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FIXED POLES OF H_2 OPTIMAL CONTROL BY MEASUREMENT FEEDBACK

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This paper is concerned with the flexibility in the closed loop pole location when solving the H_2 optimal control problem (also called the H_2 optimal disturbance attenuation problem) by proper measurement feedback. It is shown that there exists a precise and unique set of poles which is present in the closed loop system obtained by any measurement feedback solution of the H_2 optimal control problem. These "H_2 optimal fixed poles" are characterized in geometric as well as structural terms. A procedure to design H_2 optimal controllers which simultaneously freely assign all the remaining poles, is also provided.

1. INTRODUCTION

The H_2 optimal control problem, which amounts to minimizing the H_2 norm of the closed-loop transfer from a disturbance input to the output by a stabilizing controller, has been considered since the works of [10, 11] and [7], without regularity assumptions and the question of flexibility in closed-loop pole placement has appeared challenging. By making use of decompositions of the system in a particular basis (the so-called Special Coordinate Basis), [4] characterized the H_2 optimal fixed poles for the state feedback case and, in the case of measurement feedbacks, [8] studied the flexibility in closed loop poles for the following design method: select a state feedback matrix and use an observer so that the resulting compensator is solution of H_2 optimal control problem. This method exhibits some "fixed poles" which depend on the class of observers as well as on the preliminary selected state feedback matrix but there is no guaranty that they correspond to the H_2 *optimal fixed poles*, the latter being present in the closed loop system obtained by *any* solution of the H_2 optimal control problem, whatever may be the type of measurement feedback proper compensator.

The disturbance rejection problem, which amounts to canceling the closed-loop transfer between the disturbance and the output is obviously one particular case of the H_2 optimal control problem (when the optimum is zero) for which many works contributed to explain the geometric structures of the system involved in the problem

[1]This work has been done while the author was working within IRCCyN.

solvability ([1, 13]). In this context, [3] characterize, when the disturbance can be rejected by measurement feedback, the so-called *Disturbance Rejection Fixed Poles*, namely poles which are present in the closed loop system with any measurement feedback solution, whatever be the way used to find the compensator. A transformation has been proposed by [10], completed by [11], which revealed that solving the H_2 optimal control problem is equivalent to solving the disturbance rejection problem on a modified system. Starting from that transformation and the study of [3] on the Disturbance Rejection Fixed Poles, the aim of the present paper is mainly to characterize the fixed poles of the H_2 optimal control problem by measurement feedback. The characterization will be geometric as well as structural (in terms of invariant zeros) for strictly proper systems under some mild minimality assumption. Additionally, a procedure is provided to construct H_2 optimal compensators which are also "optimal" in the sense of pole placement, i.e. which freely assign all the poles except the H_2 optimal fixed poles.

The detailed formulation of the problem and the notation are stated in Section 2. In Section 3, the characterization of the Disturbance Rejection Fixed Poles and the system transformation which converts an H_2 optimal problem into a disturbance rejection problem are recalled. The main results, i.e. the H_2 optimal fixed poles characterizations and the constructive procedure for H_2 optimal compensators are presented in Section 4. We conclude in Section 5 with some possible extensions of this work.

2. PROBLEM FORMULATION

We consider linear time-invariant systems described by:

$$\Sigma \; : \; \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dh(t) \\ z(t) = Ex(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $x(t) \in \mathcal{X} \approx \mathbb{R}^n$ is the state, $u(t) \in \mathcal{U} \approx \mathbb{R}^m$ is the control input, $h(t) \in \mathcal{H} \approx \mathbb{R}^q$ is the disturbance input, $z(t) \in \mathcal{Z} \approx \mathbb{R}^r$ is the output to be controlled and $y(t) \in \mathcal{Y} \approx \mathbb{R}^p$ is the measured output. $\mathcal{B}, \mathcal{D}, \mathcal{C}$, and $\mathcal{E}$ respectively denote Im B, Im D, Ker C and Ker E, $\dot{\cup}$ stands for the union with repeated common elements.

We make the following assumptions:

A.1: $(A, [BD])$ is controllable, $\left(\begin{bmatrix} C \\ E \end{bmatrix}, A \right)$ is observable;

A.2: (A, B) is stabilizable, (C, A) is detectable.

These assumptions are not restrictive: (A.1) corresponds to a minimal description of the system with respect to all the external variables. (A.2) is necessary to control the system with stability. Nevertheless the system description (1) is not as general as in [8] where feedthrough matrices (from input or disturbance to controlled output or measurement) are present. The main reason for restricting our study to strictly proper systems like (1) is that the available geometric characterizations of the DRP fixed poles (see Section 3) have been proposed up to now just for this particular class of systems. On the other hand, for the study of such "exact" control problems, classical tricks exist (addition of some integrators) for replacing the proper system by an extended strictly proper one (see for instance, [5, 6]). However, it is not yet guaranteed that such a simple trick is applicable for the H_2 optimal control case.

The proper compensator Γ is generically described by:

$$\Gamma : \begin{cases} \dot{w}(t) = Nw(t) + My(t) \\ u(t) = Lw(t) + Ky(t) \end{cases} \tag{2}$$

where $w(t) \in \mathcal{W} \approx \mathbb{R}^\nu$ is the state of the compensator.

The compensator internally stabilizes the system if the eigenvalues of

$$A_e := \begin{bmatrix} A + BKC & BL \\ MC & N \end{bmatrix} \tag{3}$$

are all stable (i. e. are in the open left half-plane, denoted by $\mathbb{C}^-$). The resulting compensated system is denoted by $(\Sigma \times \Gamma)$, its associated transfer function matrix is denoted by

$$T_\Gamma(s) := \begin{bmatrix} E & 0 \end{bmatrix} (sI_{(n+\nu)} - A_e)^{-1} \begin{bmatrix} D \\ 0 \end{bmatrix} \tag{4}$$

and the H_2 norm of $T_\Gamma(s)$ is:

$$\|T_\Gamma(s)\|_2 := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{tr} \left(T_\Gamma(j\omega) T_\Gamma^T(-j\omega) \right) \mathrm{d}\omega \right)^{\frac{1}{2}} \tag{5}$$

where the transposition and the trace of a matrix are respectively denoted by $\cdot^T$ and $\operatorname{tr}(\cdot)$. The optimum of the H_2 norm over all internal stabilizing compensators is defined as follows:

$$\gamma_{\text{opt}} := \inf_\Gamma \left\{ \|T_\Gamma(s)\|_2 | \sigma(A_e) \subset \mathbb{C}^- \right\} \tag{6}$$

where $\sigma(\cdot)$ stands for the spectrum.

We are now able to formulate the H_2 optimal control problem (referred sometimes also as H_2 optimal disturbance attenuation problem) by measurement feedback.

Definition 1. Let a system Σ be given, the H_2 Optimal Control Problem by measurement feedback (H_2OCP) amounts to finding, if possible, a stabilizing compensator Γ such that

$$\|T_\Gamma(s)\|_2 = \gamma_{\text{opt}}. \tag{7}$$

When this problem is solvable, one also says that the optimum is attained. A compensator solution to this problem is said to be an H_2 optimal compensator.

When this problem is not solvable, one is faced to consider sub-optimal problems, which may amount to apprcaching the optimum as close as possible, or selecting compensators for which the norm is less than a prespecified number (see [11]).

The H_2OCP for the particular case when the optimum is zero corresponds to the Disturbance Rejection Problem with Stability, designed here by DRPS. Contrary to the H_2OCP for which the stability requirement affects the value of the infimum, it is pertinent to define the Disturbance Rejection Problem without requiring stability, denoted here as DRP, since the two requirements are independent.

The DRP thus corresponds to:

Definition 2. Let a system Σ be given, the Disturbance Rejection Problem by measurement feedback (DRP) amounts to finding, if possible, a compensator Γ such that $T_\Gamma(s) = 0$.

The DRPS (DRP with internal Stability) has the same definition plus the requirement $\sigma(A_e) \subset \mathbb{C}^-$.

Let us remark that the fact that $\gamma_{\text{opt}} = 0$ does not assure the DRPS or the DRP to be solvable. It only indicates that the so-called H_2 Almost Disturbance Rejection Problem with Stability is solvable. This problem introduced by [12] amounts to finding a sequence of controllers whose limit (possibly not attained) gives the optimal solution $\gamma_{\text{opt}} = 0$.

The notion of fixed poles of a Problem (which may stand for either the DRP or the H_2OCP) is precisely formulated in the following definition:

Definition 3. Let a system Σ as well as a Problem be given and let us denote Θ(Problem) the set of measurement feedback compensators Γ which are solutions of the Problem. Then, the *Problem fixed poles* are defined by:

$$\text{Problem fixed poles} := \bigcap_{\Gamma \in \Theta(\text{Problem})} \sigma(\mathrm{A_e}). \tag{8}$$

3. BACKGROUND

3.1. Geometric tools

In this section are briefly recalled some elements of the geometric approach. The interested reader which would like to have more details might refer to [14] and [1] for the basic notions and to [3] for the DRP fixed poles characterizations.

Given two subspaces $\mathcal{T} \subset \mathcal{X}$ and $\mathcal{L} \subset \mathcal{X}$, we will denote:

— $\mathcal{V}^*_{(\mathcal{T},\mathcal{L})}$: the supremal $(A, \mathcal{T})$-invariant subspace contained in $\mathcal{L}$.

— $\mathcal{S}^*_{(\mathcal{L},\mathcal{T})}$: the infimal $(\mathcal{L}, A)$-invariant subspace containing $\mathcal{T}$.

— $\mathcal{R}^*_{(\mathcal{T},\mathcal{L})} = \mathcal{V}^*_{(\mathcal{T},\mathcal{L})} \cap \mathcal{S}^*_{(\mathcal{L},\mathcal{T})}$: the supremal $(A, \mathcal{T})$-controllability subspace contained in $\mathcal{L}$.

— $\mathcal{N}^*_{(\mathcal{L},\mathcal{T})} = \mathcal{V}^*_{(\mathcal{T},\mathcal{L})} + \mathcal{S}^*_{(\mathcal{L},\mathcal{T})}$: the infimal $(\mathcal{L}, A)$-complementary observability subspace containing $\mathcal{T}$.

Let $\mathcal{V}$ be an $(A, \mathcal{B})$-invariant subspace, then $\mathcal{F}(\mathcal{V})$ denotes the set of matrices F that satisfy $(A+BF)\mathcal{V} \subset \mathcal{V}$. Let $\mathcal{S}$ be a $(\mathcal{C}, A)$-invariant subspace, then $\mathcal{G}(\mathcal{S})$ denotes the set of matrices G that satisfy $(A + GC)\mathcal{S} \subset \mathcal{S}$. A $(\mathcal{C}, A, \mathcal{B})$-pair is defined [9] to be a pair of subspaces of $\mathcal{X}$, say $(\mathcal{S},\mathcal{V})$, where $\mathcal{S}$ is a $(\mathcal{C}, A)$-invariant subspace, $\mathcal{V}$ is an $(A, \mathcal{B})$-invariant subspace and $\mathcal{S} \subset \mathcal{V}$.

Let us now recall the geometric definition of the set of invariant zeros for the triple (A, B, E), but the definition is also valid for each subsystem of Σ represented by state-input-output matrices.

Definition 4. The invariant zeros of the system (A, B, E), denoted as $\mathcal{Z}(A, B, E)$, are equal to the eigenvalues of the map induced by $(A + BF)$ in the quotient space $\mathcal{V}^*_{(\mathcal{B},\mathcal{E})}/\mathcal{R}^*_{(\mathcal{B},\mathcal{E})}$ i.e.:

$$\mathcal{Z}(A, B, E) := \sigma\left(A + BF \Big| \frac{\mathcal{V}^*_{(\mathcal{B},\mathcal{E})}}{\mathcal{R}^*_{(\mathcal{B},\mathcal{E})}}\right), \tag{9}$$

where $F \in \mathcal{F}(\mathcal{V}^*_{(\mathcal{B},\mathcal{E})})$.

Note that a dual equivalent definition is expressed in terms of $(\mathcal{C}, A)$-invariant subspaces.

Of particular importance in the present design procedure of H_2 optimal compensators, is the so-called notion of $(\mathcal{S}, \mathcal{V})$-based compensator introduced by [1]. Such a compensator is of full order $(\nu = n)$ and based on a selected $(\mathcal{C}, A, \mathcal{B})$-pair, say $(\mathcal{S},\mathcal{V})$. The matrices (K, L, M, N) of Γ (2) are obtained by:

$$\begin{cases} N = A + GC + BFL_2 \\ M = -G + BFL_1 \\ L = FL_2 \\ K = FL_1 \end{cases} \tag{10}$$

with matrices L_1 and L_2 such that:

— $L_1C + L_2 = I_n$

— $\ker L_2 \oplus (\mathcal{S} \cap \mathcal{C}) = \mathcal{S}$.

It has been shown by [1] that $\sigma(A_e) = \sigma(A + BF) \dot{\cup} \sigma(A + GC)$, which enhances the fact that this type of compensator may be seen as designed in two parts: a full order observer and an estimated state feedback.

Let us insist on the fact that, even if a special type of compensator is used here for simultaneously solving either the H_2OC or the DR Problem and achieving maximal pole assignment, the results concerning fixed poles characterizations are valid for any type of compensator Γ (generically expressed in (2)), provided only that it is proper and that the compensator input is the system measurement output.

Let us recall the basic geometric solvability condition of the DRP by dynamic measurement feedback (see [9] and [13]):

Theorem 5. Let a system Σ be given, the DRP is solvable if and only if:

$$\mathcal{S}^*_{(\mathcal{C},\mathcal{D})} \subset \mathcal{V}^*_{(\mathcal{B},\mathcal{E})}. \tag{11}$$

Let us denote $\mathcal{R}^* := \mathcal{R}^*_{(\mathcal{B},\mathcal{E})}$, $\mathcal{R}^*_c := \mathcal{R}^*_{(\mathcal{B}+\mathcal{D},\mathcal{E})}$, $\mathcal{N}^* := \mathcal{N}^*_{(\mathcal{C},\mathcal{D})}$, $\mathcal{N}^*_c := \mathcal{N}^*_{(\mathcal{C}\cap\mathcal{E},\mathcal{D})}$. [1] obtained geometric solvability conditions of the DRPS by dynamic measurement feedback involving the subspaces $\mathcal{R}^*_c$ and $\mathcal{N}^*_c$. DRPS solvability is equivalent to the fact that the set of the DRP fixed poles is stable. The knowledge of the location of the fixed poles obviously gives more dynamic information (distance to instability, minimal possible damping, ...).

Theorem 6. Let the system Σ be given and let us assume that the DRP is solvable. Then, the DRP Fixed Poles are given by:

$$\begin{aligned} \text{DRP fixed poles} \quad &= \quad \sigma\left(A+BF\Big|\frac{\mathcal{R}^*_c}{\mathcal{R}^*}\right) \\ &\dot{\cup} \quad \sigma\left(A+GC\Big|\frac{\mathcal{N}^*}{\mathcal{N}^*_c\cap\mathcal{R}^*_c}\right) \end{aligned} \tag{12}$$

where $F \in \mathcal{F}(\mathcal{R}^*_c)$, $G \in \mathcal{G}(\mathcal{N}^*_c \cap \mathcal{R}^*_c)$ and all the remaining poles of the compensated system can be placed at any (symmetric) desired location by a suitable choice of the parameters K, L, M, N in (2). Moreover, a structural characterization is given by:

$$\begin{aligned} \text{DRP fixed poles} \quad &= \quad \left\{ \mathcal{Z}(A,B,E) - \mathcal{Z}\left(A, \begin{bmatrix} B & D \end{bmatrix}, E\right) \right\} \\ &\dot{\cup} \quad \left\{ \mathcal{Z}(A,D,C) - \mathcal{Z}\left(A, D, \begin{bmatrix} C \\ E_c \end{bmatrix}\right) \right\} \end{aligned} \tag{13}$$

where E_c is defined by $\ker E_c = \mathcal{R}^*_c$.

3.2. The system transformation

Here is recalled the transformation from the original system Σ to a modified system Σ_{PQ} on which the DRPS is studied. This transformation has been introduced by [10] for strictly proper compensators and generalized in [11] to proper compensators.

Let us define P and Q, respectively, as the largest symmetric solutions[1] of the following linear matrix inequalities:

$$\Phi(X) := \begin{pmatrix} A^TX + XA + E^TE & XB \\ B^TX & 0 \end{pmatrix} \geq 0 \tag{14}$$

$$\Psi(Y) := \begin{pmatrix} AY + YA^T + DD^T & YC^T \\ CY & 0 \end{pmatrix} \geq 0 \tag{15}$$

The existence and uniqueness of P and Q are guaranteed by assumptions (A.2). The most common way to compute these matrices is to use LMI techniques (see for instance [2]). A constructive procedure to compute these matrices is also presented in [11] using decomposition in the so-called Special Coordinate Basis (SCB). E_P and D_Q are derived from:

$$\begin{bmatrix} E_P^T \\ 0 \end{bmatrix} \begin{bmatrix} E_P & 0 \end{bmatrix} = \Phi(P). \tag{16}$$

and

$$\begin{bmatrix} D_Q \\ 0 \end{bmatrix} \begin{bmatrix} D_Q^T & 0 \end{bmatrix} = \Psi(Q). \tag{17}$$

The modified system Σ_{PQ} is then defined as follows:

$$\Sigma_{PQ} \ : \ \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + D_Q h(t) \\ z(t) = E_P x(t) \\ y(t) = Cx(t) \end{cases} \tag{18}$$

The formulation of [11] involves direct feedthrough matrices but one can see that original strictly proper systems induce modified systems which are also strictly proper.

The link between the H_2 optimal control and the disturbance rejection problem is stated in the following important theorem, established by [11]:

Theorem 7. Let the system Σ be given. We have the following equivalent assertions:

(i) the compensator Γ solves the H_2OCP for system Σ,

(ii) the compensator Γ solves the DRPS for system Σ_{PQ}.

Moreover, the optimum of the H_2 norm can be computed by:

$$\gamma_{\text{opt}} = \sqrt{\text{tr}(D^TPD) + \text{tr}(E_PQE_P^T)} \tag{19}$$

$$= \sqrt{\text{tr}(D_Q^TPD_Q) + \text{tr}(EQE^T)}. \tag{20}$$

[1]largest in the sense that any X (resp. Y) such that $\Phi(X) \geq 0$ (resp. $\Psi(Y) \geq 0$) satisfies $(P - X) \geq 0$ (resp. $(Q - Y) \geq 0$)

4. MAIN RESULTS

One advantage of the following formulation is to consider at the same time the solvability test for the H_2OCP as well as the characterization of the H_2 optimal fixed poles.

From the above mentioned results, it is simple to derive the following theorem:

Theorem 8. Let the system Σ be given, then the H_2OCP is solvable if and only if the two following conditions are satisfied:

(i) The DRP is solvable for system Σ_{PQ}

(ii) The DRP fixed poles for system Σ_{PQ} are stable, i. e. they lie in the open left half-plane.

Moreover, if the H_2OCP is solvable, the H_2 optimal fixed poles for system Σ are the DRP fixed poles for system Σ_{PQ} and all the remaining poles of the compensated system can be placed at any (symmetric) desired location by a suitable choice of the parameters K, L, M, N in (2).

Proof. Thanks to Theorem 7, the H_2OCP is solvable for system Σ if and only if the DRPS is solvable for system Σ_{PQ}. As the DRPS is equivalent to the DRP plus the requirement that the DRP fixed poles are all stable, the first part of the theorem is easily proved.

Now let us assume that the problem is solvable, i. e. the optimum γ_{opt} is attained and let us denote σ_{fix} the DRP fixed poles of Σ_{PQ}. Let us remark that E, E_P, D, D_Q are not involved in the expression (3), which means that the closed loop poles of $(\Sigma_{PQ} \times \Gamma)$ and $(\Sigma \times \Gamma)$ are the same:

$$\sigma\left(\Sigma_{PQ} \times \Gamma\right) = \sigma\left(\Sigma \times \Gamma\right) = \sigma(A_e). \tag{21}$$

Thanks to this statement and Theorem 7, any H_2 optimal compensator, say Γ_{opt}, is solution to the DRPS for Σ_{PQ} and $\sigma_{\text{fix}} \subset \sigma\left(\Sigma_{PQ} \times \Gamma_{\text{opt}}\right) = \sigma\left(\Sigma \times \Gamma_{\text{opt}}\right)$.

Reversing the argument, let us choose (by the procedure in [3] for instance) one particular compensator, say Γ_{DRP}, solving the Disturbance Rejection Problem with Stability for the system Σ_{PQ} with all the poles of $\sigma\left(\Sigma_{PQ} \times \Gamma_{DRP}\right)$ freely located except the DRP fixed poles σ_{fix}. We thus have found one H_2 optimal compensator for which the closed loop poles are all freely placed except σ_{fix}. We have consequently proved that σ_{fix} represents the H_2 optimal fixed poles. □

Let us note the following properties (for which the proof is quite direct):

— $\mathcal{R}^*_{(\mathcal{B},\mathcal{E})} = \mathcal{R}^*_{(\mathcal{B},\mathcal{E}_P)} = \mathcal{R}^*$ and $\mathcal{N}^*_{(\mathcal{C},\mathcal{D})} = \mathcal{N}^*_{(\mathcal{C},\mathcal{D}_Q)} = \mathcal{N}^*$.

— $(A, [BD])$ is controllable implies that $(A, [BD_Q])$ is controllable and dually $\left(\begin{bmatrix} C \\ E \end{bmatrix}, A\right)$ is observable implies that $\left(\begin{bmatrix} C \\ E_P \end{bmatrix}, A\right)$ is observable.

These remarks enable us to precise some characterizations of the set of H_2 optimal fixed poles, denoting for that purpose $\overline{\mathcal{R}}_c^* := \mathcal{R}^*_{(\mathcal{B}+\mathcal{D}_Q,\mathcal{E}_P)}$ and $\overline{\mathcal{N}}_c^* := \mathcal{N}^*_{(\mathcal{C}\cap\mathcal{E}_P,\mathcal{D}_Q)}$ where $\mathcal{E}_P = \operatorname{Ker} E_P$, $\mathcal{D}_Q = \operatorname{Im} D_Q$ and E_{PQ} is defined by $\ker E_{PQ} = \overline{\mathcal{R}}_c^*$:

$$\begin{aligned} H_2 \text{ optimal fixed poles} &= \sigma\left(A+BF\Big|\frac{\overline{\mathcal{R}}_c^*}{\mathcal{R}^*}\right) \\ &\quad \dot{\cup}\, \sigma\left(A+GC\Big|\frac{\mathcal{N}^*}{\overline{\mathcal{N}}_c^*\cap\overline{\mathcal{R}}_c^*}\right) \end{aligned} \tag{22}$$

where $F \in \mathcal{F}(\overline{\mathcal{R}}_c^*)$, $G \in \mathcal{G}(\overline{\mathcal{N}}_c^* \cap \overline{\mathcal{R}}_c^*)$. The following characterization is in terms of invariant zeros:

$$\begin{aligned} H_2 \text{ optimal fixed poles} &= \{\mathcal{Z}(A,B,E_P) - \mathcal{Z}(A,[\,B \quad D_Q\,],E_P)\} \\ &\quad \dot{\cup}\left\{\mathcal{Z}(A,D_Q,C) - \mathcal{Z}\left(A,D_Q,\begin{bmatrix} C \\ E_{PQ}\end{bmatrix}\right)\right\}. \end{aligned} \tag{23}$$

Example. Let us sum up the different steps of the H_2 optimal compensators design procedure illustrated on the system described by:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & -2 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}.$$

1. The computation of matrices $P = 0$ and $Q = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ respecting (14) leads to the modified system Σ_{PQ} with $E_P = E$ and $D_Q = [\,0 \quad 0 \quad 1 \quad -2\,]^T$.
2. The solvability conditions of the disturbance rejection problem on Σ_{PQ} can be checked thanks to

$$\mathcal{S}^*_{(\mathcal{C},\mathcal{D}_Q)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix} \subset \mathcal{V}^*_{(\mathcal{B},\mathcal{E}_P)} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \\ -2 & 1 \end{pmatrix},$$

as well as the stability condition of the DRP fixed poles $= \{-1,-1,-3\}$, computed for instance by (23), i. e.:

$$\mathcal{Z}(A,B,E_P) = \{-2,-3\}, \quad \mathcal{Z}(A,[BD_Q],E_P) = \{-2\},$$
$$\mathcal{Z}(A,D_Q,C) = \{-1,-1\} \quad \text{and} \quad \mathcal{Z}\left(A,D_Q,\begin{bmatrix} C \\ E_{PQ}\end{bmatrix}\right) = \emptyset.$$

Then there does exist H_2 optimal compensators.

3. For a desired symmetric set $(\sigma_{\text{free1}} \dot{\cup} \sigma_{\text{free2}})$, let us choose $F \in \mathcal{F}(\overline{\mathcal{R}}_c^*)$ and $G \in \mathcal{G}(\overline{\mathcal{N}}_c^* \cap \overline{\mathcal{R}}_c^*)$ with

$$\sigma(A+BF) = \sigma_{\text{free1}} \dot{\cup} \, \sigma\left(A+BF \mid \frac{\overline{\mathcal{R}}_c^*}{\mathcal{R}^*}\right)$$

and

$$\sigma(A+GC) = \sigma_{\text{free2}} \dot{\cup} \, \sigma\left(A+GC \mid \frac{\mathcal{N}^*}{\overline{\mathcal{N}}_c^* \cap \overline{\mathcal{R}}_c^*}\right)$$

The choice $\sigma_{\text{free1}} = \{-3,-1,-1\}$ and $\sigma_{\text{free2}} = \{-3,-2\}$ induces

$$F = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ -0.3 & -0.3 \\ -1.6 & -1.1 \\ -0.5 & 0.5 \end{bmatrix},$$

4. Applying $(\mathcal{S},\mathcal{V})$-based compensator formulas, we obtain the matrices

$$K = \begin{bmatrix} 0 & 0 \\ \frac{-5}{3} & \frac{1}{3} \end{bmatrix}, \quad L = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & \frac{-1}{3} & \frac{-2}{3} & \frac{-1}{3} \end{bmatrix},$$

$$M = \begin{bmatrix} 0 & 0 \\ 0.3 & 0.3 \\ 1.6 & 1.1 \\ \frac{-7}{6} & \frac{-1}{6} \end{bmatrix}, \quad N = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -3.3 & -0.6 & -0.3 \\ 0 & -1.1 & -3.2 & -0.6 \\ 0 & \frac{1}{6} & \frac{1}{3} & \frac{-11}{6} \end{bmatrix}.$$

The closed loop transfer function matrix (inducing $\|T_\Gamma(s)\|_2 = \gamma_{\text{opt}} = \sqrt{2}$) is then

$$T_\Gamma(s) = \begin{bmatrix} \frac{2}{s+1} & 0 \\ 0 & 0 \end{bmatrix}.$$

5. CONCLUSION

We have presented some characterizations of the Fixed Poles of the H_2 Optimal Control Problem by measurement feedback as well as a constructive procedure to obtain H_2 optimal compensators while simultaneously assigning all the remaining poles to arbitrary symmetric location.

These two results are obtained by taking advantage of two previous contributions: a system transformation introduced by [10] which transforms the H_2 Optimal Control Problem into a Disturbance Rejection Problem on a modified system and the characterizations of the Fixed Poles of the Disturbance Rejection Problem by Measurement Feedback given in [3].

We have considered here strictly proper systems, then a further step would be to generalize the results to simply proper systems. The generalization does not appear trivial since the situation when direct feedthrough matrices are present (from input or disturbance to controlled output or measurement) is much more intricate, at least for the geometric approach. One possible way is to use the classical trick as introduced in [6], but it must be checked that it fully works in the present H_2 optimal control case.

Another possible extension would be to express the solvability conditions and the fixed poles characterizations directly from the data of the original system before computing the modified system.

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