KYBERNETIKA — VOLUME 38 (2002), NUMBER 5, PAGES 609–615

POLES AND ZEROS OF NONLINEAR CONTROL SYSTEMS

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During the last ten years, the concepts of "poles" and "zeros" for linear control systems have been revisited by using modern commutative algebra and module theory as a powerful substitute for the theory of polynomial matrices. Very recently, these concepts have been extended to multidimensional linear control systems with constant coefficients. Our purpose is to use the methods of "algebraic analysis" in order to extend these concepts to the variable coefficients case and, as a byproduct, to the nonlinear situation. We also provide nontrivial explicit examples.

1. INTRODUCTION

It is a matter of fact that, during the last ten years or so, the concepts of "poles" and "zeros" for linear control systems with constant coefficients have been revisited using modern commutative algebra and module theory. Among the various attempts, we quote the survey [11] and the recent intrinsic approach [2]. The main idea is to relate the definition of poles and zeros for linear control systems to the theory of modules over a commutative ring that can be adopted for linear *ordinary differential* (OD) *control systems*, that is when the input/output relations are defined by systems of ordinary differential equations. The extension of these definitions to linear multidimensional control systems has been recently obtained by means of the "*algebraic analysis*" of linear *partial differential* (PD) *control systems*, that is when the input/output relations are defined by systems of partial differential equations [9, 12]. Hence, only the extension to nonlinear control systems, in a way coherent with the direct study of [7], was remaining. In view of the underlying amount of commutative and homological algebra needed [9, 10], the purpose of this paper is only to provide an elementary sketch of the main ideas involved and we refer to [8] for more details.

2. PRELIMINARIES

The basic procedure is to associate algebraic sets with the differential modules defined by systems of partial differential equations and certain submodules defined by selecting the inputs and the outputs among the control variables. In particular, if k is a (differential) field of constants, say $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ in the applications, we may consider the (commutative) ring $D = k[d_1, \dots, d_n] = k[d]$ of differential operators in n formal derivatives and we shall set $d_i d_j = d_j d_i = d_{ij}$. The control system thus allows to introduce by residue the differential module (or D-module) M obtained by quotienting $Dy^1 + \dots + Dy^m = Dy$ by the subdifferential module generated by the control OD or PD equations when $y^1, \dots, y^m$ are the control variables. In this framework, making a partition of the control variables into inputs and outputs allows, by restriction, to obtain differential submodules respectively denoted by M_{in} and M_{out}.

Identifying D with the polynomial ring $A = k[\chi_1, \dots, \chi_n] = k[\chi]$ in n indeterminates, while taking into account the fact that M is a finitely generated noetherian module over A, we may refer to the standard localization technique in commutative algebra [1, 4, 6] that supersedes the transfer matrix approach [10] in order to introduce the *support* of M, namely $\text{supp}(M) = \{\mathfrak{p} \in \text{spec}(A) | M_{\mathfrak{p}} \neq 0\} = \{\mathfrak{p} \in \text{spec}(A) | \mathfrak{p} \supseteq \text{ann}_A(M)\} = Z(\text{ann}_A(M))$. We can therefore associate algebraic sets with M but also with modules such as M/M_{in} or M/M_{out}. We notice that, with the above definition, we have $M_{\text{in}} + M_{\text{out}} = M$ but it may also happen, more generally, that $M_{\text{in}} + M_{\text{out}} \subset M$ with a strict inclusion if, for example, one has to eliminate *latent variables* among the control variables. At that time, a basic assumption, usually done in classical OD control theory, is that M/M_{in} is a torsion module. We recall that the *torsion submodule* of M is $t(M) = \{m \in M | \exists 0 \neq a \in A, am = 0\}$ and that M is called a *torsion module* if $M = t(M)$ or is said to be *torsion-free* if $t(M) = 0$. The true reason for this assumption is that, in this case, $\text{ann}_A(M/M_{\text{in}}) \neq 0$ provides a well defined support, strictly distinct from $\text{spec}(A)$, that can be identified with the algebraic set defined over k by the radical ideal $I(M) = \text{rad}(\text{ann}_A(M))$ so that $\text{supp}(M)$ can be identified with $Z(I(M))$ where Z is used for "zero" in algebra ... and this is just the idea for introducing "zeros" in control theory when using M/M_{out} or, by symmetry, the so-called "poles" when using M/M_{in}.

Finally, the main property of the support is that, for any short exact sequence $0 \to M' \to M \to M'' \to 0$ of modules over A, one can prove that $\text{ann}_A(M) \subseteq \text{ann}_A(M') \cap \text{ann}_A(M'')$ but $\text{rad}(\text{ann}_A(M))M = \text{rad}(\text{ann}_A(M')) \cap \text{rad}(\text{ann}_A(M''))$ in such a way that $\text{supp}(M) = \text{supp}(M') \cup \text{supp}(M'')$. This specific nontrivial property of the support is crucially used for considering various chains of inclusions of submodules of M such as $M'_{\text{in}} = M_{\text{in}} + t(M), \dots$ and respective quotient modules, provided that they are torsion modules [9].

It is only at this stage that one can feel about the main challenging difficulty met in extending these ideas to the variable coefficients case, that is when $D = K[d]$ if now K is a (OD or PD) differential field with n commuting derivations $\partial_1, \dots, \partial_n$ such that, in the operator sense, one has $d_i a = a d_i + \partial_i a$ and D becomes a noncommutative ring. In this case, the above technique for introducing algebraic sets fails because it is no longer possible to introduce any polynomial ring or algebraic set as before.

The standard way to escape from this trouble is to pass from *filtred modules* to *graded modules* and we explain this procedure not at all well known by the control community where the people are mostly familiar with the constant coefficients case.

Meanwhile, the reader will understand on the examples presented later on that such an approach must bring quite a new field of research, even in OD control theory. We may use the standard filtration $D = \{D_r\}_{r\geq 0}$ by the order of operators with $D_r = \{P \in D | \mathrm{ord}(P) \leq r\}$ and $D_0 = k$ or K. We check at once the well known properties:

— $0 = D_{-1} \subseteq D_0 \subseteq D_1 \subseteq \ldots \subseteq D$.

— $\cup_{r\geq 0} D_r = D$.

— $D_r D_s \subseteq D_{r+s}$.

The induced residual filtration of M will be $M = \{M_q\}_{q\geq 0}$ and we have:

— $D_r M_q \subseteq M_{q+r}, \forall\, q,\, r \geq 0$, with equality for q large enough (Maisonobe and Sabbah [5]). The associated graded algebra $\mathrm{gr}(D) = \oplus_{r\geq 0} D_r/D_{r-1}$ is a commutative ring. In this case, if $\mu = (\mu_1, \ldots, \mu_n)$ is a multi-index with length $|\mu| = \mu_1 + \ldots + \mu_n$, we may write any $P \in D$ with $\mathrm{ord}(P) = r$ as a finite sum $P = \sum_{0\leq|\mu|\leq r} a^\mu d_\mu$ where $a^\mu \in K$ and $d_\mu = (d_1)^{\mu_1} \ldots (d_n)^{\mu_n}$. We then define the *symbol* of P with respect to the polynomial variables (covector in differential geometry) χ to be the polynomial $\sigma_\chi(P) = \sum_{|\mu|=r} a^\mu \chi_\mu$ and $\sigma_\chi(PQ) = \sigma_\chi(P)\sigma_\chi(Q),\ \forall\, P,\, Q \in D$.

Proceeding similarly for M, we may introduce the *associated graded module* $G = \mathrm{gr}(M) = \oplus_{q\geq 0} M_q/M_{q-1}$. Accordingly, G becomes a module over $\mathrm{gr}(D)$ and we may apply the commutative machinery already introduced in order to set:

Definition. $\mathrm{char}(M) = \mathrm{supp}(G)$.

Now we can associate with M two integers that can be computed effectively by using Gröbner or similar bases [3]:

— For q large enough, there exists a unique polynomial H_M, called *Hilbert polynomial* of M, such that $\dim(M_{q+r}) = H_M(q+r) = (m/d!)r^d + \ldots$. The degree $d = d(M)$ is called the *dimension* of M.

— $Z(I(G))$ is an algebraic set which is the union of irreducible algebraic sets or varieties. We may consider the maximum of the (Krull) dimensions of these varieties.

A key result is the following theorem [6]:

Theorem. (Hilbert–Serre) These two numbers coincide.

3. ANALYSIS OF LINEAR CONTROL SYSTEMS

We now come to the unexpected application of these apparently abstract results to control theory.

First of all, we have of course the way to define poles and zeros for the variable case. Meanwhile, it will bring a striking result that we shall discuss and illustrate. We notice that this result is implicitly used in the definition of causality which only involves the top degree terms in the numerator and denominator of a transfer matrix for example. Indeed, in the case $k = \mathrm{cst}(K)$ is the subfield of constants of K, if our control system is defined over k, for any differential module M we can compute $d = d(M)$ by means of the Hilbert polynomial but also by using $\mathrm{ann}_A(M)$ or $\mathrm{ann}_A(G)$ and these three numbers coincide. Accordingly, as far at least as dimensions are only concerned, one can study the graded or the filtred framework equivalently, *even though they can look like quite different.*

Example. With $m = 1$, $n = 3$, $q = 2$ and $k = \mathbb{Q}$, let us consider the following (formal) linear system:

$$\begin{cases} d_{33}y - d_{13}y - d_3y &= 0 \\ d_{23}y - d_{12}y - d_2y &= 0 \\ d_{22}y - d_{12}y &= 0. \end{cases}$$

It is easy to check that $\dim(G_q) = 3$, $\forall q \geq 1 \Rightarrow d(M) = 1$. The algebraic sets defined by the two ideals:

$$\begin{aligned} \mathrm{ann}(M) = \ & ((\chi_3)^2 - \chi_1\chi_3 - \chi_3, \\ & \chi_2\chi_3 - \chi_1\chi_2 - \chi_2, (\chi_2)^2 - \chi_1\chi_2) \\ \mathrm{ann}(G) = \ & ((\chi_3)^2 - \chi_1\chi_3, \chi_2\chi_3 - \chi_1\chi_2, \\ & (\chi_2)^2 - \chi_1\chi_2) \end{aligned}$$

are both the union of three varieties of dimension 1, even if they are quite distinct ideals indeed.

Of course, an additional difficulty (leading in fact to the definition of Gröbner bases) is that the graded approach only works if the system is *formally integrable*, that is if the given equations at order q generate all the ones existing at order $q + r$ through no more than r differentiations (prolongations), $\forall r \geq 0$.

Example. With $n = 4$, $m = 1$, $q = 1$ and $K = \mathbb{Q}(x^1, x^2, x^3, x^4)$, let us consider the nonformally integrable system:

$$\begin{aligned} & d_4y - x^3d_2y - y = 0, \quad d_3y - x^4d_1y = 0 \\ & \Longrightarrow d_2y - d_1y = 0. \end{aligned}$$

We notice at once that the ideal $(\chi_4 - x^3\chi_2, \chi_3 - x^4\chi_1)$ does not contain $\chi_2 - \chi_1$. Accordingly the dimension of the corresponding differential module is 1 and not 2 as one could imagine from pure algebra.

It must be emphasized that the lack of formal integrability is responsible for the fact that the exactness of a short sequence $0 \to M' \to M \to M'' \to 0$ of filtred modules does not necessarily imply the exactness of the associated sequence $0 \to G' \to G \to G'' \to 0$ of graded modules, but this is a very delicate question [5,8].

The preceding results can also be used in order to study the structural properties of control systems, that is properties, such as controllability, that do not depend on the choice of inputs and outputs among the control variables. The idea, still not known in multidimensional control theory today, up to our knowledge, relies on the following trick that only gets a deep meaning if one uses the extension functor as a key homological tool for studying dimensions [9].

If $P \in D$ with $P = \sum a^\mu d_\mu$, we define $\mathrm{ad}(P) = \sum (-1)^{|\mu|} d_\mu a^\mu$ after pushing all the coefficients to the left as in the standard presentation of an operator, using the commutation relations of the corresponding noncommutative ring if needed. The *adjoint operation* thus defined can be extended to operators and systems in a linear way or through integration by part as in mechanics (elasticity) or physics (electromagnetism). Accordingly, if the control system is formally defined by the linear system $\mathcal{D}y = 0 \Longleftrightarrow a_k^{\tau\mu} d_\mu y^k = 0$ where k is an index for the variables and τ is an index for the equations, we may multiply on the left by test functions λ_τ and integrate by part to get $\tilde{\mathcal{D}}\lambda = 0$ as adjoint system. If now M (respectively N) is the differential module defined by $\mathcal{D}$ (respectively $\tilde{\mathcal{D}}$), the key idea is to notice that N becomes a torsion module *if and only if* $\mathcal{D}$ is surjective, that is has no compatibility conditions on z for solving formally $\mathcal{D}y = z$. *This is the main reason for which control systems are most of the time assumed to be defined by surjective operators.* In this case, one has the following delicate theorem generalizing the well known Kalman and Hautus tests for OD systems while showing how the classification of modules depends on the dimension [9,12].

Theorem. When $\mathcal{D}$ is formally surjective as above, then M is torsion-free if $d(N) \leq n-2$ (*minor primeness*), M is reflexive if $d(N) \leq n-3, \ldots$, M is projective if $d(N) = -1$, that is $Z(I(M)) = \emptyset$ (*zero primeness*).

Corollary. An OD control system defined by a surjective operator is controllable if and only if $N = 0$.

Example. If we consider the preceding example as providing $\tilde{\mathcal{D}}$, then the corresponding $\mathcal{D}$ surely defines a torsion-free module, without any need to refer to any direct computation. Of course, all these facts cannot be imagined when $n = 1$.

4. ANALYSIS OF NONLINEAR CONTROL SYSTEMS

We are finally ready to extend these results to nonlinear control systems. For this, setting $y_q = \{y^k_\mu = d_\mu y^k | 0 \leq |\mu| \leq q, 1 \leq k \leq m\}$ and $k\{y\} = \lim_{q\to\infty} k[y_q]$, we may suppose that the system is defined over K by a prime ideal $\mathfrak{p} \subset K\{y\}$ (care to the notation) and we introduce the quotient differential field $L = Q(K\{y\}/\mathfrak{p})$. Referring to [6], we may then consider the module $M = \Omega_{L/K}$ of *Kähler differentials* as a differential module over $L[d]$ and this is just as if we were dealing with the variable coefficients case. In the differential geometric framework, a similar procedure can be followed by using vertical bundles for linearizing the system.

In actual practice, if the nonlinear system is defined by equations of the form $\Phi^\tau(y_q) = 0$, we may introduce the vertical variations $Y_q = \delta y_q$ related by equations of the form $\frac{\partial \Phi^\tau}{\partial y^k_\mu}(y_q) d_\mu Y^k = 0$ where the y_q are solutions of the above system of (non-differential) equations. The following nonlinear example exhibits features similar to the ones of the preceding examples. It is rather difficult to find such examples.

Example. With $n = 2$, $m = 1$, $q = 2$, let us consider the nonlinear formally integrable system defined by two differential polynomials, using jet notation:

$$\left\{ \begin{array}{lll} \Phi^1 \equiv & y_{12} - \frac{1}{2}(y_{11})^2 + y_1 & = 0 \\ \Phi^2 \equiv & y_{22} - \frac{1}{3}(y_{11})^3 + 3y_2 + 2y & = 0. \end{array} \right.$$

One has:

$$d_2\Phi^1 - d_1\Phi^2 + y_{11} d_1 \Phi^1 + 2\Phi^1 \equiv 0.$$

The linearized system is:

$$\left\{ \begin{array}{ll} d_{12}Y - y_{11} d_{11} Y + d_1 Y & = 0 \\ d_{22}Y - (y_{11})^2 d_{11} Y + 3 d_2 Y + 2Y & = 0 \end{array} \right.$$

and one can check that $\dim(G_q) = 1$, $\forall q \geq 2 \Rightarrow d(M) = 1$. Accordingly, the characteristic set is defined by:

$$\chi_1\chi_2 - y_{11}(\chi_1)^2 = 0, \qquad (\chi_2)^2 - (y_{11})^2(\chi_1)^2 = 0.$$

Using a primary decomposition, it is the union of two varieties, namely:

$$(\chi_2 - y_{11}\chi_1 = 0) \cup (\chi_1 = 0, \chi_2 = 0)$$

and the second variety (a point) is imbedded in the first variety having dimension 1. One can also notice the identity:

$$(\chi_2 - y_{11}\chi_1)^2 = (\chi_2)^2 - (y_{11})^2(\chi_1)^2 - 2y_{11}(\chi_1\chi_2 - y_{11}(\chi_1)^2)$$

and the radical of the preceding ideal is indeed a prime ideal only generated by $\chi_2 - y_{11}\chi_1$.

5. CONCLUSION

We hope to have convinced the reader that poles and zeros must only be considered as particular algebraic sets related to the input/output structure of the control systems. As a byproduct, other algebraic sets can also be of use, provided one can refer to intrinsically defined numbers such as dimensions. We have shown how to restrict these computations by using the symbol terms, contrary to what is done in polynomial matrix theory. Also we have been able to extend these results to nonlinear systems. More results on both linear and nonlinear systems can be found in the book [8]. The application of these results to stability problems is an important open problem for the future.

(Received February 2, 2002.)

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