KYBERNETIKA — VOLUME 38 (2002), NUMBER 5, PAGES 521-540

PARAMETRIZATION AND RELIABLE EXTRACTION OF PROPER COMPENSATORS[1]

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The polynomial matrix equation $X_l D_r + Y_l N_r = D_k$ is solved for those X_l and Y_l that give proper transfer functions $X_l^{-1} Y_l$ characterizing a subclass of compensators, contained in the class whose arbitrary element can be cascaded to a plant with the given strictly proper transfer function $N_r D_r^{-1}$ such that wrapping the negative unity feedback round the cascade gives a system whose poles are specified by D_k. The subclass is navigated and extracted through a conventional parametrization whose denominators are affine to row echelon form and the centre is in a compensator whose numerator has minimum column degrees. Applications include stabilization of linear multivariable systems.

1. INTRODUCTION

1.1. Configuration and goals

We consider the linear, time-invariant, closed-loop system $\mathrm{S}(P, C)$ in the negative unity feedback configuration shown in Figure 1, with $P : e_2 \mapsto y_2$ and $C : e_1 \mapsto y_1$ respectively an $m \times p$ plant and a $p \times m$ compensator. The input-error map of the system $\mathrm{S}(P, C)$ is

$$H_{eu}\colon\ u \mapsto e, \qquad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{1}$$

The external signals are the input u and the output y. The components of y are y_1 and y_2. Either P and C obeys a set of ordinary differential equations in polynomial matrix fraction description (mfd), obtained by the Laplace transformation with zero initial conditions. Cancellation is not permitted or at least if it is carried out the order of the equations is changed, leading to a different system, which is set up from fewer independent initial conditions [10] and hence to be avoided.

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[1] This research has been supported by a Marie Curie Fellowship of the European Community programme "Improving Human Research Potential and the Socioeconomic Knowledge Base" under contract number HPMF-CT-1999-00347. P. Zagalak acknowledges support of the Grant Agency of the Czech Republic under project 102/01/0608.

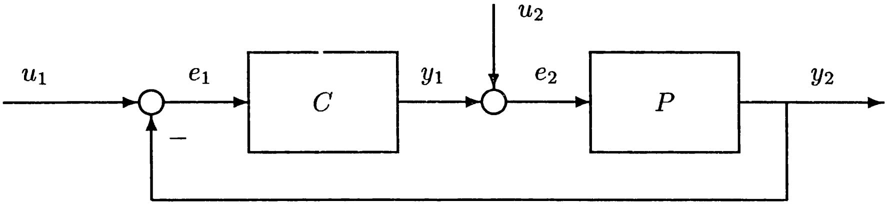

Fig. 1. Negative unity feedback configuration.

The task is to find a proper transfer function C for a given strictly proper transfer function P, and such that the closed-loop transfer function

$$H_{eu} = \begin{bmatrix} (I_p + PC)^{-1} & -P(I_m + CP)^{-1} \\ C(I_p + PC)^{-1} & (I_m + CP)^{-1} \end{bmatrix} \tag{2}$$

is proper and has poles exclusively in the open left half-plane. A rational transfer function matrix P is said to be strictly proper if the limit

$$\lim_{s \to \infty} P(s) \tag{3}$$

exists and is zero; P is said to be proper if (3) exists and is finite; P is said to be biproper if it is proper, invertible and the inverse is proper.

If P is strictly proper and C is proper, then $I_m + CP$ and $I_p + PC$ are biproper. In particular, the limit

$$\lim_{s \to \infty} C(s)P(s) \tag{4}$$

exists, is zero, and renders the rational matrix

$$B(s) = I_m + C(s)P(s) \tag{5}$$

biproper with $B(\infty) = I_m$. It follows that (2) is proper and its poles may be modified by selecting a convenient C. And this is where we turn to mfd: $X_l^{-1}Y_l$ for the compensator and $N_r D_r^{-1}$ for the plant. The polynomial matrices (X_l, Y_l) and (N_r, D_r) are respectively left and right coprime and are uniquely defined up to nonsingular multipliers on respectively the left and the right. The multipliers, called unimodular polynomial matrices, are such that their inverse is a polynomial matrix. The two mfds convert (5) into

$$D_k(s) = X_l(s)D_r(s) + Y_l(s)N_r(s). \tag{6}$$

The matrix D_k is polynomial because the right-hand side is a sum of polynomial matrices. Moreover, D_k is nonsingular and the roots of $\det D_k$ are the closed-loop poles, following the role of (5) in (2). Application related assumptions about D_k, X_l, D_r, Y_l and N_r are imposed in Definition 1.3.

The goals of this paper are: to recall the concept of data- and parameter-degree control for polynomial mfd of all proper feedback compensators whose denominator is row reduced with sufficiently large prescribed row degrees, to propose and justify an appropriate form for the conventional parametrization of such compensators, and to extract such a form in a numerically reliable way.

1.2. Concepts

A number of concepts may be introduced and exploited through the rational equation

$$I_m + X_l^{-1} Y_l N_r D_r^{-1} \quad = \quad X_l^{-1} D_k D_r^{-1}. \tag{7}$$

The concepts relate to the following lemma.

Lemma 1.1. (Proper MFD, cf. Kailath [6, p. 385].) If D is column reduced, then ND^{-1} is strictly proper (proper) if each column of N has degree less than (less than or equal to) the degree of the corresponding column of D.

The Lemma gives a simple test for strict properness (properness) of right mfds provided the denominator is column reduced. It is obvious that the test can be dualized for left mfds and given a two-sided extension: If X_l and D_r are respectively row and column reduced, then Lemma 1.1 can be applied to those right-hand sides of (7) whose D_k is a product of two polynomial matrices that are respectively row reduced with the row degrees of X_l and column reduced with the column degrees of D_r. The extension justifies the following definition.

Definition 1.2. (Row-Column Reducedness, cf. Callier and Desoer [3, p. 116].) An $m \times m$ polynomial matrix D is said to be *row-column reduced* if there exist m nonnegative integers r_i, called row powers, and m nonnegative integers k_i, called column powers, such that the limit

$$D_{\mathrm{h}} \quad = \quad \lim_{s \to \infty} \operatorname{diag}[s^{-r_i}]_{i=1}^{m} D(s) \operatorname{diag}[s^{-k_i}]_{i=1}^{m} \tag{8}$$

exists and it is nonsingular.

Statements such as "$D(s)$ is row-column reduced with row powers r_i, column powers k_i, and highest coefficient matrix D_{h}" reflect that

$$D(s) \quad = \quad \operatorname{diag}[s^{r_i}]_{i=1}^{m} D_{\mathrm{h}} \operatorname{diag}[s^{k_i}]_{i=1}^{m} + \text{terms of lower degree in } s. \tag{9}$$

The equation subject to this paper involves several specifications:

Definition 1.3. (Compensator Equation, cf. Rosenbrock and Hayton [11].) A linear polynomial matrix equation

$$X_l(s) D_r(s) + Y_l(s) N_r(s) \quad = \quad D_k(s) \tag{10}$$

is called the compensator equation (COMP) if

(i) D_r and N_r are right coprime,

(ii) D_r and D_k are square and nonsingular,

(iii) D_r is column reduced with column degrees $k_1, \ldots, k_m$ such that $N_r D_r^{-1}$ is strictly proper,

(iv) D_k is row-column reduced with row powers $r_1, \ldots, r_m$ and column powers $k_1, \ldots, k_m$.

The coprimeness in (i) ensures solvability for an arbitrary D_k and is conceptually linked to minimal realizations. The nonsingularity in (ii) follows by mfd for the plant and the closed loop. The reducedness in (iii) ensures coprimeness (solvability) at high frequencies and is inherent in high frequency behavior of $N_r(j\omega)D_r^{-1}(j\omega)$ and $X_l^{-1}(j\omega)Y_l(j\omega)$ as physically realizable systems, while (iv) has to do with similar requirements about the closed loop, cf. Lemma 2.2.

According to [8] the equation (10) is solvable if and only if a greatest common divisor of D_r and N_r is a right divisor of D_k; the solutions are related through a particular solution, say (X_{lo}, Y_{lo}), in the parametrization

$$\begin{aligned} X_l &= X_{lo} + TD_l \\ Y_l &= Y_{lo} - TN_l \end{aligned} \tag{11}$$

where the polynomial matrix T is the parameter and the polynomial matrices D_l and N_l satisfy $N_l D_r = D_l N_r$.

1.3. Literature

Our most recent and influential source of inspiration is the review by [1], see also [2], drawing on earlier results in [3]. In a conventional manner, the class of all polynomial matrices (X_l, Y_l) is recalled before those pairs are singled out which describe compensators with proper rational transfer functions $X_l^{-1}Y_l$. Sections 2.1 and 2.2 contain results from [1], with modifications to accommodate the computational procedures in the present paper.

The above results can be shown dual to those in the study by [7], our second most influential source. Despite a different D_k, which is assumed simultaneously row and column reduced with the highest coefficient matrices equal to an identity matrix, the assumptions are compatible and hence are the results. Whatever may be said, the study provides examples that should enable readers to form their own picture of the subject.

A fundamental (if not the fundamental) step in either approach derives from the sufficient condition for the general problem of pole assignment by dynamical output feedback [11], alternatively proved by [17] using linear polynomial matrix equations. The latter proof is constructive and is subsequently used in [7].

2. PROPER COMPENSATORS

2.1. Existence

Sufficient conditions for the existence of proper compensators are reviewed, leading to Lemma 2.2 whose closed-loop role for the compensators is analogous to the open-loop role of Lemma 1.1 for the plant.

Lemma 2.1. (Candidate Denominators) Let COMP have a particular solution such that $\delta_{\mathrm{ri}}[Y_l] \leq r_i$ for all $i = 1, \ldots, m$. Then $X_l(s)$ is row reduced with $\delta_{\mathrm{ri}}[X_l] = r_i$ for all $i = 1, \ldots, m$ and it is such that

$$X_{l\mathrm{hr}} D_{r\mathrm{hc}} \;=\; D_{k\mathrm{h}} \tag{12}$$

where $X_{l\mathrm{hr}}$, $D_{r\mathrm{hc}}$ and $D_{k\mathrm{h}}$ are the highest coefficient matrices of respectively the row reduced $X_l(s)$, the column reduced $D_r(s)$, and the row-column reduced $D_k(s)$.

Proof. Let (X_l, Y_l) be the solution whose existence we assume. Consider (10) with D_k, (D_r, N_r) and (X_l, Y_l) of respectively the form (9) and its single-sided versions, displaying respectively the highest coefficient matrices $D_{k\mathrm{h}}$, $(D_{r\mathrm{hc}}, N_{r\mathrm{hc}})$ and $(X_{l\mathrm{hr}}, Y_{l\mathrm{hr}})$. The transfer function $N_r D_r^{-1}$ is proper by assumption and hence the constant matrix $N_{r\mathrm{hc}}$ is zero. To match the constant matrix $D_{k\mathrm{h}}$, no contribution is recorded from $Y_l N_r$ and a pair of equalities is established: the row degrees of X_l equal the corresponding row powers of D_k and the highest row-column power coefficient matrix of $X_l D_r$ equals that of D_k. The latter equality is (12) and it provides a nonsingular $X_{l\mathrm{hr}}$ by the nonsingularity of $D_{r\mathrm{hc}}$ and $D_{k\mathrm{h}}$. □

A set of sufficiently large row powers of D_k ensures a proper $X_l^{-1} Y_l$ to exist.

Lemma 2.2. (Proper Compensators: Existence) Let μ be the highest power of s in a coprime left mfd

$$D_l^{-1} N_l \;=\; N_r D_r^{-1} \tag{13}$$

of the strictly proper plant P. If for all $i = 1, \ldots, m$

$$r_i \;\geq\; \mu - 1 \tag{14}$$

then COMP admits a particular solution (X_l, Y_l) such that $X_l^{-1} Y_l$ is proper.

Proof. Particular solutions are generated one from another through the choice of the $m \times p$ parameter T in (11). To determine a proper particular solution from an arbitrary but fixed particular solution, say (X_{lp}, Y_{lp}) from (X_{lo}, Y_{lo}), note that the right multiple by D_l^{-1} of $Y_{lo} = Y_{lp} + T D_l$ uniquely defines T and Y_{lp} when the rational $Y_{l_o} D_l^{-1}$ is decomposed into strictly proper and polynomial part, that is, $Y_{l_p} D_l^{-1}$ and T. The strict-properness of $Y_{l_o} D_l^{-1}$ implies $\delta_{\mathrm{cj}}[Y_{lo}] < \delta_{\mathrm{cj}}[D_l]$ for all $j = 1, \ldots, p$. Since for all $i = 1, \ldots, m$

$$\delta_{\mathrm{ri}}[Y_{lo}] \leq \max_{i=1,\ldots,m} \delta_{\mathrm{ri}}[Y_{lo}] = \max_{j=1,\ldots,p} \delta_{\mathrm{cj}}[Y_{lo}] < \max_{j=1,\ldots,p} \delta_{\mathrm{cj}}[D_l] = \max_{i=1,\ldots,p} \delta_{\mathrm{ri}}[D_l] \overset{\mathrm{def}}{=} \mu$$

it follows that $\delta_{\mathrm{ri}}[Y_{lo}] < \mu$, that is, $\delta_{\mathrm{ri}}[Y_{lo}] \leq \mu - 1$. Combine the latter condition with the assumption (14) to obtain $\delta_{\mathrm{ri}}[Y_{lo}] \leq r_i$ and apply Lemma 2.1 to show that X_{lo} is row reduced with $\delta_{\mathrm{ri}}[X_{lo}] = r_i$ for all $i = 1, \dots, m$. By Lemma 1.1, the particular solution (X_{lo}, Y_{lo}) is proper because X_{lo} is row reduced and $\delta_{\mathrm{ri}}[Y_{lo}] \leq \delta_{\mathrm{ri}}[X_{lo}]$ for all $i = 1, \dots, m$. □

The conservatism in the characterization of the sufficiently large row powers of D_k may be decreased by the choice of a convenient $D_l^{-1} N_l$.

Lemma 2.3. (Observability Index) If a coprime left mfd for the plant is chosen such that D_l is row reduced, then μ is minimal with respect to all coprime left mfds for the plant.

Proof. Let the polynomial matrices D_l and $\bar{D}_l$ be row reduced with row degrees arranged in order, say ascending. According to [15] if $D_l = U\bar{D}_l$ for some unimodular U, then the row degrees of D_l equal the corresponding row degrees of $\bar{D}_l$. It follows that $\delta_{\mathrm{rp}}[D_l] = \delta_{\mathrm{rp}}[\bar{D}_l] = \mu$. The rest is trivial. □

Such a minimization of the sufficiently large row powers of D_k doesn't protect us from choosing D_k whose row powers are unnecessarily high.

Example 2.4. Consider a 2 × 2 plant with proper transfer function in the mfd

$$P(s) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s^2+1 & 1 \\ 0 & s+1 \end{bmatrix}^{-1}.$$

To calculate the greatest observability index it is preferable to use the method in Section 3, but it is possible to find a left mfd with row-reduced denominator in hand and to check the highest power of s; this is much easier for our simple mfd, for example take

$$P(s) = \begin{bmatrix} 0 & s+1 \\ s^2+1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & s-1 \end{bmatrix}.$$

Either way, $\mu = 2$. According to Lemma 2.2 an admissible D_k is row-column reduced with column powers $(k_1, k_2) = (2, 1)$ and row powers $(r_1, r_2) = (1, 1)$. If D_k is row-column reduced with $(k_1, k_2) = (2, 1)$ and $(r_1, r_2) = (0, 0)$, then Lemma 2.2 is not applicable, yet compensators with proper transfer function may exist. For example consider

$$D_k(s) = \begin{bmatrix} s^2+1 & 1 \\ 1 & s+2 \end{bmatrix}$$

and verify the existence of the compensator

$$C(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

2.2. Parametrization

To generate all compensators with proper transfer function, a proper compensator may be put at the centre of a conventional parametrization whose parameter matrix is restricted conformally [1, 7].

Lemma 2.5. (Proper Compensators: Parametrization) Let (X_{lp}, Y_{lp}) be a particular solution to COMP such that $X_{lp}^{-1}Y_{lp}$ is proper. Then all pairs (X_l, Y_l) with proper $X_l^{-1}Y_l$ are specified by

$$\begin{aligned} X_l &= X_{lp} - TN_l \\ Y_l &= Y_{lp} + TD_l \end{aligned} \tag{15}$$

where T is an $m \times p$ polynomial matrix parameter such that

$$\delta_{ij}[T] \leq r_i - \delta_{\mathrm{rj}}[D_l] \tag{16}$$

for all $i = 1, \dots, m$, $j = 1, \dots, p$.

Proof. Arbitrary proper particular solutions may be generated one from another, say (X_l, Y_l) from (X_{lp}, Y_{lp}), by a convenient choice of the parameter T in (15). As an extension to the proof of Lemma 2.2, the second equation in (15) reveals that $\delta_{\mathrm{ri}}[Y_l] \leq r_i$ if and only if $\delta_{\mathrm{ri}}[TD_l] \leq r_i$. To translate the latter condition in condition (16), consider a row-reduced D_l in the left-sided version of (8). □

2.3. Homogeneous and particular solutions

It is the control system specifications that define best descriptions: they do not define what the overall best description is, but the best description in view of the limitations imposed. With no limitations beyond "proper compensator and strictly proper plant" we pursue common system theoretical and computational concepts: absence of zeros at infinity, minimal order compensators; minimal basis of rational vector spaces, and numerical linear algebra methods for rank determination leading to solution of uniquely defined linear systems.

The homogeneous system may be viewed as a conversion between right and left mfd, $N_r D_r^{-1}$ being the reference. Various forms of left-coprime $D_l^{-1}N_l$ obey

$$D_l^{-1}N_l = N_r D_r^{-1}, \tag{17}$$

in relation to the homogeneous matrix polynomial equation

$$-N_l(s)D_r(s) + D_l(s)N_r(s) = 0 \tag{18}$$

with N_l and D_l unknown polynomial matrices that are respectively $p \times m$ and $p \times p$ in dimension and such that D_l is nonsingular.

The notion of left minimal basis (LMB) is instrumental. According to [4] we define polynomial matrices E and F such that

$$E = \begin{bmatrix} -N_l & D_l \end{bmatrix} \tag{19}$$

$$F = \begin{bmatrix} D_r \\ N_r \end{bmatrix} \tag{20}$$

$$E(s)F(s) = 0. \tag{21}$$

Because D_r and N_r are right coprime, the $(m+p) \times m$ matrix F has full column rank m in the entire complex plane. The $p \times (m+p)$ matrix E is contained in the left null space of F. This null space is customarily described by E that obeys (19) - (21) and is also of

(i) full row rank p in the entire complex plane and

(ii) row reduced.

Then E is said to be a minimal basis of the left null space of F, or a LMB for short.

Our interest is with $\begin{bmatrix} -N_l & D_l \end{bmatrix}$ in LMB. The one we have in mind is sparse and uniquely derives from a unique form of D_l in the sense that if D_l and $\bar{D}_l$ satisfy $D_l = U\bar{D}_l$ for some unimodular U, then the form of D_l equals that of $\bar{D}_l$.

Definition 2.6. (Polynomial Row–Echelon Form, cf. Popov [9].) A $p \times p$ nonsingular polynomial matrix D_l is said to be in polynomial row-echelon form if

(i) D_l is row reduced with the row degrees arranged in ascending order

$$\delta_{\mathrm{r1}}[D_l] \leq \delta_{\mathrm{r2}}[D_l] \leq \cdots \leq \delta_{\mathrm{rp}}[D_l] \overset{\mathrm{def}}{=} \delta[D_l]$$

(ii) For row i, there is an index p_i, called a pivot index, such that

- (a) $\delta_{ip_i}[D_l] = \delta_{\mathrm{ri}}[D_l]$ and the element is monic,
- (b) $\delta_{ij}[D_l] < \delta_{\mathrm{ri}}[D_l]$ if $j > p_i$,
- (c) if $\delta_{\mathrm{ri}}[D_l] = \delta_{\mathrm{rj}}[D_l]$ and $i < j$, then $p_i < p_j$,
- (d) $\delta_{ip_i}[D_l] < \delta_{\mathrm{ri}}[D_l]$ if $i \neq j$.

For reasons mentioned above, the polynomial row-echelon form is considered useful for the description of the solution $(X_l, Y_l) = (-N_l, D_l)$ of the homogeneous form of COMP due to (18), and corresponding to a left-coprime description $D_l^{-1}N_l$ of the plant.

Definition 2.7. (Homogeneous Solution) Consider the polynomial matrix solutions to the homogeneous form COMP. A pair (X_l, Y_l) is said to be desirable if

(i) X_l and Y_l are left coprime,

(ii) Y_l is nonsingular,

(iii) Y_l is in polynomial row-echelon form.

Particular solutions include a special class whose description is similar in nature to the polynomial row-echelon form description of the plant. The class is distinct by the column degrees of the polynomial matrix Y_l.

Lemma 2.8. (Column Degrees) Let the assumptions of Lemmas 2.2 and 2.3 hold. Then COMP admits a particular solution (X_l, Y_l) such that $X_l^{-1}Y_l$ is proper and

$$\delta_{\mathrm{cj}}[Y_l(s)] \leq \delta_{\mathrm{rj}}[\Pi D_l(s)] - 1 \tag{22}$$

for all $j = 1, \ldots, p$ and a convenient permutation matrix Π.

Proof. Let us choose the plant description $D_l^{-1}N_l$ in the form obeying Definition 2.7. Then D_l is simultaneously row and column reduced by Definition 2.6. As shown in the initial part of the proof to Lemma 2.2, the transfer function $Y_l D_l^{-1}$ is strictly proper and hence

$$\delta_{\mathrm{cj}}[Y_l] \leq \delta_{\mathrm{cj}}[D_l] - 1. \tag{23}$$

Finally, (22) follows from the simultaneous row- and column-reducedness and the row degree invariance[2] of D_l. □

For reasons mentioned above, the following form is considered a useful description to be put at the centre of the parametrization.

Definition 2.9. Consider the class of COMP solutions such that $X^{-1}Y$ is proper. A pair (X, Y) is said to be desirable if the column degrees of Y_l are minimal with respect to the class.

3. VERIFICATION OF EXISTENCE

3.1. Applicability and methodology

A judicious choice of D_k, the polynomial matrix on the right-hand side of COMP, requires a reliably determined greatest observability index of the plant. The same applies when the row powers of D_k are too high, cf. Section 4.2.

To compute the greatest observability index, the plant may be efficiently realized in state space, where orthogonal similarity transformations to Hessenberg form may be applied to identify a set of integers whose sum equals the index. Details are given in the next two sections.

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[2] See the proof to Lemma 2.3 for reference and other use to the row degree invariance.

3.2. Implementation

The plant described by mfd $N_r D_r^{-1}$ admits state-space realization in controller form [6], with computational expenses depending upon the structure of $D_{r\text{hc}}$, the highest coefficient matrix of the column-reduced $D_r(s)$. If $D_{r\text{hc}}$ equals an identity matrix, or a permutation of an identity matrix, then there is no computation; if $D_{r\text{hc}}$ is a triangular matrix, or a permutation of a triangular matrix, then a permuted backsubstitution will do; otherwise a general triangular factorization by Gaussian elimination with partial pivoting is an adequate tool to obtain the realization, except in cases of ill-conditioning where orthogonal methods give an added measure of reliability [5].

The realization, say (A, B, C), may be transformed to observer Hessenberg form [12], whose structure reveals the greatest observability index as shown in the next section. For the Hessenberg form, let U_1 be an orthogonal transformation compressing the columns of C and let ρ_1 be the rank of C; then A_1, C_1, X_1, Y_1 and Z_1 are matrices of appropriate dimensions defined by

$$CU_1 = [\ \underbrace{Z_1}_{\rho_1}\ \ \underbrace{0}_{\tau_1}\ ] \qquad U_1^* A U_1 = \left[\begin{array}{cc} Y_1 & C_1 \\ X_1 & A_1 \end{array} \right] \begin{array}{l} \}\ \rho_1 \\ \}\ \tau_1 \end{array} \tag{24}$$

where Z_1 has full column rank ρ_1. Applied to (24), a similarity transformation of the type block $\text{diag}(I_{\rho_1}, U_2)$ only effects A_1, C_1 and X_1. If C_1 has neither zero rank nor full column rank, then we can use U_2 to compress the columns of C_1 and repeat a partitioning of the type (24) on C_1U_2 and $U_2^* A_1 U_2$. The algorithm continues this recursion until a matrix C_k is obtained with full column rank — a corollary to Definition 1.3 is that $\tau_k = 0$ is the one and only stopping rule – reducing the pencil in the "staircase" form

$$\left[\begin{array}{c} C \\ sI_n - A \end{array} \right] = \left[\begin{array}{ccccc} Z_1 & & & & \\ sI_{\rho_1} - Y_1 & -Z_2 & & & \\ & sI_{\rho_2} - Y_2 & \ddots & & \\ & & \ddots & -Z_k & \\ & \mathsf{X} & & sI_{\rho_k} - Y_k & \\ & & & & sI_{\tau_k} - A_k \end{array} \right] . \tag{25}$$

The blanks denote zeros. The elements denoted x, as well as the matrix Y_1, need not be computed for the purpose we have in mind. The Z_i have full column rank by construction, which implies that the shaded submatrix has full column rank for any value of s. According to the Popov–Hautus test the shaded submatrix describes the observable part of (C, A).

Because $N_r D_r^{-1}$ is coprime by assumption, the controller form is observable [6] and hence the rows and columns are void that intersect at $sI_{\tau_k} - A_k$.

3.3. Greatest observability index

The greatest observability index of the plant may be determined from the cardinality of the set $(\rho_1, \ldots, \rho_k)$, specified in the previous section.

Lemma 3.1. (Plant: Left Denominator) Let a strictly proper transfer function with mfd $N_rD_r^{-1}$ be realized in observer Hessenberg form with k the number of full column rank blocks. Then there exists a left-coprime mfd $D_l^{-1}N_l = N_rD_r^{-1}$ with D_l row reduced and such that $\delta[D_l] = k$(highest degree of all elements of D_l).

Proof. Let $(A, B, C) \equiv N_rD_r^{-1}$ be in observer Hessenberg form. From the identity

$$D_l^{-1}(s)N_l(s) \;=\; C(sI_n - A)^{-1}B \;=\; N_r(s)D_r^{-1}(s)$$

we extract

$$\begin{bmatrix} D_l(s) & W(s) \end{bmatrix} \begin{bmatrix} C \\ sI_n - A \end{bmatrix} \;=\; 0 \tag{26}$$

with the polynomial matrix $W(s)$ determined through

$$-W(s)B \;=\; N_l(s).$$

To exploit the Hessenberg structure we consider block column partitioning

$$W(s) \;=\; \left[\; W_2(s) \mid W_3(s) \mid \cdots \mid W_k(s) \;\right].$$

For improved visualization, the detailed version of (26) may be considered such that for $i = 2, \ldots, k$ the matrices Z_i are scaled to the last ρ_i columns of I_p. To prove the existence of a solution to (26) such that k, $k-1, \ldots, 0$ are the highest powers of s in $D_l(s)$, $W_2(s), \ldots, W_k(s)$ we may backsubstitute

$$\begin{bmatrix} D_l(s) & W_2(s) & \cdots & W_k(s) \end{bmatrix} \left[\begin{array}{c|c} Z_1 & \\ sI_{\rho_1} - Y_1 & \ddots \\ & \ddots \quad -Z_k \\ \times & sI_{\rho_k} - Y_k \end{array}\right] \;=\; 0$$

to obtain

$$W_k \;\overset{\text{def}}{=}\; \begin{bmatrix} 0 \\ I_{\rho_k} \end{bmatrix}$$

$$W_{k-1} \;=\; \begin{bmatrix} 0 & 0 \\ I_{\rho_{k-1}-\rho_k} & 0 \\ 0 & sI_{\rho_k} - Y_k \end{bmatrix}$$

$$W_{k-2} \;=\; \begin{bmatrix} 0 & 0 \\ I_{\rho_{k-2}-\rho_{k-1}} & 0 \\ 0 & M_{k-1}(s) \end{bmatrix}$$

$$\cdots$$

with M_{k-1} a unique $\rho_{k-1} \times \rho_{k-1}$ matrix polynomial of second degree. □

4. EXTRACTION

4.1. Representation and solution: methodology

Both the homogeneous COMP and COMP can be represented in various real linear systems which are underdetermined or square and their dimensions derive from μ, the greatest observability index of the plant. The systems are in the form

$$\left[\; X_0 \quad X_1 \quad \cdots \quad X_x \quad Y_0 \quad Y_1 \quad \cdots \quad Y_y \;\right] S \;=\; \left[\; \Phi_0 \quad \Phi_1 \quad \cdots \quad \Phi_\phi \;\right] \tag{27}$$

where $(X_0, X_1, \ldots, X_x)$ and $(Y_0, Y_1, \ldots, Y_y)$ are sets of unknown constant matrices generating the matrix polynomials

$$X(s) \;=\; X_0 + X_1 s + \cdots + X_x s^x, \qquad Y(s) \;=\; Y_0 + Y_1 s + \cdots + Y_y s^y. \tag{28}$$

$X(s)$ has degree x and number of columns m, while $Y(s)$ has degree y and number of columns p. The matrix S is associated with the given matrix polynomials $D_r(s)$ and $N_r(s)$, represented in constant matrix sets $(D_0, D_1, \ldots, D_d)$ and $(N_0, N_1, \ldots, N_n)$ which are displayed as shifted block structures

$$S \;=\; \left[\begin{array}{cccccccc} D_0 & \cdots & \cdots & D_d & & & & \\ & D_0 & \cdots & \cdots & D_d & & & \\ & & \cdots & \cdots & \cdots & \cdots & & \\ & & & D_0 & \cdots & \cdots & D_d & \\ \hline N_0 & \cdots & N_n & & & & & \\ & N_0 & \cdots & N_n & & & & \\ & & \cdots & \cdots & \cdots & & & \\ & & & N_0 & \cdots & N_n & & \end{array}\right], \tag{29}$$

called resultants, cf. [14] for example. Specifications for $(\Phi_0, \Phi_1, \ldots, \Phi_\phi)$ are subject to Section 4.2.

A crucial observation is that the upper $l = m(1 + x)$ rows of S are linearly independent by the column reducedness of $D_r(s)$, while the remaining rows include those that are linearly dependent. The linearly dependent rows may be determined in a consecutive search. For $i = l + 1,\ l + 2,\ \ldots$ we search, with sufficient accuracy, for the first row of S that depends linearly on the preceding rows. By the shifted block structure of S, if the ith row of S depends linearly on the preceding rows, also the rows $i + p,\ i + 2p,\ \ldots$ depend linearly on the preceding rows. The row i is called a *primary* dependent row, while the rows $i + p,\ i + 2p,\ \ldots$ are called *nonprimary dependent rows.* Having the primary dependent row recorded we delete it altogether with all the nonprimary dependent rows that are associated with it from S.

The procedure is continued until all rows of (29) have been examined, converting (27) into the square nonsingular system

$$\left[\; X_0 \quad X_1 \quad \cdots \quad X_x \quad \hat{Y} \;\right] \hat{S} \;=\; \left[\; \Phi_0 \quad \Phi_1 \quad \cdots \quad \Phi_\Phi \;\right] \tag{30}$$

with the new quantities denoted by hats. The system can be uniquely solved through inversion and the matrix $\left[\; Y_0 \quad Y_1 \quad \cdots \quad Y_y \;\right]$ can be recovered by inserting specific columns at appropriate positions of $\hat{Y}$.

4.2. Homogeneous and particular solutions

Descriptions of the homogeneous and particular solutions may be sought in sparse forms. Such forms rely on a minimum number of parameters, which is desirable for analysis and design. In connection with Lemma 2.5, the forms allow to efficiently characterize all solutions to a particular problem, which is of course appealing.

Lemma 4.1. (Homogeneous Solution) Consider COMP in homogeneous form. Let μ be the greatest observability index of the plant. Then the linear system (30) such that

(i) $x = \mu - 1$,

(ii) $y = \mu$,

(iii) row i of $\begin{bmatrix} \Phi_0 & \Phi_1 & \cdots & \Phi_\Phi \end{bmatrix}$ is the negative of the ith primary dependent row for all $i = 1, 2, \ldots, p$

has a unique solution whose inspection gives the polynomial matrix pair $(-N_l, D_l)$ such that $D_l^{-1} N_l = N_r D_r^{-1}$ is left coprime with D_l in polynomial row-echelon form.

Proof. The solution to (30) is unique because $\hat{S}$ has full row-rank by construction; the solution exists because the right-hand side consists of primary dependent rows. $\begin{bmatrix} Y_0 & Y_1 & \cdots & Y_y \end{bmatrix}$ is set up from $\hat{Y}$ column-wise: columns of I_p are inserted at the positions corresponding to the primary dependent rows and zero $p \times 1$ columns at the positions corresponding to the nonprimary dependent rows of S. □

Lemma 4.2. (Particular Solution) Consider COMP. Let ϕ be the highest degree among the elements of D_k and the condition $r_i \geq \mu - 1$ hold for $i = 1, 2, \ldots, m$. Then the linear system (30) such that

(i) $x = \phi - \max\{k_1, \ldots, k_m\}$,

(ii) $y = \mu - 1$,

(iii) $(\Phi_0, \Phi_1, \ldots, \Phi_\phi)$ are the coefficients of the matrix polynomial $D_k(s)$

has a unique solution whose inspection gives the polynomial matrix pair (X_{lm}, Y_{lm}) such that $X_{lm}^{-1} Y_{lm}$ is proper with the least column degrees of Y_{lm}.

Proof. Analogy to the proof of Lemma 4.1. $\begin{bmatrix} Y_0 & Y_1 & \cdots & Y_y \end{bmatrix}$ is set up from $\hat{Y}$ column-wise: zero $p \times 1$ columns are inserted at the positions corresponding to the dependent rows of S. □

4.3. Recycled rank determination

Compensators whose (McMillan) degree equals that obtained by a conventional method, such as state space, are of particular interest and this is not only because the rank determination problem subject to Section 4.1 can be recycled:

Corollary 4.3. (Recycled Rank Determination) Consider COMP. Let ϕ be the highest degree among the elements of D_k and the condition $r_i \geq \mu - 1$ hold for $i = 1, 2, \ldots, m$ such that

$$\max\{r_1, \ldots, r_m\} \;=\; \mu - 1\,. \tag{31}$$

Then the coefficient matrix $\hat{S}$ in the linear system (30) subject to Lemma 4.1 equals that subject to Lemma 4.2.

Proof. A straightforward application of Definition 1.2. $\square$

5. LOW-LEVEL IMPLEMENTATION

5.1. Simplification by inspection

The structured linear system for the extraction of proper compensators entails zero columns whose explicit formation is undesirable for computations. The positions of such columns are a priori known, given the column powers $k_1, \ldots, k_m$.

If $\max\{r_1, \ldots, r_m\} = \mu - 1$, then the desired particular and homogeneous solutions require S whose number of rows is respectively $(m+p)\mu$ and $(m+p)\mu + p$. In either case there are $(\max\{k_1, \ldots, k_m)\} + \mu)m$ columns of which $k_1 + \cdots + k_m + \mu m$ are nonzero. The nonzero columns are linearly independent, as shown in [16, Theorem 7.3.30, p. 243].

If $\max\{r_1, \ldots, r_m\} > \mu - 1$, then the number of rows and columns is increased by an integer multiple of m, the number of zero columns is intact, and the nonzero columns are linearly independent.

5.2. Transformations

The above full-rank underdetermined system either has no solution or has an infinity of solutions. Not all solutions relate to proper compensators and not all proper compensators suit direct extraction. An indirect extraction is available through Lemma 2.5. The row search in Section 4.1, the column compressions in Section 3.2, as well as the solution of square system in Section 3.2 — all may be implemented as orthogonal transformations, whose stability is guaranteed and unsurpassed when it comes to producing a meaningful solution in cases of ill-conditioning.

Householder reflections are orthogonal transformations that are exceedingly useful for a grand scale annihilation of all but the first component of a vector by properly choosing the reflection plane. A small example illustrates the general idea. Consider a 5×4 system and assume that Householder matrices H_1 and H_2 have been computed so that

$$SH_1H_2 \;=\; \begin{bmatrix} \times & 0 & 0 & 0 \\ \times & \times & 0 & 0 \\ \times & \times & \boxtimes & \boxtimes \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}. \tag{32}$$

We examine the 2-norm of the highlighted vector. If the norm is sufficiently big, then the vector is part of a linearly independent row and we determine a 2 × 2 Householder matrix $\tilde{H}_3$ such that

$$\begin{bmatrix} \boxtimes & \boxtimes \end{bmatrix} \tilde{H}_3 = \begin{bmatrix} \times & 0 \end{bmatrix}. \tag{33}$$

If $H_3 = \operatorname{diag}(I_2, \tilde{H}_3)$, then

$$SH_1H_2H_3 = \begin{bmatrix} \times & 0 & 0 & 0 \\ \times & \times & 0 & 0 \\ \times & \times & \times & 0 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \tag{34}$$

and the following step focuses on the 1 × 1 vector at the $(4,4)$ position of (34). If, on the contrary, the norm is sufficiently small then the vector is part of a linearly dependent row, $H_3 = I_4$, and the following step focuses on the 1 × 2 vector at positions $(4,3)-(4,4)$ of (34). After at most 5 such steps we obtain a list of linearly dependent rows and a lower quasi-triangular $SH_1 \cdots H_5$, which is column compressed by construction. The structure and implementation of Householder matrices, as well as many relevant details, may be found in [5].

5.3. Summary

The design of proper compensators for a strictly proper plant may rely on COMP, a special form of Diophantine equation. Assuming the right-hand side is chosen to give compensators whose (McMillan) degree equals that obtained via conventional methods, such as state space, the design steps are:

COMP setup (Definition 1.3)

(1) compute the greatest observability index μ,

(2) choose D_k with row powers $r_1 = \cdots = r_m = \mu - 1$ and column powers $k_1, \ldots, k_m$,

Representation and transformation

(3) set up the structured system and omit the zero columns,

(4) transform the system and record the positions of the linearly dependent rows,

Particular solution (Definition 2.9)

(5) backsubstitute a triangular system,

(6) insert zero vectors,

Homogeneous solution (Definition 2.7)

(7) backsubstitute the above system for a different right-hand side,

(8) insert columns of I_p and zero columns,

All proper compensators

(9) parametrize as described in Lemma 2.5.

Details on choosing the right-hand side matrix D_k are beyond the scope of this paper and remain an open problem. The above procedure can be modified to accommodate D_k that give compensators whose (McMillan) degree is higher than that obtained by conventional methods. The modification requires an additional structured system; recycling is inapplicable.

6. EXAMPLE

As a continuation to Example 2.4, a simple illustrative example, consider the plant

$$P(s) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s^2+1 & 1 \\ 0 & s+1 \end{bmatrix}^{-1},$$

whose greatest observability index is $\mu = 2$. Choose the right-hand side

$$D_k(s) = \begin{bmatrix} s^3-6s^2+11s-6 & 4s^2+3s+2 \\ 0 & s^2-2s+1 \end{bmatrix},$$

which is row-column reduced with row powers $(1,1)$ and column powers $(2,1)$, in compliance with Lemma 2.2.

Associated with both the homogeneous and particular solution, the system (27) takes on the form

$$[\, X_0 \;\; X_1 \;\; Y_0 \;\; Y_1 \;\; Y_2 \,] \left[\begin{array}{cccccccc} 1&1&0&0&1&0&0&0 \\ 0&1&0&1&0&0&0&0 \\ \hline 0&0&1&1&0&0&1&0 \\ 0&0&0&1&0&1&0&0 \\ \hline 1&1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ \hline 0&0&1&1&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ \hline 0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&1&0&0 \end{array}\right] = \left[\begin{array}{cc|cc|cc|cc} -6 & 2 & 11 & 3 & -6 & 4 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 \end{array}\right].$$

There is a single column of zeros, the rightmost column of the coefficient matrix, whose existence is implied by $k_1 - k_2 = 1$ and which is to be deleted prior to transformation. Householder reflections reduce the system to quasi-triangular form with the coefficient matrix

$$\left[\begin{array}{rrrrrrr}
-1.7321 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.5774 & -1.2910 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.7746 & -1.5492 & 0.0000 & 0.0000 & 0 & 0.0000 \\
0 & -0.7746 & -0.2582 & -1.1547 & 0 & 0.0000 & 0 \\
\hline
-1.1547 & -0.2582 & 0.1291 & 0.1443 & 0.7500 & 0.0000 & 0.0000 \\
-0.5774 & -0.5164 & 0.2582 & 0.2887 & 0.1667 & -0.4714 & 0 \\
0 & -0.7746 & -0.9037 & -0.1443 & -0.0833 & 0.2357 & 0.7071 \\
0 & -0.7746 & -0.2582 & -0.2887 & -0.1667 & 0.4714 & -0.0000 \\
\hline
-0.5774 & 0.2582 & -0.1291 & -1.0104 & -0.5833 & -0.4714 & 0.0000 \\
* & * & * & * & * & * & *
\end{array}\right]$$

and the right hand side

$$\left[\begin{array}{rrrrrrr}
5.7735 & -6.4550 & -6.4550 & -0.2887 & 2.5000 & -7.0711 & 7.0711 \\
-0.5774 & 1.0328 & 0.7746 & 0.0000 & 0.6667 & -1.8856 & 0.0000
\end{array}\right].$$

The 8th row is identified as the primary dependent row. As a result, row 10 is a nonprimary dependent row and hence exempt from computation. This is denoted by $*$. The remaining primary dependent row is row 9.

The extraction of the particular solution starts with the backsubstitution of the system with coefficient matrix and right-hand side respectively given by

$$\left[\begin{array}{rrrrrrr}
-1.7321 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.5774 & -1.2910 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.7746 & -1.5492 & 0.0000 & 0.0000 & 0 & 0.0000 \\
0 & -0.7746 & -0.2582 & -1.1547 & 0 & 0.0000 & 0 \\
\hline
-1.1547 & -0.2582 & 0.1291 & 0.1443 & 0.7500 & 0.0000 & 0.0000 \\
-0.5774 & -0.5164 & 0.2582 & 0.2887 & 0.1667 & -0.4714 & 0 \\
0 & -0.7746 & -0.9037 & -0.1443 & -0.0833 & 0.2357 & 0.7071
\end{array}\right],$$

$$\left[\begin{array}{rr|rr|rr|r}
5.7735 & -6.4550 & -6.4550 & -0.2887 & 2.5000 & -7.0711 & 7.0711 \\
-0.5774 & 1.0328 & 0.7746 & 0.0000 & 0.6667 & -1.8856 & 0.0000
\end{array}\right].$$

The backsubstitution gives

$$\left[\begin{array}{rr|rr|rr|r}
-6.0000 & -12.0000 & 1.0000 & 4.0000 & -0.0000 & 20.0000 & 10.0000 \\
-0.0000 & -3.0000 & 0.0000 & 1.0000 & -0.0000 & 4.0000 & 0.0000
\end{array}\right],$$

allowing to conclude the extraction by inserting the zero 2 × 1 vector at the position corresponding to the (deleted) 8th primary dependent row. The result,

$$\left[\begin{array}{cccc} X_0 & X_1 & Y_0 & Y_1 \end{array}\right] = \left[\begin{array}{rr|rr|rr|rr}
-6 & -12 & 1 & 4 & 0 & 20 & 10 & 0 \\
0 & -3 & 0 & 1 & 0 & 4 & 0 & 0
\end{array}\right],$$

describes the compensator transfer function

$$C(s) \quad = \quad \begin{bmatrix} s-6 & 4s-12 \\ 0 & s-3 \end{bmatrix}^{-1} \begin{bmatrix} 10s & 20 \\ 0 & 4 \end{bmatrix}.$$

The extraction of the homogeneous solution starts with the backsubstitution of the system with a coefficient matrix as above and a right-hand side made-up of the primary dependent rows taken with a negative sign,

$$-\left[\begin{array}{rr|rr|rr|r} 0 & -0.7746 & -0.2582 & -0.2887 & -0.1667 & 0.4714 & -0.0000 \\ -0.5774 & 0.2582 & -0.1291 & -1.0104 & -0.5833 & -0.4714 & 0.0000 \end{array}\right].$$

The backsubstitution gives

$$\left[\begin{array}{rr|rr|rr|r} -0.0000 & -1.0000 & -0.0000 & -0.0000 & 0.00 & 1.00 & 0.00 \\ -1.0000 & 1.0000 & 0.0000 & -1.0000 & 1.00 & -1.00 & -0.00 \end{array}\right],$$

allowing to conclude the extraction by inserting columns of I_2 and the 2×1 column of zeros at the positions corresponding to respectively the primary dependent rows and the nonprimary dependent row. The result,

$$\left[\begin{array}{ccccc} X_0 & X_1 & Y_0 & Y_1 & Y_2 \end{array}\right] = \left[\begin{array}{rr|rr|rr|rr|rr} 0 & -1 & -0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \end{array}\right],$$

describes the plant transfer function

$$P(s) \quad = \quad \begin{bmatrix} 0 & s+1 \\ s^2+1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & s-1 \end{bmatrix}.$$

7. CONCLUSIONS

In a polynomial matrix context with data- and parameter-degree control it is possible to characterize all proper feedback compensators whose left denominator is row reduced with sufficiently large prescribed row degrees. The characterization via conventional parametrization [1], [7] admits a variety of forms which are distinct by the compensator at the parametrization's centre and by the affine terms that enable to navigate off the centre. In this paper, an appropriate centre and affine terms are proposed and extracted with system theoretical and numerical issues in mind. In effect, all proper feedback compensators are extracted.

The numerical extraction is based on ideas that may be traced to [14] and is performed as backsubstitution of a square system which is obtained through concatenation of orthogonal transformations. To our knowledge the method is novel in the context of COMP. The compensator at the centre and the affine terms match each other in relation to polynomial row-echelon form and are expected to enhance the applicability of the parametrization. On the numerical level, orthogonal transformations are recognized for reformulation of the problem in a new coordinate system which is more appropriate for solving the problem, and this without affecting its sensitivity. Next, they can be performed in a numerically stable manner because numerical errors resulting from previous steps are maintained in norm throughout subsequent steps.

The polynomial mfd parametrization can be shown consistent with a similar result over $\mathbb{R}\mathcal{H}_\infty$ in [13, Section 5.2]. Hence it should be useful for appropriate plant feedback stabilization in optimization or tracking contexts, see also [7], [3, Section 7.3]. It is obvious that the results can be dualized for the case that the plant is given as a left polynomial matrix fraction.

ACKNOWLEDGEMENT

The assistance of Professor Frank M. Callier in improving the text readability is gratefully acknowledged.

(Received April 2, 2002.)

REFERENCES

[1] F. M. Callier: Proper feedback compensators for a strictly proper plant by solving polynomial equations. In: Proc. MMAR 2000. Miedzyzdroje 2000, pp. 55–59.

[2] F. M. Callier: Polynomial equations giving a proper feedback compensator for a strictly proper plant. In: Proc. 1st IFAC Symposium on System Structure and Control, Prague 2001.

[3] F. M. Callier and C. A. Desoer: Multivariable Feedback Systems. Springer, New York 1982.

[4] G. D. Forney: Minimal bases of rational vector spaces, with applications to multivariable linear systems. SIAM J. Control *13* (1975), 493–520.

[5] G. H. Golub and C. F. Van Loan: Matrix Computations. North Oxford Academic, Oxford 1989.

[6] T. Kailath: Linear Systems. Prentice Hall, Englewood Cliffs, N. J. 1980.

[7] V. Kučera and P. Zagalak: Proper solutions of polynomial equations. In: Proc. IFAC World Congress, Pergamon, Oxford 1999, pp. 357–362.

[8] C. C. MacDuffee: The Theory of Matrices. Springer, Berlin 1933.

[9] V. M. Popov: Some properties of the control systems with irreducible matrix transfer functions. (Lecture Notes in Mathematics 144.) Springer, Berlin 1969, pp. 169–180.

[10] H. H. Rosenbrock: State-space and Multivariable Theory. Nelson, London 1970.

[11] H. H. Rosenbrock and G. E. Hayton: The general problem of pole assignment. Internat. J. Control *27* (1978), 837–852.

[12] P. M. Van Dooren: The generalized eigenstructure problem in linear system theory. IEEE Trans. Automat. Control *AC-26* (1981), 111–129.

[13] M. Vidyasagar: Control System Synthesis. MIT Press, Cambridge, MA 1985.

[14] S. H. Wang and E. J. Davison: A minimization algorithm for the design of linear multivariable systems. IEEE Trans. Automat. Control *AC-18* (1973), 220–225.

[15] J. Wedderburn: Lectures on Matrices. American Mathematical Society, Providence, R.I. 1934.

[16] W. A. Wolovich: Linear Multivariable Systems. Springer, New York 1974.
[17] P. Zagalak and V. Kučera: The general problem of pole assignment: a polynomial approach. IEEE Trans. Automat. Control *AC-30* (1985), 286–289.

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