ON THE STABILIZABILITY OF SOME CLASSES OF BILINEAR SYSTEMS IN $\mathbb{R}^3$

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In this paper, we consider some classes of bilinear systems. We give sufficient condition for the asymptotic stabilization by using a positive and a negative feedbacks.

1. INTRODUCTION

Stabilizability of bilinear systems of the form

$$\dot{x} = Ax + uBx \quad (1)$$

(where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $A$, $B$ are constant real matrices ($n \times n$)) has widely studied in the past years by many authors (see e.g. [1–13]). In [4], the authors give a necessary and sufficient condition, algebraically computable, for the global stabilization of the planar bilinear systems

$$\left\{ \begin{array}{l}
\dot{z} = \tilde{A}z + v\tilde{B}z \\
z \in \mathbb{R}^2, \quad v \in \mathbb{R} \quad \text{and} \quad \tilde{A}, \quad \tilde{B} \in M(2, \mathbb{R})
\end{array} \right. \quad (2)$$

It turns out that the stabilizability by homogeneous feedback is equivalent to the asymptotic controllability to the origin which is equivalent to the stabilizability. Moreover, they show that there exists a large class of planar bilinear systems that are not $C^1$ stabilizable but stabilizable by means of homogeneous feedback of the form $v(z) = \frac{Q_1(z)}{Q_2(z)}$, where $Q_1$ is a quadratic form and $Q_2$ is a positive-definite quadratic form.

For the three dimensional case, in [3] the authors deal with a particular class of bilinear systems of form (1) with $A$ diagonal and $B$ skew symmetric. For these systems a necessary and sufficient condition for global asymptotic stabilization by constant feedback and a sufficient condition for stabilization by a family of linear feedbacks are given. Another interesting problem is considered in the literature. The question is: does the local asymptotic stabilizability of (1) imply the global asymptotic stabilizability? More precisely let us assume that there exists a feedback law (locally defined) $u : x \mapsto u(x)$ such that the closed system $\dot{x} = Ax + u(x)Bx \quad (\Sigma)$
is locally asymptotically stable about the origin. Does there exist a feedback law (globally defined) \( \tilde{u}(x) \) which makes the origin of (1) globally asymptotically stable?

To the closed-loop system (\( \Sigma \)) a positive-definite function \( V \) (locally defined) is associated, such that \( \dot{V}(x) \) (the derivative of \( V \) along the trajectories of system (\( \Sigma \))) is negative definite.

Hammouri and Marques [8], proved that local asymptotic stabilizability implies global asymptotic stabilizability under some assumption on the level surfaces of the Lyapunov function related to system (\( \Sigma \)). Andriano [1] assert, that the answer to the above question is yes without any assumption on the level surfaces of the Lyapunov function. In [5] the authors clarify the result of Andriano given in [1].

This work is a contribution to the study of stabilization of bilinear systems by homogeneous feedback. The results concern single-input bilinear systems of the form

\[
\dot{x} = Ax + uBx
\]

where \( x \in \mathbb{R}^3 \), \( u \in \mathbb{R} \) and \( TA, TB \) two matrices supposed have a same eigenvector (\( TA \) denotes the transpose of matrix \( A \)). In a suitable basis matrices \( A \) and \( B \) can be written as

\[
A = \begin{pmatrix}
a_{(1,1)} & 0 & 0 \\
a_{(2,1)} & a_{(2,2)} & a_{(2,3)} \\
a_{(3,1)} & a_{(3,2)} & a_{(3,3)}
\end{pmatrix}, \quad B = \begin{pmatrix}
b_{(1,1)} & 0 & 0 \\
b_{(2,1)} & b_{(2,2)} & b_{(2,3)} \\
b_{(3,1)} & b_{(3,2)} & b_{(3,3)}
\end{pmatrix}.
\]

We define matrices \( \tilde{A} \) and \( \tilde{B} \) as

\[
\tilde{A} = \begin{pmatrix}
a_{(2,2)} & a_{(2,3)} \\
a_{(3,2)} & a_{(3,3)}
\end{pmatrix} = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix}
b_{(2,2)} & b_{(2,3)} \\
b_{(3,2)} & b_{(3,3)}
\end{pmatrix}.
\]

We suppose more that system (2) is not stabilizable by a constant feedback and \( \tilde{B} \) is not diagonalizable.

In this paper we show how to compute the homogeneous feedback of the system (3) when the planar bilinear system (2) is stabilizable by a positive and negative feedback.

The paper is organized as follows. In Section 2, for the convenience of the reader we recall two results of constant use in the sequel.

Section 3: In the case where the eigenvalues of \( B \) associate to the common eigenvector of \( TA \) and \( TB \) is zero we give a necessary and sufficient condition for the stabilizability of system (3), the feedback is given explicitly. Next we prove that if the planar bilinear system is globally asymptotically stable (GAS) by a feedback \( v(z) \) such that \( b_{1,1}v(z) < 0 \) then system (2) is GAS.

In Section 4, we suppose that the system (2) is not stabilizable by a constant feedback and \( \tilde{B} \) is not diagonalizable. As an application of the last result of the Section 2, we give a necessary and sufficient condition, algebraically computable, for the global stabilization of the planar bilinear systems by a positive and negative feedback.
2. TWO RESULTS ON STABILIZATION

We recall the following theorem, because we need these results to prove that we can stabilize some bilinear system by a positive and negative feedback (see Theorems 5, 6, 7 and 8).

**Theorem 1.** Consider the two-dimensional system,

\[ T[\dot{z}_1, \dot{z}_2] = [f_1(z_1, z_2), f_2(z_1, z_2)] \]

where \( T[f_1, f_2] \) is Lipschitz continuous and is homogeneous of degree \( p \). Then the system is asymptotically stable if and only if one of the following is satisfied:

(i) The system does not have any one-dimensional invariant subspaces and

\[
I = \int_0^{2\pi} \frac{\cos \theta f_1(\cos \theta, \sin \theta) + \sin \theta f_2(\cos \theta, \sin \theta)}{\cos \theta f_2(\cos \theta, \sin \theta) - \sin \theta f_1(\cos \theta, \sin \theta)} \, d\theta
\]

\[
= \int_{-\infty}^{+\infty} \frac{f_1(1, s)}{f_2(1, s) - sf_1(1, s)} \, ds < 0
\]

or

(ii) The restriction of the system to each of its one-dimensional invariant subspaces is asymptotically stable.

For a proof see [2, 7].

In the sequel we use constantly a result of asymptotic stability using positive-semidefinite function. The theorem can be found in [9] or [10], we use the formulation of [10]. Consider the differential equation

\[
(\Gamma) \begin{cases} 
\dot{x} = X(x) \\
X(0) = 0 
\end{cases}
\]

where \( X \) is a smooth vector field on \( \mathbb{R}^n \). For a differentiable function \( V \), we denote the action of \( X \), considered as a differential operator, on \( V \) by \( XV \), which is defined by

\[
XV(x) = \frac{d}{dt} V(X_t(x))_{t=0}
\]

\( X_t(x_0) \) is the solution of (\( \Gamma \)) starting at \( x_0 \), i.e. \( \frac{d}{dt} X_t(x_0) = X(X_t(x_0)) \) and \( X_0(x_0) = x_0. \)
**Theorem 2.** We suppose that there exists a function \( V \in C^1(\mathbb{R}^n, \mathbb{R}) \) such that

1. \( V(x) \geq 0 \) for all \( x \in \mathbb{R}^n \) and \( V(0) = 0 \)
2. \( \dot{V}(x) = XV(x) \leq 0. \)

We denote by \( \mathcal{L} \) the largest positively invariant set of \( X \) contained in \( M = \{ x \in \mathbb{R}^n : \dot{V}(x) = 0 \} \).

If the origin is asymptotically stable with respect to the system \((\Gamma)\) restricted to \( \mathcal{L} \), then the origin is asymptotically stable.

3. **MAIN RESULT**

Consider a single input bilinear system \((3)\), we suppose that the \(TB\) and \(TA\) have a same eigenvector.

We recall that, in a suitable basis of \(\mathbb{R}^3\), the matrices \(A, B\) take the following forms

\[
A = \begin{pmatrix}
a_{(1,1)} & 0 & 0 \\
a_{(2,1)} & a_{(2,2)} & 0 \\
a_{(3,1)} & a_{(3,2)} & a_{(3,3)} \\
\end{pmatrix}, \quad B = \begin{pmatrix}
b_{(1,1)} & 0 & 0 \\
b_{(2,1)} & b_{(2,2)} & b_{(2,3)} \\
b_{(3,1)} & b_{(3,2)} & b_{(3,3)} \\
\end{pmatrix}
\]

and \(\tilde{A}, \tilde{B}\) as follows

\[
\tilde{A} = \begin{pmatrix}
a_{(2,2)} & a_{(2,3)} \\
a_{(2,3)} & a_{(3,3)} \\
\end{pmatrix}, \quad \tilde{B} = \begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix}
\]

For the sake of clarity, we set \(x = (x_1, x_2, x_3) = (x_1, z)\) where \(z = (x_2, x_3)\). We denotes \(a_{(1,1)} = \alpha\) and \(\beta = b_{(1,1)}\)

**Theorem 3.** In the case when \(\beta = 0\) we can assume that:

The bilinear system \((3)\) is GAS if and only if \(\alpha < 0\) and the planar bilinear system \(\dot{z} = \tilde{A}z + \tilde{B}z, \ (2)\) is GAS.

If system \((2)\) is GAS by the feedback law \(v(z) = Q_1(z) / Q_2(z)\) (where \(Q_1\) is a quadratic form and \(Q_2\) is a positive-definite quadratic form), then the feedback \(u(x) = Q_1(z) / (Q_2(z) + x_1^2)\) stabilizes the system \((3)\).

**Proof.** The system \((3)\) takes the form

\[
\begin{cases}
\dot{x}_1 = \alpha x_1 \\
\dot{x}_2 = x_1 \left( \frac{a_{(2,1)}}{a_{(3,1)}} \right) + u x_1 \left( \frac{b_{(2,1)}}{b_{(3,1)}} \right) + \tilde{A} \left( \begin{array}{c} x_2 \\ x_3 \end{array} \right) + u \tilde{B} \left( \begin{array}{c} x_2 \\ x_3 \end{array} \right).
\end{cases}
\]

\[(4)\]

It is clear that it is necessary for the stabilizability of \((4)\) that \(\alpha < 0\) and the system \(\dot{z} = \tilde{A}z + vBz\) is stabilizable.

We suppose now that these two conditions are satisfied. Let \(v(z) = Q_1(z) / Q_2(z)\) be a stabilizing homogeneous feedback for \(\dot{z} = \tilde{A}z + vBz\), where \(Q_1\) is a quadratic form.
and $Q_2$ is a positive-definite quadratic form. We define $u(x_1, x_2, x_3) = \frac{Q_1(x)}{Q_2(x) + x_1^2}$ on $\mathbb{R}^3$. This feedback is homogeneous and $C^\infty$ in $\mathbb{R}^3 \setminus \{0\}$. We denote by $X(x) = \dot{A}(x) + u(x) B(x)$ the vectors field of the closed-loop system. We shall prove, by using Theorem 2, that this system is asymptotically stable. It is clear that the vectors field $X$ and $Y$ where $Y(x) = \left(Q_2(\dot{x}) + ||x||^2\right) X(x)$ have the same orbits. We choose $V(x) = x_1^2$, $V$ is clearly positive-semi-definite on $\mathbb{R}^3$. Since $\alpha < 0$, then

$$YV(x) = \dot{V}(x) = \alpha (Q_2(x) + ||x||^2)||x||^2 \leq 0.$$  

Let $M$ be the set $M = \{x \in \mathbb{R}^3 : \dot{V}(x) = 0\}$. It is clear that $M$ is $\{0\} \times \mathbb{R}^2$ in $\mathbb{R}^3$. Then the vectors field $Y$ is reduced on $M$ to $\dot{z} = Q_2(\dot{z})(\dot{A}z + u(0, z)Bz$.

Since $Q_2(z)$ is positive-definite and $\dot{z} = \dot{A}z + \nu(z)Bz$ is asymptotically stable, then $Y$ and hence $X$, is asymptotically stable.

In the case when $\beta \neq 0$ and without loss of generality we can suppose that $a_{(1,1)} = a < 0$. □

**Theorem 4.** If the planar bilinear system $\dot{z} = \dot{A}z + \nu Bz$ (2) is globally asymptotically stabilizable by a feedback law of the form $\nu(z) = \frac{Q_1(z)}{Q_2(z)}$ such that $\beta \nu(z) \leq 0$ $\forall z \in \mathbb{R}^2 - \{(0,0)\}$ then the feedback $u(x) = \frac{Q_1(x)}{Q_2(x) + x_1^2}$ stabilizes the system (3).

**Proof.** It is straightforward that the the closed-loop system (3) with the feedback

$$u(x) = \frac{Q_1(x)}{Q_2(x) + x_1^2}$$

$$\begin{cases} 
\dot{x}_1 = \alpha x_1 + u\beta x_1 \\
\dot{x}_2 = x_1 \left(\begin{array}{c} a_{(2,1)} \\
a_{(3,1)} \end{array}\right) + ux_1 \left(\begin{array}{c} b_{(2,1)} \\
b_{(3,1)} \end{array}\right) + \dot{A} \left(\begin{array}{c} x_2 \\
x_3 \end{array}\right) + u\dot{B} \left(\begin{array}{c} x_2 \\
x_3 \end{array}\right) 
\end{cases}$$

(5)

is GAS. □

The proof is organized as the proof of the preceding theorem. Since this system is in triangular forms (see [10]), and $\dot{z} = \dot{A}z + u(0, z)Bz$ is GAS, then the equation (5) is GAS.

4. STABILIZABILITY OF PLANAR BILINEAR SYSTEM BY A POSITIVE AND NEGATIVE FEEDBACK

In this section we consider the planar bilinear system $\dot{z} = \dot{A}z + \nu Bz$ (2) which it not be stabilizable by a constant feedback. We suppose that matrix $\dot{B}$ is not diagonalizable.

As an application of Theorem 4, in this section we construct a positive and negative feedbacks who's stabilize the planar bilinear systems (2).
4.1. $\tilde{B}$ have no real eigenvalues

In a suitable basis of $\mathbb{R}^2$, the matrix $\tilde{B}$ takes the following form

$$\tilde{B} = \begin{pmatrix} \nu & \mu \\ -\mu & \nu \end{pmatrix}.$$ 

For the sake of clarity, we set $z = (z_1, z_2)$, $\tilde{z}_1 = e_1 z_1 + e_2 z_2$ and $\tilde{z}_2 = -e_2 z_1 + e_1 z_2$, where $e_1 = (a - d) - \sqrt{(b + c)^2 + (a - d)^2}$ and $e_2 = (b + c)$.

According to the assumption that system (2) is not stabilizable by a constant feedback, it is the classification of planar bilinear systems we are speaking of [4], we have

(i) $\text{Tr}(\tilde{A}) > 0$, $\text{Tr}(\tilde{B}) = 0$ and $(b + c)^2 - 4ad > 0$.

**Theorem 5.** If the condition (i) is satisfied then for $t_1 > 0$ and $t_2 > 0$ large enough and $\tilde{d} = \sqrt{\frac{t_1}{t_2}} + \tilde{a} > 0$ the positive feedback law

$$v_1(z_1, z_2) = \frac{t_1 \tilde{z}_1^2 + (\tilde{d} - \tilde{a}) \tilde{z}_1 \tilde{z}_2 + t_2 \tilde{z}_2^2}{\mu(\tilde{z}_1^2 + \tilde{z}_2^2)} + \frac{c-b}{2\mu}$$

and the negative feedback law

$$v_2(z_1, z_2) = -\frac{t_1 \tilde{z}_1^2 + (\tilde{a} - \tilde{d}) \tilde{z}_1 \tilde{z}_2 + t_2 \tilde{z}_2^2}{\mu(\tilde{z}_1^2 + \tilde{z}_2^2)} + \frac{c-b}{2\mu}$$

where

$$\tilde{a} = (ae_1^2 + (c+b)e_1 e_2 + de_2^2)/(e_1^2 + e_2^2)$$

and

$$\tilde{d} = (ae_2^2 - (c+b)e_1 e_2 + de_1^2)/(e_1^2 + e_2^2)$$

stabilize the system (2).

**Proof.** We consider the closed-loop system (2) by the feedback $v_1(z_1, z_2)$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} Z_1(z_1, z_2) \\ Z_2(z_1, z_2) \end{pmatrix} = (\tilde{z}_1^2 + \tilde{z}_2^2) \left[ \tilde{A} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + v_1(z_1, z_2) \tilde{B} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right].$$

Since the function $(\tilde{z}_1^2 + \tilde{z}_2^2)$ is positive-definite then there is equivalence between asymptotic stability of the vector field $Z = (Z_1, Z_2)$ and the closed-loop bilinear system (2), defining the function $F$ as follows

$$F(z_1, z_2) = z_1 Z_2(z_1, z_2) - z_2 Z_1(z_1, z_2)$$

a simple computation gives

$$F(z_1, z_2) = -(t_1 \tilde{z}_1^2 + t_2 \tilde{z}_2^2)(\tilde{z}_1^2 + \tilde{z}_2^2).$$

From Theorem 1 one can deduce that the vector field $Z$ is GAS if and only if

$$I = \int_{-\infty}^{+\infty} \frac{Z_1(1, s)}{F(1, s)} \, ds < 0.$$
It is easy to verify that
\[
\tilde{a} \tilde{d} < 0 \quad I = -\frac{\pi}{t_2} \frac{t_1}{t(t+1)} \quad \text{where} \quad t = \sqrt{\frac{t_1}{t_2}}.
\]

Since \( t_1 \) and \( t_2 \) have been chosen large enough such that \( v_1(z_1, z_2) > 0 \) and \( \tilde{d} + \tilde{a} > 0 \), then \( I < 0 \).

From the fact that the vector field \( Z \) is GAS, we can assume that the differential equation
\[
\begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix} = \begin{pmatrix}
a & -b \\
-c & d
\end{pmatrix} \begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix} - \frac{b-c}{\mu(\xi_1^2 + \xi_2^2)} \begin{pmatrix}
0 & -\mu \\
\mu & 0
\end{pmatrix} \begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix}
\]
\[
\xi_1 = e_1 \xi_1 - e_2 \xi_2 \quad \text{and} \quad \xi_2 = e_2 \xi_1 + e_1 \xi_2 \quad \text{is GAS. Under a linear change in the state space of the form} \quad z_1 = \xi_1 \quad \text{and} \quad z_2 = -\xi_2 \quad \text{the differential equation} \quad (6) \quad \text{becomes}
\]
\[
\dot{z} = \bar{A}z + v_2(z_1, z_2)\bar{B}z \quad \text{which it GAS.}
\]

\( \square \)

4.2. Case where the eigenvalues of \( B \) are real without \( B \) being diagonalizable

In a suitable basis of \( \mathbb{R}^2 \), matrices \( \bar{A} \) and \( \bar{B} \) take the following forms
\[
\bar{A} = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \quad \bar{B} = \begin{pmatrix}
\lambda & 1 \\
0 & \lambda
\end{pmatrix}.
\]

According to the assumption that the system (2) is not stabilizable by a constant feedback, it is the classification of planar bilinear systems we are speaking of [4], we have

(i) \( \text{Tr}(\bar{A}) > 0, \text{Tr}(\bar{B}) = 0 \) and \( \text{Tr}(\bar{A}\bar{B}) = c \neq 0 \);

(ii) \( \text{Tr}(\bar{A}) = \text{Tr}(\bar{B}) = 0 \) and \( \text{Tr}(\bar{A}\bar{B}) = c \neq 0 \).

Without loss of generality, we can suppose that \( \text{Tr}(\bar{A}\bar{B}) = c > 0 \).

4.2.1. Case when \( \text{Tr}(\bar{A}) > 0 \)

We will treated separately the two subcases: \( 4bc + (a-d)^2 < 0 \) and \( 4bc + (a-d)^2 \geq 0 \).

The sub case when \( 4bc + (a-d)^2 \geq 0 \).
Theorem 6. If the condition (i) is satisfied then the negative feedback

\[ v(z) = -b - \frac{(a-d)^2}{4c} - \frac{(d+a)^2 P(z)}{4cQ(z)} \]

where

\[ Q(z) = (cz_1 - (d+2a)z_2)^2 + \frac{29}{4}(d+a)^2 z_2^2 \]
\[ P(z) = 26c^2 z_1^2 + (88ac + 140cd)z_1 z_2 + \left( \frac{849}{2}d^2 + 709ad + \frac{621}{2}a^2 \right) z_2^2 \]

stabilizes the system (2).

Proof. Suppose that, the condition (i) is satisfied. We consider the Lyapunov function,

\[ V(z) = \left( c^2 z_1^2 - \frac{dc + 5ac}{2} z_1 z_2 - \left( \frac{17}{2}a^2 + \frac{39}{2}da + 10d^2 \right) z_2^2 \right)^2 + \frac{133}{4} \left( \frac{d^2 - a^2}{2} z_2^2 + c(d+a)z_1 z_2 \right)^2 \]

for the system (2), and the feedback law

\[ v(z) = -b - \frac{(a-d)^2}{4c} - \frac{(d+a)^2 P(z)}{4cQ(z)} \]

One can verify that, \( V \) is positive-definite and the feedback law is homogeneous of degree zero. A simple computation gives

\[ \dot{V}(z) = \frac{-(a+d)D(z)R(z)}{Q(z)} < 0 \quad \forall \ z \neq 0 \]

where \( R(z) = (cz_1 + \frac{d-a}{2} z_2)^2 + \frac{19}{2}(a+d)^2 z_2^2 \), and

\[ D(z) = \left( c^2 z_1^2 - (cd + 3ac) z_1 z_2 - \left( \frac{33}{4}a^2 + \frac{39}{2} ad + \frac{41}{2} d^2 \right) z_2^2 \right)^2 + 33 \left( \frac{d^2 - a^2}{2} z_2^2 + c(d+a)z_1 z_2 \right)^2. \]

This prove that the feedback \( v(z) \) stabilizes the system (2).

Since \( P(z) - Q(z) = 25 \left( cz_1 + \frac{(71d+46a)}{25} z_2 \right)^2 + \frac{21461}{100} (a+d)^2 z_2^2 \), is positive-definite then

\[ \frac{P(z)}{Q(z)} > 1, \quad \forall \ z \neq 0. \]

Tacking into account the fact that \( 4bc + (a-d)^2 \geq 0 \) and \( \frac{P(z)}{Q(z)} > 1 \) then

\[ v(z) < -\frac{(d+a)^2}{4c} - b - \frac{(a-d)^2}{4c} < 0. \]
Proposition 1. The system (2) is not stabilizable by a positive feedback.

Proof. Consider the linear change of coordinates whose transformation matrix is given by

\[ P = \begin{pmatrix} 1 & \frac{d-a}{2c} \\ 0 & 1 \end{pmatrix}. \]

The matrix \( \tilde{B} \) keep its initial form and the matrix \( \tilde{A} \) becomes

\[ \tilde{A} = \begin{pmatrix} \frac{a+d}{2} & b + \frac{(a-d)^2}{4c} \\ c & \frac{a+d}{2} \end{pmatrix}. \]

In the new basis it is easy to verify that the set \( H = \{ (z_1, z_2) \in \mathbb{R}^2 \text{ such that } z_1 > 0, z_2 = 2 \} \) is invariant by the open loop system \( \dot{z} = \tilde{A}z + v\tilde{B}z \) where \( v \) lie in \( \mathbb{R}^+ \).

The sub case where \( 4bc + (a-d)^2 < 0 \). Under a change in input state of the form \( \tilde{v} = \left( \frac{2c}{\sqrt{-4bc-(a-d)^2}} \right)v \) and if we consider the linear change of coordinates whose transformation matrix is given by

\[ P = \begin{pmatrix} 1 & \frac{d-a}{2c} \\ 0 & \frac{\sqrt{-4bc-(a-d)^2}}{2c} \end{pmatrix}. \]

The matrix \( \tilde{B} \) keep its initial form and the matrix \( \tilde{A} \) becomes

\[ \tilde{A} = \begin{pmatrix} \frac{a+d}{2} & -\tilde{c} \\ \tilde{c} & \frac{a+d}{2} \end{pmatrix} \]

where \( \tilde{c} = \frac{\sqrt{-4bc-(a-d)^2}}{2} \).

In the new basis, we prove the following result.

Theorem 7. If the condition (i) is satisfied, then for \( -t > 0 \) large enough the negative feedback

\[ \tilde{v}(z) = \left( \frac{a+d}{2} \right) \left( \frac{-z_1^2 + \left( -\frac{10c}{a+d} + p \right)z_1z_2 + (t+7)z_2^2}{2c} \right) \]

where

\[ p = -15\frac{(a+d)}{c} - 8\frac{(a+d)^2 + c^2}{c(a+d)^2} \]

stabilizes the system (2).

Proof. Suppose that, the condition (i) is satisfied. We consider the closed-loop system (2) by the feedback \( \tilde{v}(z) \)

\[
\begin{pmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{pmatrix}
= \begin{pmatrix}
Z_1(z_1, z_2) \\
Z_2(z_1, z_2)
\end{pmatrix}
= \begin{pmatrix}
\frac{2c}{a+d}z_1^2 - 5z_1z_2 + \left( \frac{7a + 7d}{2c} \right)z_2^2 \\
\tilde{A} \begin{pmatrix}
z_1 \\
z_2
\end{pmatrix} + \tilde{v}(z_1, z_2)\tilde{B} \begin{pmatrix}
z_1 \\
z_2
\end{pmatrix}
\end{pmatrix}.
\]
Since the function \((\frac{2c}{a+d}z_1^2 - 5z_1z_2 + (\frac{7a+7d}{2c})z_2^2)\) is positive-definite then there is equivalence between asymptotic stability of the vector field \(Z = (Z_1, Z_2)\) and the closed-loop bilinear system (2), defining the function \(F\) as follows:

\[
F(z_1, z_2) = z_1Z_2(z_1, z_2) - z_2Z_1(z_1, z_2)
\]
a simple computation gives

\[
F(z_1, z_2) = \left(\frac{2c^2}{a+d}\right) z_1^4 - 5cz_1^3z_2 + \left(4a + 4d + \frac{2c^2}{a+d}\right) z_1^2z_2^2 - \frac{pa + pd}{2} z_1^3z_2^2 - \frac{ta + td}{2} z_2^4.
\]

It is clear that for \(-t > 0\) large enough \(F\) is a positive-definite function. From Theorem 1 one can deduce that the vector field \(Z\) is GAS if and only if \(I = \int_{-\infty}^{+\infty} \frac{Z_1(1,y)}{F(1,y)} \, dy < 0\). It is easy to verify that

\[
I = \int_{-\infty}^{+\infty} \frac{Z_1(1,y)}{F(1,y)} \, dy = \frac{a + d}{8} \int_{-\infty}^{+\infty} \left(-\frac{2c}{a+d} - (8 + 2(\frac{2c}{a+d})^2)y + (\frac{28a+28d}{2c} + p)y^2\right) \, dy.
\]

Since \(p = -15(\frac{a+d}{c}) - 8(\frac{a+d}{c})^2 + c^2\), then we can verify that \(-\left(\frac{2c}{a+d}\right) - (8 + 2(\frac{2c}{a+d})^2)y + (\frac{28a+28d}{2c} + p)y^2 < 0 \ \forall y \in \mathbb{R}\). Consequently, the proof of theorem follows from Theorem 1.

**Theorem 8.** If the condition (i) is satisfied, then for \(t > 0\) large enough the positive feedback

\[
\bar{u}(z) = \frac{(2\bar{c}(a + d) + \bar{c}^3(a + d)/2 + 8\bar{c}t)\bar{z}_1^2 - 8(a + d)z_1\bar{z}_2 - (16(a + d)/\bar{c} + 8\bar{c}t)\bar{z}_2^2}{(\bar{c}^2(a + d)/2)\bar{z}_1^2 + 8\bar{z}_2^2}
\]

stabilizes the system (2).

**Proof.** Suppose that, the condition (i) is satisfied. We consider the closed-loop system (2) by the feedback \(\bar{u}(z)\)

\[
\begin{pmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{pmatrix} = \begin{pmatrix}
Z_1(z_1, z_2) \\
Z_2(z_1, z_2)
\end{pmatrix} = \begin{pmatrix}
\bar{c}^2 \\
8(a + d)^2 z_1^2 + t z_2^2
\end{pmatrix} \begin{pmatrix}
\bar{A}
\end{pmatrix} \begin{pmatrix}
z_1 \\
z_2
\end{pmatrix} + \bar{v}(z_1, z_2)\bar{B} \begin{pmatrix}
z_1 \\
z_2
\end{pmatrix}.
\]

Since the function \(\frac{\bar{c}^2}{8(a + d)^2}z_1^2 + tz_2^2\) is positive-definite, then there is equivalence between asymptotic stability of the vector field \(Z = (Z_1, Z_2)\) and the closed-loop bilinear system (2), defining the function \(F\) as follows

\[
F(z_1, z_2) = z_1Z_2(z_1, z_2) - z_2Z_1(z_1, z_2)
\]
a simple computation gives

\[
F(z_1, z_2) = \left(\frac{a + d}{2}\right) \left(\frac{\bar{c}}{a + d}z_1 + z_2\right)^2 \left(\frac{-\bar{c}}{a + d}z_1^2 + 2z_1z_2 - 2\frac{a + d}{\bar{c}}z_2^2\right).
\]
It is easy to see that, if $(1, \xi)$ verify $F(1, \xi) = \xi X_1(1, \xi) - X_2(1, \xi) = 0$ then there exists $\nu \in \mathbb{R}$ such that $(X_1(1, \xi), X_2(1, \xi)) = \nu(1, \xi)$.

In our case we have $\xi = \frac{-\bar{v}}{a+d}$ and $\nu = \frac{\bar{v}+(a+d)\xi/2}{\xi} = -(a+d)/2 < 0$. Consequently, the proof of theorem follows from Theorem 1. \hfill \Box

4.2.2. Case where $\text{Tr}(\hat{A}) = \text{Tr}(\hat{B}) = 0$

Under the assumptions that $\text{Tr}(\hat{A}) = \text{Tr}(\hat{B}) = 0$ and $\text{Tr}(\hat{A} \hat{B}) = c > 0$, and in suitable basis of $\mathbb{R}^2$, the matrices $\hat{A}$ and $\hat{B}$ can be written as $\hat{A} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, $\hat{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ where $c > 0$ consider the linear change of coordinates whose transformation matrix is given by

$$P = \begin{pmatrix} 1 & \frac{-a}{c} \\ 0 & 1 \end{pmatrix}.$$ 

The matrix $\hat{B}$ keep its initial form and the matrix $\hat{A}$ becomes

$$\tilde{A} = \begin{pmatrix} 0 & b + \frac{a^2}{c} \\ c & 0 \end{pmatrix}.$$ 

In the new basis and in the case where $b + a^2/c < 0$ there is equivalence between the stabilizability of system $\dot{z} = (\hat{A} + v\hat{B})z$ by a positive feedback and the stabilizability of the system $\dot{z} = (\hat{B} + v\hat{A})z$ (7) where $v \in \mathbb{R}_+$. Moreover we can assume that there is equivalence between the stabilizability of system $\dot{z} = (\hat{A} + v\hat{B})z$ by a negative feedback and the stabilizability of the system $\dot{z} = (-\hat{B} + v\hat{A})z$ (8) where $v \in \mathbb{R}_+$. The stabilizability problem of systems (7) and (8) was treated in the subsection 4.1.

In the case when $b + a^2/c > 0$ we consider the change of feedback

$$\tilde{v}(z) = v(z) + \frac{a^2 + bc}{c} + c$$

the system (2) becomes

$$\dot{z} = (\tilde{A} + \tilde{v}\tilde{B})z \quad \text{where} \quad \tilde{A} = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}.$$ 

The characteristic polynomial of matrix $\tilde{A}$ is equal to $X^2 + c^2$, so matrix $\tilde{A}$ admits a first integral, namely the positive-definite function

$$V(z_1, z_2) = \frac{1}{2}(z_1^2 + z_2^2).$$

Moreover, the rank of the family $\{\tilde{B}z, \text{ad}\tilde{A}\tilde{B}z, \ldots\}$ is equal to two on $\mathbb{R}^2 \setminus \{0\}$, hence for any positive constant $\delta$ the feedback law

$$\tilde{v}(z) = -\frac{L_{\tilde{B}}V(z)}{\delta V(z_1, z_2)} = -\frac{z_1 z_2}{\delta (z_1^2 + z_2^2)}.$$
stabilizes the system (2) (the proof is a modification of the result of [6] with the feedback rendered homogeneous; the proof is exactly the same). It follows for \( \delta > 0 \) large enough the negative feedback

\[
v(z) = -\frac{z_1 z_2}{\delta (z_1^2 + z_2^2)} - \frac{a^2 + bc}{c} - c.
\]

**Proposition 2.** The system (2) is not stabilizable by a positive feedback.

**Proof.** In the new basis it is easy to verify that the set \( H = \{(z_1, z_2) \in \mathbb{R}^2 \text{ such that } z_1 \geq 2, z_2 \geq 2\} \) is invariant by the open loop system \( \dot{z} = Az + vBz \) where \( v \) lie in \( \mathbb{R}^+ \).

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