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NONREGULAR DECOUPLING WITH STABILITY OF TWO-OUTPUT SYSTEMS[1]

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In this paper we present a solution to the decoupling problem with stability of linear multivariable systems with 2 outputs, using nonregular static state feedback. The problem is tackled using an algebraic-polynomial approach, and the main idea is to test the conditions for a decoupling compensator with stability to be feedback realizable. It is shown that the problem has a solution if and only if Morse's list I_2 is greater than or equal to the infinite and unstable structure of the proper and stable part of the stable interactor of the system. A constructive procedure to find a state feedback, which achieves decoupling with stability, is also presented.

1. INTRODUCTION

The row-by-row decoupling of linear multivariable systems by static state feedback has been extensively studied since the 1960's. This problem has been solved for the case of systems with the same number of inputs and outputs, or square systems (see for instance [5] and [1]), which is usually referred to as the regular decoupling problem. The regular decoupling problem with stability by static state feedback has been solved in [8] and [10].

Regarding the decoupling of systems with more inputs than outputs, or nonregular decoupling problem, even though there exist solutions for particular cases, namely, systems with 2 outputs [2, 7], and systems whose essential orders are all equal [3], the problem remains unsolved in its full generality.

The aim of this paper is to study the nonregular decoupling problem with stability for linear systems with two outputs. We present a solution to this problem in terms of structural information of the system. It is shown that a linear system with 2 outputs and 3 or more inputs is decouplable with stability if and only if Morse's list I_2 [9] is greater than or equal to the infinite and unstable structure of the proper and stable part of the stable interactor of the system. The problem is tackled using

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an algebraic-polynomial approach, and the main idea is to test the conditions for a decoupling compensator with stability to be feedback realizable.

The problem statement is presented in Section 2. In Section 3, we introduce the main ingredients in the study of this problem, namely: the stable interactor, the extended system, the extended stable interactor, and feedback realization of precompensators. The main result is presented in Section 4, which relies also on technical results presented in Appendix 1 and Appendix 2. The problem statement and preliminaries will be presented in a general setting for linear systems with p outputs and m inputs, and the assumption of $p = 2$ will be made evident until Section 4. An example is presented in Section 5, which illustrates the procedure to obtain a nonregular state feedback, which decouples with stability a 2-output system. Finally, we end up with some conclusions.

2. PROBLEM STATEMENT

We consider in this work linear multivariable controllable systems described by

$$(A, B, C) \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

where $x \in I\!R^n$, $u \in I\!R^m$ and $y \in I\!R^p$ are, respectively, the state, input and output vectors of the system. Further, in Section 4 we will restrict ourselves to systems (A, B, C) with 2 outputs and 3 or more inputs, i. e., $p = 2$ and $m \geq 3$.

The system (A, B, C) is said to be row by row decouplable with stability by static state feedback if there exists a state feedback

$$(F, G) : u(t) = Fx(t) + Gv(t)$$

where $F \in I\!R^{m\times n}$ and $G \in I\!R^{m\times p}$ are constant matrices, rank $G = p$ (nonregular static state feedback), and $v(t)$ is a new input vector, such that the input $v_i(t)$ controls the output $y_i(t)$, $i = 1, \dots, p$, without affecting the other outputs, and the closed-loop system $(A + BF, BG, C)$ is internally stable, i. e., the eigenvalues of the matrix $A + BF$ are located in the open left half complex plane.

From the input-output point of view, the previous formulation is equivalent to the existence of a state feedback (F, G) such that the transfer function $T_{F,G}(s)$ of the closed-loop system $(A + BF, BG, C)$ is of the form

$$T_{FG}(s) = C(sI - A - BF)^{-1}BG = \operatorname{diag}\{w_1(s), \dots, w_p(s)\} =: W(s) \qquad (1)$$

and the closed-loop system $(A + BF, BG, C)$ is internally stable, which implies also that $w_i(s) \neq 0$, $i = 1, \dots, p$, are strictly proper and stable rational functions.

We can suppose without loss of generality that the system (A, B, C) is stable; if not, there always exists a preliminary state feedback, which stabilizes the system, since we are considering it to be controllable. Thus, the transfer function matrix of the system, $T(s) = C(sI - A)^{-1}B$, is a strictly proper and stable rational matrix.

3. PRELIMINARIES

3.1. The stable interactor

Let $u(t) = Fx(t) + Gv(t)$ be a regular static state feedback applied on the stable system (A, B, C), such that the closed-loop system $(A + BF, BG, C)$ is internally stable. The closed-loop transfer function is given by

$$T_{FG}(s) = C(sI - A - BF)^{-1}BG.$$

After some manipulations we obtain

$$T_{FG}(s) = C(sI - A)^{-1}B[I - F(sI - A)^{-1}B]^{-1}G = T(s)R(s)$$

where $T(s) = C(sI - A)^{-1}B$ is the transfer function of the system (A, B, C), and

$$R(s) := [I - F(sI - A)^{-1}B]^{-1}G.$$

Since the closed-loop system is supposed to be stable, then $R(s)$ must be clearly a proper and stable rational matrix. Further, from

$$R^{-1}(s) = G^{-1}[I - F(sI - A)^{-1}B] = \frac{1}{\det(sI - A)}G^{-1}[I - F\ \mathrm{Adj}(sI - A)B]$$

it can be seen that $R^{-1}(s)$ is also proper and stable, since (A, B, C) is stable. Then, we have the following result.

Remark 1. Let $T(s)$ be the transfer function of the stable system (A, B, C). Then, the effect of a regular static state feedback $u(t) = Fx(t) + Gv(t)$ which preserves internal stability can be represented in transfer function terms as a biproper and bistable matrix postmultiplying $T(s)$.

This can be considered as the matrix interpretation of the fact that we are neither allowed to introduce unstable poles nor to cancel out unstable zeros in order to keep the internal stability of the closed-loop system.

At this stage, it is important to consider the information of the system (A, B, C) which remains invariant under the action of biproper and bistable compensation, and consequently, invariant under the action of a regular state feedback which preserves internal stability. This information is contained in the stable interactor (or π-interactor) of the system, defined below ([4, 10]).

In the study of problems involving stability from an algebraic point of view, it is important to consider the properties of the ring of proper and stable rational functions $\mathbb{R}_{ps}(s)$. This set is known to be a Euclidean ring [11], the degree of a proper and stable rational function $f(s) \in \mathbb{R}_{ps}(s)$, hereafter denoted $\deg_{ps} f(s)$, taken as the number of infinite plus unstable zeros of $f(s)$.

Lemma 1. Let $T(s)$ be the transfer function of (A, B, C). Then, there exist a biproper and bistable matrix $B(s) \in \mathbb{R}_{ps}^{m\times m}(s)$ and a nonsingular lower triangular matrix $\Phi_s^{-1}(s) \in \mathbb{R}_{ps}^{p\times p}(s)$, unique up to units of the ring $\mathbb{R}_{ps}(s)$, such that

$$T(s)B(s) = \begin{bmatrix} \Phi_s^{-1}(s) & 0 \end{bmatrix}, \tag{2}$$

where

$$\Phi_s^{-1}(s) = \begin{bmatrix} \varphi_{11}(s) & & (0) \\ \vdots & \ddots & \\ \varphi_{p1}(s) & \dots & \varphi_{pp}(s) \end{bmatrix}, \tag{3}$$

and the rational functions $\varphi_{ij}(s) \in \mathbb{R}_{ps}(s)$ satisfy, for $i > j$,

$$\varphi_{ij}(s) = 0, \quad \text{or} \quad \deg_{ps} \varphi_{ij}(s) < \deg_{ps} \varphi_{ii}(s),$$

and they are of the form

$$\varphi_{ii}(s) = \frac{\alpha_{ii}(s)}{\pi^{n_{ii}}}$$

$$\varphi_{ij}(s) = \frac{\alpha_{ij}(s)}{\pi^{n_{ij}}},$$

where $\alpha_{ii}(s) \in \mathbb{R}[s]$ is a polynomial with only unstable roots (antistable polynomial), $\pi = s + \beta$ is a stable term and $\alpha_{ij}(s) \in \mathbb{R}[s]$ is a polynomial.

Notice that the proper and stable rational matrix $\Phi_s^{-1}(s)$ is actually the column Hermite form of $T(s)$ over the ring of proper and stable rational functions $\mathbb{R}_{ps}(s)$.

The rational matrix $\Phi_s(s)$, which is the inverse of $\Phi_s^{-1}(s)$, is known as the stable interactor of the system, and it contains the information of the system that is invariant under a regular state feedback which preserves internal stability. In particular, it contains the infinite zeros and unstable zeros of the system, information that plays a key role in the decoupling problem with stability. While the classical system interactor $\Phi(s)$ is a polynomial matrix with certain properties [12], the stable interactor $\Phi_s(s)$ is in general a nonsingular lower triangular rational matrix having only unstable poles. That $\Phi_s(s)$ has only unstable poles can be seen from the fact that the numerator of the determinant of $\Phi_s^{-1}(s)$ is the product of the antistable polynomials $\alpha_{ii}(s)$, $i = 1, \dots, p$. Observe also that if the system (A, B, C) has no unstable zeros, then $\Phi_s(s)$ is a polynomial matrix.

Let $T(s) \in \mathbb{R}_{ps}^{p\times m}(s)$ be the transfer function of (A, B, C) and let $\Phi_s(s)$ be its stable interactor. Factorize $\Phi_s(s)$ as

$$\Phi_s(s) = \Gamma_s(s) \operatorname{diag}\left\{\frac{1}{g_1(s)}, \cdots, \frac{1}{g_p(s)}\right\} \tag{4}$$

where $g_i(s)$, $i = 1, \dots, p$, are proper and stable rational functions of the least possible degree, such that the elements of the ith column of $\Gamma_s(s)$ have no unstable or infinite poles. In other words, $g_i(s) \in \mathbb{R}_{ps}(s)$ are the least degree proper and stable rational functions such that $\Gamma_s(s)$ is a proper and stable rational matrix.

The integers

$$n_{ie,s} := \deg_{ps}\, g_i(s), \qquad i = 1, \ldots, p,$$

are called the s-essential orders of the system (A, B, C) and the matrix $\Gamma_s(s)$ is called the proper and stable part of the stable interactor $\Phi_s(s)$ (see [4]).

The non null degrees of the proper and stable rational functions in the Smith form of $\Gamma_s(s)$ over $I\!R_{ps}(s)$, denoted $\{\delta_i\}$, will be called the infinite and unstable structure of $\Gamma_s(s)$.

3.2. The extended system and the extended stable interactor

Besides the information about the infinite and unstable zeros of the system, in the nonregular decoupling problem with stability is also important the information about Morse's list I_2 [9]. To make this information appear, it is necessary to define a so-called extended system.

Consider

$$T(s) = C(sI - A)^{-1}B = N(s)D^{-1}(s),$$

where $N(s)$, $D(s)$ is a right coprime matrix fraction description of the system (A, B, C). Let $U(s)$ be a unimodular matrix such that

$$N(s)U(s) = \begin{bmatrix} Q(s) & 0 \end{bmatrix}$$

where $Q(s) \in I\!R^{p\times p}[s]$ is a nonsingular polynomial matrix (which can be considered lower triangular without loss of generality), and define

$$K(s) := \begin{bmatrix} Q(s) & 0 \\ 0 & I_{m-p} \end{bmatrix} U^{-1}(s).$$

The matrix $U(s)$ can be chosen such that $T_e(s) = K(s)D^{-1}(s)$ is strictly proper. Since we are supposing the system to be stable, it follows that $T_e(s)$ is a strictly proper and stable rational matrix. The matrix $T_e(s)$ is called an extension of $T(s)$, and a realization of $T_e(s)$ with the same order of the system (A, B, C) is called an extended system [6].

Lemma 2. (Ruiz et al [10]) Let $T(s)$ be the transfer function of the system (A, B, C), and let $T_e(s)$ be the transfer function of its associated extended system. Then there exist a biproper and bistable matrix $B_1(s)$ and a nonsingular lower triangular matrix $\Phi_{es}^{-1}(s)$, such that

$$T_e(s)B_1(s) = \Phi_{es}^{-1}(s), \tag{5}$$

and the stable interactor $\Phi_{es}(s)$ of $T_e(s)$ has the form

$$\Phi_{es}(s) = \begin{bmatrix} \Phi_{1,s}(s) & 0 \\ \Phi_{2,s}(s) & \Phi_{3,s}(s) \end{bmatrix} \tag{6}$$

where $\Phi_{1,s}(s)$ is the stable interactor of $T(s)$, and the matrix $\Phi_{3,s}(s)$ is given by

$$\Phi_{3,s} = \text{diag}\ \{\pi^{\sigma_1}, \ldots, \pi^{\sigma_{m-p}}\} \tag{7}$$

where $\{\sigma_i\}$ is Morse's list I_2 of the system.

3.3. Feedback realization of precompensators

A given proper compensator $C(s)$ is said to be feedback realizable on the system (A, B, C) if there exists a state feedback (F, G) such that

$$C(s) = [I - F(sI - A)^{-1}B]^{-1}G.$$

The following result states the conditions for a full column rank proper compensator to be realizable by a nonregular static state feedback, which preserves internal stability.

Lemma 3. [10] Let the matrices $N_1(s)$ and $D(s)$ be a right coprime matrix fraction description of the system (A, B, I_n), i.e., $T(s) = CN_1(s)D^{-1}(s)$, and let $C(s) \in I\!\!R_{ps}^{m\times m}(s)$ be a proper and stable compensator. Then $C(s)$ is realizable on (A, B, C) by a static state feedback which preserves internal stability if and only if there exists a biproper and bistable matrix $V(s) \in I\!\!R_{ps}^{m\times m}$ such that

— $V(s)C(s) = \begin{bmatrix} I_p \\ 0 \end{bmatrix}$, and

— $V(s)D(s)$ is a polynomial matrix.

It is clear that if we propose a compensator such that the compensated system is decoupled with stability, then solving the decoupling problem with stability by state feedback amounts to find the conditions for this compensator to be feedback realizable. This idea will be used in the proof of Theorem 1. First, we have the following result, which can be deduced from the previous Lemma 3.

Lemma 4. The system (A, B, C) is decouplable with stability by a nonregular static state feedback, such that the transfer function of the closed-loop system is given by

$$T_{FG}(s) = C(sI - A - BF)^{-1}BG = \operatorname{diag}\{g_1(s), \ldots, g_p(s)\}$$

where $\{g_i(s)\}$ are the proper and stable rational functions in (4), if and only if there exists a biproper and bistable matrix $V(s) \in I\!\!R_{ps}^{m\times m}(s)$ such that

— $V(s)\begin{bmatrix} \Gamma_s(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} I_p \\ 0 \end{bmatrix}$, and (8)

— $V(s)\Phi_{es}(s)K(s)$ is a polynomial matrix, (9)

where $\Gamma_s(s)$ is the proper and stable part of the system interactor $\Phi_{1,s}(s)$, and $X(s)$ is a proper and stable matrix such that $\begin{bmatrix} \Gamma_s(s) \\ X(s) \end{bmatrix}$ is column biproper and bistable.

Regarding the feedback realization of precompensators, the following result shows that there always exists a state feedback such that $\Phi_{es}^{-1}(s)$ is the closed-loop transfer function of a extended system.

Lemma 5. The matrix $B_1(s)$ appearing in (5) is realizable by regular state feedback.

Proof. With $K(s)$, $D(s)$ being a right coprime matrix fraction description of $T_e(s)$, and from Lemma 3, the matrix $B_1(s)$ will be proved to be feedback realizable if the product $B_1^{-1}(s)D(s)$ is polynomial.

From (5) we have that

$$T_e(s)B_1(s) = K(s)D^{-1}(s)B_1(s) = \Phi_{es}^{-1}(s).$$

Then, it follows that

$$B_1^{-1}(s)D(s) = \Phi_{es}(s)K(s)$$

is polynomial, since $\Phi_{es}(s)$ is a rational matrix with only unstable poles, $B_1(s)$ is biproper and bistable, and $D(s)$ is stable. □

The fact that the product $\Phi_{es}(s)K(s)$ is polynomial will be also used in the procedure to find a realizable compensator that decouples a system with stability.

4. MAIN RESULT

In this section we present as main result the necessary and sufficient conditions for a linear multivariable system with 2 outputs and more than 3 inputs to be decouplable with stability. Roughly speaking, the problem is solvable if and only if Morse's list I_2 is big enough to compensate the unstable and infinite zero structure of the proper and stable part of the stable interactor of the system.

Theorem 1. Let (A, B, C) be a linear multivariable system with 2 outputs and 3 or more inputs m. Let $\{\sigma_1, \sigma_2, \ldots, \sigma_{m-2}\}$ be Morse's list I_2 of the system, and δ_1 be the infinite and unstable structure of the proper and stable part of the stable interactor. Then, the system (A, B, C) is decouplable with stability if and only if

$$\delta_1 \leq \sum_{i=1}^{m-2} \sigma_i. \tag{10}$$

Proof. The necessity of the result is proved as follows: From Lemma 4, there exists a biproper and bistable matrix $V(s) = \begin{bmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{bmatrix}$ fulfilling conditions (8) and (9). Since $\begin{bmatrix} \Gamma_s(s) \\ X(s) \end{bmatrix}$ is column biproper and bistable, then there exist proper and stable matrices $R_{12}(s)$ and $R_{22}(s)$ such that $\begin{bmatrix} \Gamma(s) & R_{12}(s) \\ X(s) & R_{22}(s) \end{bmatrix}$ is biproper and bistable. Then, from

$$\begin{bmatrix} I_p & 0 \\ V_{21}(s) & V_{22}(s) \end{bmatrix} \begin{bmatrix} \Gamma(s) & R_{12}(s) \\ X(s) & R_{22}(s) \end{bmatrix} = \begin{bmatrix} \Gamma(s) & R_{12}(s) \\ 0 & I_{m-p} \end{bmatrix}$$

it can be seen that the infinite and unstable structure of $V_{22}(s)$ is equal to δ_1.

From this, and since the product

$$\begin{bmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{bmatrix} \begin{bmatrix} \Phi_{1,s}(s)Q(s) & 0 \\ \Phi_{2,s}(s)Q(s) & \Phi_{3,s}(s) \end{bmatrix}$$

is polynomial, and $\Phi_{3,s} = \operatorname{diag}\{\pi^{\sigma_1}\dots,\pi^{\sigma_{m-2}}\}$, then we have that (10) holds.

For the sufficiency, given that $\delta_1 \leq \sum_{i=1}^{m-2}\sigma_i$, we will show how to obtain a state feedback which decouples the system with stability. First, the case of systems with 3 inputs will be considered, and afterwards it will be shown how to reduce the case of more than 3 inputs to the case of systems with 3 inputs.

Consider that (A, B, C) has 2 outputs and 3 inputs. For these systems we have only one element in Morse's list I_2, namely σ_1, and (10) becomes

$$\delta_1 \leq \sigma_1.$$

The extended interactor in this case has the general form

$$\Phi_{es}(s) = \begin{bmatrix} \frac{\alpha_{11}(s)}{\pi^{n_1}} & 0 & 0 \\ \frac{\alpha_{21}(s)}{\pi^{n_{21}}} & \frac{\alpha_{22}(s)}{\pi^{n_2}} & 0 \\ \frac{\alpha_{31}(s)}{\pi^{n_{31}}} & \frac{\alpha_{32}(s)}{\pi^{n_{32}}} & \frac{1}{\pi^{\sigma_1}} \end{bmatrix}^{-1} \tag{11}$$

$$= \begin{bmatrix} \frac{\pi^{n_1}}{\alpha_{11}} & 0 & 0 \\ -\frac{\alpha_{21}}{\alpha_{11}\alpha_{22}}\pi^{n_1+n_2-n_{21}} & \frac{\pi^{n_2}}{\alpha_{22}} & 0 \\ \frac{\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}}\pi^{\sigma_1+n_1+n_2-n_{21}-n_{32}} - \frac{\alpha_{31}}{\alpha_{11}}\pi^{\sigma_1+n_1-n_{31}} & -\frac{\alpha_{32}}{\alpha_{22}}\pi^{\sigma_1+n_2-n_{32}} & \pi^{\sigma_1} \end{bmatrix}$$

where α_{11}, α_{22} are antistable polynomials, and because of the properties of $\Phi_{es}^{-1}(s)$ in Lemma 1 we have that

— $n_{21} < n_2$,

— $n_{31} < \sigma_1$, $n_{32} < \sigma_1$,

— $n_1 \leq n_2$.

Observe also that:

i) The lowest power of π in the elements $(2,1)$ and $(3,1)$ of $\Phi_{es}(s)$ is greater than n_1 (since $n_{21} < n_2$, $n_{32} < \sigma_1$, and $n_{31} < \sigma_1$).

ii) The lowest power of π in the element $(3,2)$ of $\Phi_{es}(s)$ is greater than n_2 (since $n_{32} < \sigma_1$).

The matrix $K(s)$ is given by

$$K(s) = \begin{bmatrix} Q(s) & 0 \\ 0 & 1 \end{bmatrix} U^{-1}(s) = \begin{bmatrix} q_{11}(s) & 0 & \\ q_{21}(s) & q_{22}(s) & \\ & & 1 \end{bmatrix} U^{-1}(s). \qquad (12)$$

Since $\alpha_{11}(s)$ contains the unstable zeros of the first row of $T(s)$ and $q_{11}(s)$ contains the stable and unstable zeros of the first row of $T(s)$, then $\alpha_{11}(s)$ divides $q_{11}(s)$, denoted as $\alpha_{11}(s) \mid q_{11}(s)$. The same holds for $\alpha_{22}(s)$ and $q_{22}(s)$, i. e., $\alpha_{22}(s) \mid q_{22}(s)$.

Finding a biproper and bistable matrix $V(s)$ that satisfies condition (8) is not so difficult, the problem is that this matrix must also satisfy the polynomiality condition (9). Based on the previously stated forms and properties of the extended interactor $\Phi_{es}(s)$ and matrix $K(s)$, a general procedure is provided in Appendix 1 to find a biproper and bistable matrix $V(s)$ (see (18)) and a decoupling precompensator satisfying conditions (8) and (9) for systems with 2 outputs and 3 inputs.

Once the biproper and bistable matrix $V(s)$ and the decoupling precompensator satisfying (8) and (9) have been found following the procedure in Appendix 1, then a nonregular state feedback which decouples the system with stability is obtained from a constant solution X, Y, with X nonsingular, to the polynomial matrix equation

$$XD(s) + YN_1(s) = V(s)\Phi_{es}(s)K(s),$$

as

$$F = -X^{-1}Y, \;\; G = X^{-1} \begin{bmatrix} I_2 \\ 0 \end{bmatrix},$$

where $N_1(s)$, $D(s)$ is a normal external description of the system.

To complete the proof of Theorem 1, in Appendix 2 it is shown how the case of a system (A, B, C) with 2 outputs and more than 3 inputs can be reduced to the previously considered case of a system with 2 outputs and 3 inputs using nonregular state feedback, thus solving the decoupling problem with stability for linear multivariable system with 2 outputs. □

5. EXAMPLE

Let the system (A, B, C) be given by

$$A = \begin{bmatrix} -3 & -3 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -4 & -6 & -4 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 & 0 & -1 & -2 \\ 1 & 2 & 1 & 0 & 0 & 1 & -2 & -1 & -2 \end{bmatrix},$$

and whose transfer function is

$$\begin{aligned} T(s) &= C(sI-A)^{-1}B \\ &= \begin{bmatrix} \dfrac{1}{s+1} & 0 & 0 \\ \dfrac{s^6+6s^5+15s^4+20s^3+14s^2+8s+1}{(s+1)^7} & \dfrac{s-2}{(s+1)^4} & \dfrac{s-2}{(s+1)^8} \end{bmatrix}. \end{aligned}$$

After some computations, the stable interactor of the system (A, B, C) and the associated extended stable interactor for $\pi = s+1$ result to be

$$\Phi_s(s) = \begin{bmatrix} s+1 & 0 \\ -\dfrac{(s+1)^4}{s-2} & \dfrac{(s+1)^4}{s-2} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{s-2}{(s+1)^3} & 0 \\ -1 & 1 \end{bmatrix}}_{\Gamma_s(s)} \begin{bmatrix} \dfrac{s-2}{(s+1)^4} & 0 \\ 0 & \dfrac{s-2}{(s+1)^4} \end{bmatrix}^{-1},$$

$$\Phi_{es}(s) = \begin{bmatrix} s+1 & 0 & 0 \\ -\dfrac{(s+1)^4}{s-2} & \dfrac{(s+1)^4}{s-2} & 0 \\ s(s+1)^2 & 0 & (s+1)^4 \end{bmatrix},$$

where it can be seen that

$$\delta_1 = 3, \quad \sigma_1 = 4$$

and since $\delta_1 < \sigma_1$, then the system is decouplable with stability.

Following the procedure from Appendix 1, the decoupling compensator with stability $C(s)$ (state feedback realizable) and the biproper and bistable matrix $V(s)$ satisfying conditions (8) and (9) are found as

$$C(s) = \begin{bmatrix} \dfrac{s-2}{(s+1)^3} & 0 \\ -\dfrac{s^4+4s^3+6s^2+4s+2}{(s+1)^4} & 1 \\ \dfrac{2s^2+1}{(s+1)^2} & 0 \end{bmatrix},$$

$$V(s) = \begin{bmatrix} \dfrac{1}{4}\dfrac{2s+5}{(s+1)^2} & 0 & \dfrac{1}{4}\dfrac{(s+2)(2s^2+6s+7)}{(s+1)^3} \\ \dfrac{1}{4}\dfrac{2s^2+3s+5}{(s+1)^3} & 1 & \dfrac{1}{4}\dfrac{2s^4+12s^3+29s^2+33s+18}{(s+1)^4} \\ \dfrac{2s^2+1}{(s+1)^2} & 0 & -\dfrac{s-2}{(s+1)^3} \end{bmatrix}.$$

From a constant solution to a polynomial matrix equation, we obtain the state feedback (F, G),

$$F = \begin{bmatrix} 2 & 5/2 & 1/2 & 0 & 0 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & -7/2 & -5/2 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ -1 & 1 \\ 2 & 0 \end{bmatrix},$$

which produces the closed-loop decoupled and stable system $(A+BF, BG, C)$, whose transfer function is

$$T_{FG}(s) = C(sI - A - BF)BG = \begin{bmatrix} \dfrac{s-2}{(s+1)^4} & 0 \\ 0 & \dfrac{s-2}{(s+1)^4} \end{bmatrix}.$$

6. CONCLUSIONS

In this paper a solution to the nonregular decoupling problem with stability for linear systems with 2 outputs has been presented. The structural solution is stated in terms of Morse's list I_2, and the infinite and unstable structure of the proper and stable part of the stable interactor of the system. A constructive procedure to find a state feedback, which achieves decoupling with stability has also been presented.

Even though this is a partial result in the sense that only applies to linear systems with 2 outputs and 3 or more inputs, to our knowledge it is the first result that provides necessary and sufficient conditions in the case of nonregular decoupling with stability.

APPENDIX 1

Computation of a biproper and bistable matrix and a decoupling precompensator satisfying conditions (8) *and* (9) *for a linear system* (A, B, C) *with 2 outputs and 3 inputs.*

Given that $\delta_1 \leq \sigma_1$, the following procedure allows to find a biproper and bistable matrix $V(s)$ and a decoupling precompensator

$$C(s) = \begin{bmatrix} \Gamma_s(s) \\ X(s) \end{bmatrix}$$

satisfying conditions (8) and (9) for a linear system (A, B, C) with 2 outputs and 3 inputs.

Consider the forms and properties of the extended stable interactor $\Phi_{es}(s)$ and matrix $K(s)$ given by (11) and (12). A first choice for the feedback realizable decoupling compensator is

$$C_0(s) = \begin{bmatrix} \Gamma_s(s) \\ X(s) \end{bmatrix} = \begin{bmatrix} \frac{\alpha_{22}}{\pi^{\delta_1}} & 0 \\ -\frac{\alpha_{21}}{\pi^k} & 1 \\ 1 & 0 \end{bmatrix} \tag{13}$$

where $\Gamma_s(s)$ is the proper and stable part of the stable system interactor, $X(s)$ is a proper and stable matrix such that $C_0(s)$ is column biproper and bistable, and $k = \deg \alpha_{21}(s)$.

In such a case, the simplest biproper and bistable matrix satisfying condition (8) is the following

$$V_0(s) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & \frac{\alpha_{21}}{\pi^k} \\ 1 & 0 & -\frac{\alpha_{22}}{\pi^{\delta_1}} \end{bmatrix}. \tag{14}$$

Next, we have to prove the polynomiality condition (9). Observe that the polynomiality of the matrix $V(s)\Phi_{es}(s)K(s)$ is equivalent to the polynomiality of the matrix $V(s)\Phi_{es}(s)M(s)$, where

$$M(s) := \begin{bmatrix} Q(s) & 0 \\ 0 & I_{m-p} \end{bmatrix},$$

since

$$\Phi_{es}(s)K(s) = \Phi_{es}(s)M(s)U^{-1}(s)$$

and $U(s)$ is a unimodular matrix.

It can be seen that the only elements in the matrix $V_0(s)\Phi_{es}(s)M(s)$ that could be no polynomials are the $(2,1)$ and $(3,1)$ entries, the first one containing terms of the form

$$\alpha_{21}(s)p_4(s)\pi^{-k+\sigma_1+n_1-n_{32}-n_{21}+n_2}$$

$$\alpha_{21}(s)p_5(s)\pi^{-k+\sigma_1+n_1-n_{31}}$$

and the second one containing terms of the form

$$\alpha_{22}(s)p_4(s)\pi^{-\delta_1+\sigma_1+n_1-n_{32}-n_{21}+n_2}$$

$$\alpha_{22}(s)p_5(s)\pi^{-\delta_1+\sigma_1+n_1-n_{31}}$$

where

$$p_4(s) = \frac{q_{11}(s)\alpha_{21}(s)\alpha_{32}(s)}{\alpha_{11}(s)\alpha_{22}(s)}, \text{ and}$$

$$p_5(s) = -\frac{q_{11}(s)\alpha_{31}(s)}{\alpha_{11}(s)}$$

Depending on the values of n_1 and δ_1 we can have one of the following two cases:

— If

$$n_1 \geq \delta_1$$

then the entries $(2,1)$ and $(3,1)$ of the matrix $V_0(s)\Phi_{es}(s)M(s)$ are also polynomials, and the proposed compensator C_0 and the biproper and bistable matrix $V_0(s)$ satisfy conditions (8) and (9); end of the search.

— If

$$n_1 < \delta_1$$

then we have to find a pair of unimodular matrices $U_R(s)$ and $U_L(s)$ such that

$$V(s) := U_L(s)V_0(s)U_R(s)$$

and

$$C(s) := U_R^{-1}(s)C_0(s) \tag{15}$$

satisfy conditions (8) and (9).

For simplicity, let us make the change of variables

$$\pi = s + \beta \longrightarrow s = \pi - \beta.$$

The polynomials entries $(2,1)$ and $(3,1)$ of the matrix $\Phi_{es}(\pi)M(\pi)$, denoted respectively as $(\Phi_{es}M)_{21}(\pi)$ y $(\Phi_{es}M)_{31}(\pi)$ have the general form

$$(\Phi_{es}M)_{21}(\pi) = \tau_0\pi^{n_1+1} + \tau_1\pi^{n_1+2} + \cdots + \tau_{r1}\pi^{w_1}$$
$$(\Phi_{es}M)_{31}(\pi) = \theta_0\pi^{n_1+1} + \theta_1\pi^{n_1+2} + \cdots + \theta_{r2}\pi^{w_2}$$

where θ_i and τ_j are real numbers, and w_1, w_2 are integers such that $w_1 \geq n_1 + 1$, $w_2 \geq n_1 + 1$. Then, we can factorize these polynomials as

$$(\Phi_{es}M)_{21}(\pi) = u_{21}(\pi)\pi^{n_1} + \tilde{\varphi}_{21}(\pi)$$
$$(\Phi_{es}M)_{31}(\pi) = u_{31}(\pi)\pi^{n_1} + \tilde{\varphi}_{31}(\pi)$$

where

$$u_{21}(\pi) = \sum_{i=1}^{\delta_1-n_1-1} \tau_i\pi^i$$
$$u_{31}(\pi) = \sum_{i=1}^{\delta_1-n_1-1} \theta_i\pi^i$$
$$\tilde{\varphi}_{21}(\pi) = \tau_{\rho_1}\pi^{\delta_1} + \tau_{\rho_1+1}\pi^{\delta_1+1} + \cdots + \tau_{r1}\pi^{w_1}$$
$$\tilde{\varphi}_{31}(\pi) = \theta_{\rho_2}\pi^{\delta_1} + \theta_{\rho_2+1}\pi^{\delta_1+1} + \cdots + \theta_{r2}\pi^{w_2}.$$

We will suppose that the system has no stable zeros, thus implying that $\alpha_{11}(s) = q_{11}(s)$ and $\alpha_{22}(s) = q_{22}(s)$. Observe that this is not an important restriction since stable zeros do not affect the conditions for decoupling with stability. Then, the product $\Phi_{es}(\pi)M(\pi)$ can be written as

$$\Phi_{es}(\pi)M(\pi) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ u_{21} & 1 & 0 \\ u_{31} & 0 & 1 \end{bmatrix}}_{U_R^{-1}(s)} \underbrace{\begin{bmatrix} \pi^{n_1} & 0 & 0 \\ \tilde{\varphi}_{21} & \pi^{n_2} & 0 \\ \tilde{\varphi}_{31} & \varphi_{32} & \pi^{\sigma_1} \end{bmatrix}}_{\Omega(s)}$$

Let us propose now

$$V_1(\pi) = V_0(\pi)U_R(\pi) = \begin{bmatrix} -u_{31} & 0 & 1 \\ -u_{21} - \alpha_{21}\pi^{-k}u_{31} & 1 & \alpha_{21}\pi^{-k} \\ 1 + \alpha_2\pi^{-\delta_1}u_{31} & 0 & -\alpha_2\pi^{-\delta_1} \end{bmatrix}, \tag{16}$$

where $V_1(\pi)\Phi_{es}(\pi)M(\pi)$ is a polynomial matrix.

Then we will have that

$$C(\pi) = U_R^{-1}(\pi)C_0(\pi) = \begin{bmatrix} \alpha_2\pi^{-\delta_1} & 0 \\ u_{21}\alpha_2\pi^{-\delta_1} - \alpha_{21}\pi^{-k} & 1 \\ 1 + \alpha_2\pi^{-\delta_1}u_{31} & 0 \end{bmatrix} \tag{17}$$

is such that

$$V_1(\pi)C(\pi) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

However, it can be noticed that $V_1(\pi)$ is not biproper because of the entries $(1,1)$ and $(2,1)$ of $V_1(\pi)$. To overcome this difficulty, let

$$U_L(\pi) = \begin{bmatrix} 1 & 0 & \omega_{13}(\pi) \\ 0 & 1 & \omega_{23}(\pi) \\ 0 & 0 & 1 \end{bmatrix}$$

be a unimodular matrix, where

$$\omega_{13}(\pi) = \frac{u_{31}}{1 + \alpha_2\pi^{-\delta_1}u_{31}}$$

$$\omega_{23}(\pi) = \frac{u_{21} + \alpha_{21}\pi^{-k}u_{31}}{1 + \alpha_2\pi^{-\delta_1}u_{31}}.$$

Thus, the matrix

$$V(\pi) = U_L(\pi)V_1(\pi) \tag{18}$$

is biproper and bistable, and satisfies

$$V_1(\pi)C(\pi) = \begin{bmatrix} I_2 \\ 0 \end{bmatrix}.$$

From $V(\pi)$ and $C(\pi)$, and using $\pi = s + \beta$, we obtain the matrices $V(s)$ and $C(s)$, which are respectively the biproper and bistable matrix and the decoupling precompensator satisfying conditions (8) and (9).

APPENDIX 2

A nonregular feedback reduction of a 2-output system with more than 3 inputs to the case of 3-input channels.

The purpose of this section is to show that a system with 2 outputs and more than 3 inputs can be reduced to the case of 2 outputs and 3 inputs using nonregular static state feedback, concatenating thus all the structural information of Morse's list I_2 in only one index σ.

Proposition 1. Let $T(s)$ be the transfer function of the system (A, B, C) with 2 outputs and m inputs, $m > 3$, and Morse's list $I_2 = \{\sigma_1, \sigma_2, \dots, \sigma_{m-2}\}$. Then, there exists a nonregular static state feedback, such that the transfer matrix of the closed-loop system has 3 inputs and Morse's list I_2 with an unique index $\sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_{m-2}$.

Proof. Since the matrix $B_1(s)$ in (5) is state feedback realizable, we can suppose without loss of generality that the transfer matrix of the extended system is $\Phi_{es}^{-1}(s)$. Then, $T_e(s)$ may be written as follows,

$$T_e(s) = \Phi_{es}^{-1}(s) = K(s)D^{-1}(s) \tag{19}$$

where $(K(s), D(s))$ is a right coprime polynomial factorization of the extended system.

Let us consider first the case of 4 inputs, $m = 4$, in which case the inverse of the extended interactor may be written as

$$\Phi_{es}^{-1}(s) = \begin{bmatrix} \tilde{\varphi}_{11}(s) & 0 & 0 & 0 \\ \tilde{\varphi}_{21}(s) & \tilde{\varphi}_{22}(s) & 0 & 0 \\ \tilde{\varphi}_{31}(s) & \tilde{\varphi}_{32}(s) & \pi^{-\sigma_1} & 0 \\ \tilde{\varphi}_{41}(s) & \tilde{\varphi}_{42}(s) & 0 & \pi^{-\sigma_2} \end{bmatrix}. \tag{20}$$

To prove Proposition 1, we find first a precompensator $\tilde{C}(s)$ realizable by regular static state feedback and then we complete it with a constant nonregular input matrix gain G.

Consider the following precompensator,

$$\tilde{C}(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\tilde{\varphi}_{41}(s) & -\tilde{\varphi}_{42}(s) & 1 & -\pi^{-\sigma_2} \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}. \tag{21}$$

In the product $\tilde{C}^{-1}(s)\Phi_{es}(s)$ the nontrivial terms are the $(3,1)$ and $(3,2)$ entries, which can be reduced respectively to $\varphi_{31}(s)$, and $\varphi_{32}(s)$ due to the fact that $\Phi_{es}(s)\Phi_{es}^{-1}(s) = I$. Then, we have that

$$\tilde{C}^{-1}(s)\Phi_{es}(s) = \begin{bmatrix} \varphi_{11}(s) & 0 & 0 & 0 \\ \varphi_{21}(s) & \varphi_{22}(s) & 0 & 0 \\ \varphi_{31}(s) & \varphi_{32}(s) & \pi^{\sigma_1} & -1 \\ \varphi_{41}(s) & \varphi_{42}(s) & 0 & \pi^{\sigma_2} \end{bmatrix}. \tag{22}$$

Now, from this expression and from (19) it can be seen that the product,

$$\tilde{C}^{-1}(s)D(s) = \tilde{C}^{-1}(s)\Phi_{es}(s)K(s)$$

is polynomial. Moreover, since $\tilde{C}(s)$ is by construction a biproper and bistable matrix, then it is realizable by regular static state stabilizing feedback.

To complete the desired nonregular feedback it is convenient to introduce a nonregular constant input matrix gain G given by,

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this way the transfer matrix of the closed-loop system $T_{eCG}(s)$ is given by

$$T_{eCG}(s) = \begin{bmatrix} \tilde{\varphi}_{11}(s) & 0 & 0 \\ \tilde{\varphi}_{21}(s) & \tilde{\varphi}_{22}(s) & 0 \\ \tilde{\varphi}_{31}(s) + \tilde{\varphi}_{41}(s)\pi^{-\sigma_1} & \tilde{\varphi}_{32}(s) + \tilde{\varphi}_{42}(s)\pi^{-\sigma_1} & \pi^{-\sigma_1-\sigma_2} \end{bmatrix}$$

Clearly, the corresponding extended interactor $\tilde{\Phi}_{es}(s)$ is

$$\tilde{\Phi}_{es}(s) = \begin{bmatrix} \varphi_{11}(s) & 0 & 0 \\ \varphi_{21}(s) & \varphi_{22}(s) & 0 \\ \varphi_{31}(s)\pi^{\sigma_2} + \varphi_{41}(s) & \varphi_{32}(s)\pi^{\sigma_2} + \varphi_{42}(s) & \pi^{\sigma_1+\sigma_2} \end{bmatrix}.$$

Observe that using nonregular feedback amounts to cancel some inputs, in the previous case the third one, and as a consequence in order to get the new extended interactor one has to eliminate a virtual output after closing the loop. Also notice that $\tilde{\Phi}_{es}(s)$ has the structure of an extended interactor, having only one term corresponding to the list I_2 of value $\sigma_1 + \sigma_2$.

The above procedure can be repeated as many times as needed for the general case where the number of inputs $m > 4$ and as result one would obtain an extended interactor with a unique index in list I_2 of value $\sigma_1 + \sigma_2 + \ldots + \sigma_{m-2}$ as claimed. □

The proof of Proposition 1 is in some sense the generalization of an equivalent statement for the problem without stability requirements and using a precompensation approach, which is more suitable in the context of the present work (see [7]).

The above result can be interpreted as a nonregular static state feedback action on the triplet (A, B, C) of the system in such a way that the resulting closed-loop system will have only one controllability chain in the maximal controllability output nulling subspace $\mathcal{R}^*$, which is the result of concatenating all the controllability chains inside it.

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REFERENCES

[1] J. Descusse and J. M. Dion: On the structure at infinity of linear square decoupled systems. IEEE Trans. Automat. Control *AC-27* (1982), 971–974.

[2] J. Descusse, J. F. Lafay, and M. Malabre: Solution of the static-state feedback decoupling problem for linear systems with two outputs. IEEE Trans. Automat. Control *AC-30* (1985), 914–918.

[3] J. Descusse, J. F. Lafay, and M. Malabre: Solution to Morgan's problem. IEEE Trans. Automat. Control *33* (1988), 732–739.

[4] J. M. Dion and C. Commault: The minimal delay decoupling problem: Feedback implementation with stability. SIAM J. Control Optim. *26* (1988), 66–82.

[5] P. L. Falb and W. A. Wolovich: Decoupling in the design and synthesis of multivariable control systems. IEEE Trans. Automat. Control *AC-12* (1967), 651–659.

[6] A. Herrera: Sur le decouplage des systemes lineaires par des lois statiques non regulieres. PhD Thesis, Université de Nantes, Ecole Centrale Nantes 1991.

[7] A. Herrera, J. A. Torres, and J. Ruiz-León: The nonregular Morgan's problem: A polynomial solution for the case of two outputs. In: Proc. European Control Conference (ECC'93), Groningen 1993, pp. 2275–2278.

[8] J. C. Martínez G. and M. Malabre: The row by row decoupling problem with stability: A structural approach. IEEE Trans. Automat. Control *39* (1994), 2457–2460.

[9] A. S. Morse: Structural invariants of linear multivariable systems. SIAM J. Control *11* (1973), 446–465.

[10] J. Ruiz-León, P. Zagalak, and V. Eldem: On the Morgan problem with stability. Kybernetika *32* (1996), 425–441.

[11] M. Vidyasagar: Control System Synthesis: A Factorization Approach. MIT Press, Cambridge, MA 1985.

[12] W. A. Wolovich and P. L. Falb: Invariants and canonical forms under dynamic compensation. SIAM J. Control Optim. *14* (1976), 996–1008.

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