the family $\langle f \rangle^0_{\tilde{r},\tilde{q}}$. Every system in $\langle f \rangle^0_{\tilde{r},\tilde{q}}$ has $\tilde{r}$-inputs and $\tilde{q}$-outputs and it is parameterised by a set of sequences $(\beta,\ \gamma) \in \Psi_{F,G}$ defining the subsets of inputs and outputs that has to be considered. Every element $S(A, B_\beta, C_\gamma, D_{\beta,\gamma}) \in \langle f \rangle^0_{\tilde{r},\tilde{q}}$ does not necessarily have a structure that is desirable, as far as other properties. In fact, $S_{\beta,\gamma}$ may be either uncontrollable, and/or unobservable and other properties may not hold true. This family $\langle f \rangle^0_{\tilde{r},\tilde{q}}$ may then be used as the starting point for additional investigations and conditions based on properties of Hankel matrices are given which also guarantee controllability and observability for the resulting system. An additional advantage of the current procedure is that the sufficient conditions for avoiding strong degeneracy also lead to systems which have no infinite zeros and thus to models with a simple structure. Searching for conditions, which lead to systems with minimum phase characteristics, as well as making the search for minimal subsystems more systematic are problems for future research.

APPENDIX

Proof of Proposition 3.2. (i) Note that

$$P'(s) = \begin{bmatrix} I_n & 0 \\ C(sI-A)^{-1} & I_q \end{bmatrix} \begin{bmatrix} sI_n - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} I_n & (sI-A)^{-1}B \\ 0 & I_r \end{bmatrix} \quad (A.1)$$
$$= \begin{bmatrix} sI_n - A & 0 \\ 0 & -C(sI_n - A)^{-1}B + D \end{bmatrix}.$$

Thus $P'(s)$ and $P(s)$ are equivalent and thus

$$\tau = \text{rank}_{\Re(s)}\{P(s)\} = \text{rank}_{\Re(s)}\{P'(s)\} = n + \text{rank}_{\Re(s)}\{H(s)\} = n + \rho.$$

(ii) From the above we have

$$\begin{aligned} \eta &= r+n-\tau = r+n-(n+\rho) = r-\rho = \dim N_r\{H(s)\} \\ \theta &= q+n-\tau = q+n-(n+\rho) = q-\rho = \dim N_\ell\{H(s)\}. \end{aligned}$$ □

Proof of Proposition 3.3. (i) If $q \geq r$ and the system is not input regular, then $N_r\{P(s)\} \neq \{0\}$ and thus $\tau < n+r$ which implies degeneracy. The $q \leq r$ case follows similarly.

(ii) From part (i) it follows that $\tau < n+r$, $\tau < n+q$ and thus $\tau < \min(n+r, n+q)$ and this implies degeneracy.

(iii) Consider the case $q \geq r$ and $\tau_\ell < r$, then $-\tau_\ell > -r$ and $q - \tau_\ell > q - r$. This condition implies that there exists a set of $q - \tau_\ell$ linearly independent vectors $\{\underline{v}_i, i = 1, \ldots, q - \tau_\ell\}$ such that

$$\underline{v}_i^t \begin{bmatrix} C & D \end{bmatrix} = \underline{0}^t. \tag{A.2}$$

The above is equivalent to

$$[\underline{0}^t, \underline{v}_i^t] \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} = \underline{0} \tag{A.3}$$

and thus also to

$$[\underline{0}^t, \underline{v}_i^t] \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} I_n & (sI-A)^{-1}B \\ 0 & I_r \end{bmatrix} = \underline{0} \tag{A.4}$$

or

$$[\underline{0}^t, \underline{v}_i^t] \begin{bmatrix} sI_n - A & 0 \\ -C & -H(s) \end{bmatrix} = \underline{0} \tag{A.5}$$

or

$$\underline{v}_i^t H(s) = \underline{0}^t. \tag{A.6}$$

Since there are $q - \tau_\ell$ constant independent vectors in $N_\ell\{H(s)\}$ it follows that $\dim N_l\{H(s)\} = q - \rho \geq q - \tau_l > q - r$ and thus $\rho < r$ and $\rho < q$, i.e. $\rho < \min(r, q)$. □

Proof of Proposition 4.1. (a) By conditions (3.10a), (3.10b) and the fact that $\tau_r \leq r$ and $\tau_l \leq q$, Part (a) readily follows.

(b) The number of nonzero cmi is $n - t_r = \tau_r - \rho$ and such indices exist only when $n - t_r = \tau_r - \rho > 0$. In the case where $\tau_r = r$ then clearly $t_r = 0$. Part (c) follows along similar lines.

In the following we consider the case where $q \geq r$ and we shall assume that (4.2) holds true, i. e. we have at least a nonzero cmi. This implies that there exists a right pair $\underline{x}(s)$,$\underline{u}(s)$, where

$$\begin{aligned} \underline{x}(s) &= \underline{x}_0 + s\underline{x}_1 + \cdots + s^{k-1}\underline{x}_{k-1} && \text{(A.7)} \\ \underline{u}(s) &= \underline{u}_0 + s\underline{u}_1 + \cdots + s^{k-1}\underline{u}_{k-1} && \text{(A.8)} \end{aligned}$$

such that

$$\begin{aligned} &(sI - A)\underline{x}(s) = B\underline{u}(s) \\ &C\underline{x}(s) + D\underline{u}(s) = \underline{0} \qquad \text{(A.9)} \end{aligned}$$

□

P r o o f o f P r o p o s i t i o n 4.2. Substituting the expressions of $\underline{x}(s)$, $\underline{u}(s)$ from (A.7), (A.8) into (A.9) we have

$$\begin{aligned} &(sI - A)(\underline{x}_0 + s\underline{x}_1 + \cdots + s^{k-1}\underline{x}_{k-1}) = B(\underline{u}_0 + s\underline{u}_1 + \cdots + s^k\underline{u}_k) \\ &C(\underline{x}_0 + sx_1 + \cdots + s^{k-1}\underline{x}_{k-1}) + D(\underline{u}_0 + su_1 + \cdots + s^1\underline{u}_k) = \underline{0}. \end{aligned}$$

By equating coefficients of equal powers, it follows that

$$\begin{aligned} &\underline{x}_{k-1} = B\underline{u}_k \\ &\underline{x}_{k-2} = AB\underline{u}_k + Bu_{\underline{k}-1} && \text{(A.10)} \\ &\vdots \\ &\underline{x}_0 = A^{k-1}B\underline{u}_k + A^{k-2}B\underline{u}_{k-1} + \cdots + AB\underline{u}_2 + B\underline{u}_1 \\ &\underline{0} = A^kB\underline{u}_k + A^{k-1}B\underline{u}_{k-1} + \cdots + A^2B\underline{u}_2 + ABu_1 + B\underline{u}_0 && \text{(A.11)} \end{aligned}$$

and

$$\begin{aligned} &C\underline{x}_0 + D\underline{u}_0 = \underline{0} = CA^{k-1}B\underline{u}_k + CA^{k-2}B\underline{u}_{k-1} + \ldots + CAB\underline{u}_2 + CB\underline{u}_1 + D\underline{u}_0 \\ &C\underline{x}_1 + D\underline{u}_1 = \underline{0} = CA^{k-2}B\underline{u}_k + CA^{k-3}B\underline{u}_{k-2} + \ldots + CAB\underline{u}_3 + CB\underline{u}_2 + D\underline{u}_1 \\ &\vdots \\ &C\underline{x}_{k-1} + D\underline{u}_{k-1} = \underline{0} = CB\underline{u}_k + D\underline{u}_{k-1} && \text{(A.12)} \\ &D\underline{u}_k = \underline{0}. \end{aligned}$$

By combining the above the result follows. □

P r o o f o f L e m m a 4.1. (i) If $D \neq 0$ and $\operatorname{rank}(D) = \delta$, there exists a pair of transformations $Q \in \Re^{q\times q}$, $R \in \Re^{r\times r}$, $|Q|$, $|R| \neq 0$ such that

$$\begin{bmatrix} I_n & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & R \end{bmatrix} = \begin{bmatrix} sI - A & -BR \\ -QC & -QDR \end{bmatrix} = Q'P(s)R' = P'(s)$$

where

$$QDR = \begin{bmatrix} I_\delta & 0 \\ 0 & 0 \end{bmatrix} = D', \quad QC = C', \quad BR = B'.$$

By partitioning C', B' according to the partitioning of D' we have

$$P'(s) = \begin{bmatrix} sI - A & -B'_\delta & \vdots & B'_{r-\delta} \\ -C'_\delta & -I_\delta & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -C'_{q-\delta} & 0 & \vdots & 0 \end{bmatrix} = \begin{bmatrix} s\widetilde{E} - \widetilde{A} & \vdots & -\widetilde{B} \\ \cdots & \cdots & \cdots \\ -\widetilde{C} & \vdots & 0 \end{bmatrix}. \tag{A.13}$$

The zero structure of $P'(s)$ [12] is defined by the zero pencil $\widetilde{Z}(s) = s\widetilde{N}\widetilde{E}\widetilde{M} - \widetilde{N}\widetilde{A}\widetilde{M}$, where $\widetilde{N}$ is a $(n-r+2\mu)\times(n+\delta)$ left annihilator of $\widetilde{B}$ and $\widetilde{M}$ is a $(n+\delta)\times(n-q+2\delta)$ right annihilator of $\widetilde{C}$. Clearly, $\widetilde{Z}(s)$ has dimensions $(n-r+2\mu)\times(n-q+2\delta)$ and $n-r+2\mu \geq n-q+2\delta$. For such a pencil the maximal possible value of a right index is when $\varepsilon_{\max}+1 = n-q+2\delta$, i. e. smallest of the two dimensions; this follows by inspection of the possible structure of the Kronecker form of $\widetilde{Z}(s)$ [3].

Part (ii) follows from Part (i) for $\delta = 0$. □

Proof of Theorem 4.1. (i) From Proposition 4.2, it follows that if $\operatorname{rank}(D) = r$, then from the last of (4.6) we have that $D\underline{u}_k = \underline{0}$. Clearly, this implies $\underline{u}_k = \underline{0}$ and this in turn (from (4.6)) yields $\underline{u}_{k-1} = \underline{0}$; again we have $\underline{u}_{k-1} = \underline{0}$ and by obvious induction, $\underline{u}_k = \underline{0}$ for all $k = 0, 1, 2, \ldots$. It is now clear that since there is no $\underline{u}(s)$ and thus no $\underline{x}(s)$ satisfying (4.5), the system is non-degenerate.

(ii) By condition (4.6) if there is a right index $\varepsilon < \tau$ then M_ε has a right kernel and from the structure of M_k for $\forall k \geq \varepsilon$ we shall also have $N_r\{M_n\} \neq \{0\}$. Since τ is the maximal possible value of a right index, if $N_r\{M_\tau\} = \{0\}$, then also for $\forall k \geq \tau \quad N_r\{M_k\} = \{0\}$, since otherwise we are led to a contradiction (existence of a right index greater than τ). This completes the proof. □

Proof of Corollary 4.1. (i) Clearly, we have that there exists a 0-right index if the matrix $[B^t, 0]^t$ or equivalently B looses rank. However, if $\operatorname{rank}(CB) = r$, then it is necessary that $\operatorname{rank}(B) = r$, because, otherwise $\exists \underline{v} : \underline{v} \neq \underline{0}$ and $B\underline{v} = \underline{0} \to CB\underline{v} = \underline{0}$ and this leads to a contradiction. Thus, there is no 0-right index. Following similar arguments to those in the proof of the Theorem, it follows that there is no other index of any value k.

Part (ii) follows along similar lines. □

Proof of Theorem 4.2. For nondegenerate systems, the systems matrix pencil $P(s)$ is right regular and thus, if:

$$P(s) = \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} = s\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} = sF - G \tag{A.14}$$

the infinite elementary divisors (ied) are characterised by the properties of the right nullities of the following sequence of Toeplitz matrices defined on the pair (F, G)

[10, 11]:

$$T_\infty^1 = F, \quad T_\infty^2 = \begin{bmatrix} F & 0 \\ G & F \end{bmatrix}, \quad \ldots, \quad T_\infty^k = \begin{bmatrix} F & 0 & \ldots & 0 & 0 \\ G & F & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & F & 0 \\ 0 & 0 & \ldots & G & F \end{bmatrix}. \tag{A.15}$$

If we denote by $\eta_k \overset{\Delta}{=} \eta_r\left(T_\infty^k\right)$ the right nullities of the T_∞^k matrices and by $\gamma_i \overset{\Delta}{=} \eta_r\left(Q_i\right)$, then we have the following relationships:

— For $k = 1 \quad \eta_r\left(T_\infty^1\right) = r = \eta_1$

— For $k = 2$ we have

$$T_\infty^2 = \begin{bmatrix} F & 0 \\ G & F \end{bmatrix} = \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline A & B & I & 0 \\ C & D & 0 & 0 \end{array}\right] \xrightarrow{\text{equivalent}} \left[\begin{array}{ccc|c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & D & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \widetilde{T}_\infty^2$$

and thus

$$\eta_r\left(T_\infty^2\right) = \eta_r\left(\widetilde{T}_\infty^2\right) = \eta_r\left(Q_0\right) + r = \eta_r(D) + r = \eta_2 = \gamma_0 + r.$$

For the general T_∞^k, by using elementary column and row operations it is readily shown that for $k > 2$ we have that T_∞^k may be reduced to the following equivalent matrix

$$\widetilde{T}_\infty^k = \left[\begin{array}{c|ccccc|c} \begin{matrix} I & & \\ & \ddots & \\ & & I \end{matrix} & & & 0 & & & \\ \hline & D & 0 & \cdots & 0 & & 0 \\ & CB & D & \cdots & 0 & & \\ 0 & & & & & & \\ & \vdots & \vdots & \ddots & \vdots & & \\ & CA^{k-3}B & CA^{k-4}B & \cdots & D & & \\ \hline & & 0 & & & & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc:c:c} I & & & & \\ & \ddots & & 0 & \\ & & & & 0 \\ & & I & & \\ \hdashline & 0 & & Q_{k-2} & \\ \hdashline & & 0 & & 0 \end{array}\right]$$

and thus

$$\eta_r \left(T_\infty^k\right) = \eta_r \left(\widetilde{T}_\infty^k\right) = \eta_r \left(Q_{k-2}\right) + r = \eta_2 = \gamma_{k-2} + r. \tag{A.16}$$

Clearly [11], $\{\eta_k\}$ is a piecewise arithmetic progression and thus also the $\{\gamma_i\}$ sequence. The singular points of the $\{\eta_k\}$, or $\{\gamma_i\}$ sequences define the degrees of ied of $P(s)$. The relationship between degrees of ied and orders of infinite zeros of $H(s)$ (Part (ii)) is a known result established in [20]. □

Proof of Proposition 5.1. The subsystem $S(A, B_\alpha, C_\beta, D_{\alpha,\beta})$ has dimension of its state space equal to $\delta_M(H)$. If the corresponding transfer function $H_{\alpha,\beta}(s)$ has $\delta_M\left(H_{\alpha,\beta}\right) < \delta_M\left(H\right)$, then clearly it is not minimal. If $\delta_M\left(H_{\alpha,\beta}\right) = \delta_M\left(H\right)$, then by Lemma 5.1 the result is established. □

Proof of Proposition 6.1. If $\operatorname{rank}[C, D] = \operatorname{rank}[C]$, then any selection $\beta \in \Omega_F$ produces some D_β submatrix and $\operatorname{rank}[C, D_\beta] = \operatorname{rank}[C, D]$. Thus, any choice of $\gamma \in \Omega_G$ based on the properties of rank of C leads to system $S_{\beta,\gamma}$ that is input, output regular. □

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REFERENCES

[1] P. J. Antsaklis and A. N. Michel: Linear Systems. McGraw-Hill, New York 1997

[2] G. D. Forney: Minimal bases of rational vector spaces with applications to multivariable systems. SIAM J. Control *13* (1975), 493–520.

[3] G. Gantmacher: Theory of Matrices. Volume 2. Chelsea, New York 1959.

[4] A. Georgiou and C. A. Floudas: Structural analysis and synthesis of feasible control systems: Theory and applications. Chem. Eng. J. *67* (1989), 600–618.

[5] R. Govind and G. J. Powers: Control systems synthesis strategies. AIChE J. *28* (1982), 60–73.

[6] T. Kailath: Linear Systems. Prentice Hall, Englewood Cliffs, N.J. 1980.

[7] N. Karcanias: Global process instrumentation: Issues and problems of a system and control theory framework. Measurement *14* (1994), 103–113.

[8] N. Karcanias: Control problems in global process instrumentation: A structural approach. In: Proc. ESCAPE-6, Comput. Chem. Eng. *20* (1996), 1101–1106.

[9] N. Karcanias and C. Giannakopoulos: Necessary and sufficient conditions for zero assignment by constant squaring down. Linear Algebra Appl. *122–124* (1989), 415–446.

[10] N. Karcanias and G. E. Hayton: State-space and transfer function invariant infinite zeros: A unified approach. In: Proc. 1981 Joint Automatic Control Conference, Univ. of Virginia, Charlottesville 1981, Paper TA–4C.

[11] N. Karcanias and G. Kalogeropoulos: On the Segre, Weyr characteristics of right (left) regular pencils. Internat. J. Control *44* (1986), 991–1015.

[12] N. Karcanias and B. Kouvaritakis: The output zeroing problem and its relationship to the invariant zero structure. Internat. J. Control *30* (1979), 395–415.

[13] M. Marcus and H. Minc: A Survey of Matrix Theory and Matrix Inequalities. Allyn and Bacon, Boston 1964.

[14] M. Mitrouli and N. Karcanias: Computation of the GCD of polynomials using Gaussian transformations and shifting. Internat. J. Control *58* (1993), 211–228.

[15] M. Morari: Effect of design on the controllability of chemical plants. In: Proc. IFAC Workshop on Interaction between Process Design and Process Control, Imperial College 1992, pp. 3–16.

[16] M. Morari and G. Stephanopoulos: Studies in the synthesis of control structures for chemical processes: Part II: Structural aspects and the synthesis of alternative feasible control schemes. AIChE J. *26* (1980), 232–246.

[17] J. E. Rijnsdorp: Integrated Process Control and Automation. Elsevier, Amsterdam 1991.

[18] H. H. Rosenbrock: State–Space and Multivariable Theory. Nelson, London 1970.

[19] S. Skogestad and I. Postlethwaite: Multivariable Feedback Control. Wiley, Chichester 1996.

[20] A. I. G. Vardulakis and N. Karcanias: Relation between strict equivalence invariants and structure at infinity of matrix pencils. IEEE Trans. Automat. Control *AC–28* (1983), 99, 514–516.

[21] M. E. Warren and A. E. Eckberg: On the dimensions of controllability subspaces: A characterisation via polynomial matrices and Kronecker invariants. SIAM J. Control Optim. *13* (1975), 434–445.

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