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ON GOODNESS-OF-FIT FOR THE ABSENCE OF MEMORY MODEL

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Logrank-type and Kolmogorov-type goodness-of-fit tests for the absence of memory model are proposed when the accelerated experiments are done under step-stresses. The power of the test against the approaching alternatives is investigated. The theoretical results are illustrated with simulated data.

1. INTRODUCTION

In survival analysis the most used model describing the influence of the explanatory variables on the lifetime distribution is the *proportional hazards* (PH) or Cox model, introduced by D. Cox [3]. We are interested in applicability of this model in accelerated life testing when units are tested under higher than usual stresses and inference about reliability in usual stress conditions are made

For constant in time stresses the PH is formulated as follows: suppose that under different constant in time stresses $x \in E_1$ the hazard rates are proportional to a baseline hazard rate:

$$\alpha_x(t) = r(x)\, \alpha_0(t).$$

For $x \in E_1$ the survival functions have the form

$$S_x(t) = S_0^{r(x)}(t) = \exp\{-r(x)A_0(t)\},$$

where

$$S_0(t) = \exp\left\{-\int_0^t \alpha_0(u)\,\mathrm{d}u\right\}, \quad A_0(t) = \int_0^t \alpha_0(u)\,\mathrm{d}u = -\ln S_0(t).$$

In the statistical literature the following formal generalization of the PH model to time-varying stresses is used: the proportional hazards (PH) model holds on a set of stresses E if for all $x(\cdot) \in E$

$$\alpha_{x(\cdot)}(t) = r\{x(t)\}\, \alpha_0(t).$$

This model is not natural when units are aging under usual constant stress. Indeed, denote by x_t a constant in time stress equal to the value of the time-varying stress

$x(\cdot)$ at the moment t. Then

$$\alpha_{x_t}(t) = r\{x(t)\}\,\alpha_0(t),$$

which implies that

$$\alpha_{x(\cdot)}(t) = \alpha_{x_t}(t). \tag{1}$$

For *any* t the hazard rate under the time-varying stress $x(\cdot)$ at the moment t *does not depend on the values of the stress* $x(\cdot)$ *before the moment* t but only on the value of stress at this moment. It is not natural when the hazard rates are not constant under constant stresses, i. e. when failure times under constant stresses are not exponential under constant stresses.

The equality (1) defines a model which means that the hazard rate under any time-varying stress at any moment t does not depend on the values of stress before this moment.

Let us call this model the *absence of memory* (AM) model. This model is wider than the PH model because it does not specify relations between survival distributions under different constant stresses. The PH model is a submodel of it.

The AM model (and the PH model)is not natural for aging units and it's application should be carefully studied. A formal goodness-of-fit test would be useful.

The most used time-varying stresses in accelerated life testing are the step-stresses: units are placed on test at an initial low stress and if they do not fail in a predetermined time t_1, the stress is increased. If they do not fail in a predetermined time $t_2 > t_1$, the stresses is increased once more, and so on.

Let us consider a set E_m of step-stresses of the form

$$x(\tau) = \begin{cases} x_1, & 0 \le \tau < t_1, \\ x_2, & t_1 \le \tau < t_2, \\ \cdots & \cdots \\ x_m, & t_{m-1} \le \tau < t_m. \end{cases} \tag{2}$$

Set $t_0 = 0$.

If the AM model holds on E_m and $x(\cdot) \in E_m$ then

$$\alpha_{x(\cdot)}(t) = \alpha_{x_i}(t), \quad \text{if} \quad t \in [t_{i-1}, t_i), \ (i = 1, 2, \ldots, m). \tag{3}$$

The AM model can be written in terms of the cumulative hazards $A_{x(\cdot)}$ and A_{x_i}: for $t \in [t_{i-1}, t_i)$ $(i = 1, 2, \ldots, m)$

$$A_{x(\cdot)}(t) = A_{x_i}(t) - A_{x_i}(t_{i-1}) + \mathbf{1}_{\{i \ge 2\}} \sum_{j=1}^{i-1} (A_{x_j}(t_j) - A_{x_j}(t_{j-1})). \tag{4}$$

A very possible alternative to this model is the generalized Sedyakin (GS) model (Bagdonavičius [2]):

$$\alpha_{x(\cdot)}(t) = g\left(x(t), S_{x(\cdot)}(t)\right),$$

which means that the hazard rate under any time-varying stress at any moment t depends not only on the value of the stress at this moment but also on the probability of survival until t.

If the GS model holds on E_m and $x(\cdot) \in E_m$ then

$$A_{x(\cdot)}(t) = A_{x_i}(t - t_{i-1} + t^*_{i-1}), \quad \text{if} \quad t \in [t_{i-1}, t_i) \; (i = 1, 2, \ldots, m), \tag{5}$$

where t^*_i can be found by solving the equations

$$A_{x_1}(t_1) = A_{x_2}(t_1{}^*), \ldots, A_{x_i}(t_i - t_{i-1} + t^*_{i-1}) = A_{x_{i+1}}(t^*_i) \; (i = 1, \ldots, m-1). \tag{6}$$

2. LOGRANK–TYPE TEST STATISTIC FOR THE AM MODEL

Suppose that a group of n_0 units is tested under the step-stress (1) and m groups of $n_1, \cdots, n_m$ units are tested under constant in time stresses $x_1 \cdots, x_m$, respectively.

Suppose that $x_1 < \cdots < x_m$. We write $x(\cdot) < y(\cdot)$ if $S_{x(\cdot)}(t) > S_{y(\cdot)}(t)$ for all $t > 0$.

The units are observed time t_m given for the experiment.

The idea of goodness-of-fit is based on comparing two estimators $\hat{A}^{(1)}_{x(\cdot)}$ and $\hat{A}^{(2)}_{x(\cdot)}$ of the cumulative hazard rate $A_{x(\cdot)}$. One estimator can be obtained from the experiment under step-stress (1) and another from the experiments under stresses $x_1, \cdots, x_m$ by using the equalities (2).

Denote by $N_i(t)$ and $Y_i(t)$ the number of observed failures in the interval $[0, t]$ and the number of units at risk just prior the moment t, respectively, for the group of units tested under the stress x_i and $N(t)$, $Y(t)$ the analogous numbers for the group of units tested under the stress $x(\cdot)$.

Set

$$\alpha_i = \alpha_{x_i}, \quad \alpha = \alpha_{x(\cdot)}, \quad A_i = A_{x_i}, \quad A = A_{x(\cdot)} \quad (i = 1, \ldots, m).$$

The first estimator $\hat{A}^{(1)}$ of the cumulative hazard A is the Nelson–Aalen estimator (see Andersen et al [1]) obtained from the experiment under the step-stress (1):

$$\hat{A}^{(1)}(t) = \int_0^t \frac{\mathrm{d}N(v)}{Y(v)}.$$

The second is suggested by the AM model (formula (3)) and is obtained from the experiments under the constant stresses:

$$\hat{A}^{(2)}(t) = \hat{A}_i(t) - \hat{A}_i(t_{i-1}) + \mathbf{1}_{\{i \geq 2\}} \sum_{j=1}^{i-1} (\hat{A}_j(t_j) - \hat{A}_j(t_{j-1})), \; t \in [t_{i-1}, t_i) \; (i = 1, \ldots, m), \tag{7}$$

where

$$\hat{A}_i(t) = \int_0^t \frac{\mathrm{d}N_i(v)}{Y_i(v)} \quad (i = 1, \ldots, m).$$

The first test is based on the logrank-type statistic

$$T_n = T_n(t_m), \quad \text{where} \quad T_n(t) = \int_0^t K(v) \, \mathrm{d}\{\hat{A}^{(1)}(t) - \hat{A}^{(2)}(t)\}; \tag{8}$$

here K is the weight function.

Similarly as in the case of classical logrank tests (see Harrington and Fleming [4]), we shall consider the weight functions of the following type: for $v \in [t_{i-1}, t_i)$

$$K(v) = \frac{1}{\sqrt{n}} \frac{Y(v)Y_i(v)}{Y(v)+Y_i(v)}\, g\left(\frac{Y(v)+Y_i(v)}{n}\right),$$

where $n = \sum_{i=0}^{m} n_i$ and g is a nonnegative bounded continuous function with bounded variation on $[0,1]$.

3. ASYMPTOTIC DISTRIBUTION OF THE LOGRANK TEST STATISTIC

Assumptions A.

a) The hazard rates α_i are positive and continuous on $(0,\infty)$;

b) $A_i(t_m) < \infty$;

c) $n \to \infty, \quad n_i/n \to l_i, \quad l_i \in (0,1)$.

Under Assumptions A (see Andersen et al [1]) for any $t \in (0, t_m]$ the estimators $\hat{A}_i$ and $\hat{A}^{(1)}$ are uniformly consistent on $[0,t]$, and

$$\sqrt{n}(\hat{A}_i - A_i) \xrightarrow{\mathcal{D}} U_i, \quad \sqrt{n}(\hat{A}^{(1)} - A) \xrightarrow{\mathcal{D}} U \tag{9}$$

on $D[0,t]$, the space of cadlag functions on $[0,t]$ with Skorokhod metric. Here U and $U_1, \cdots, U_m$ are independent Gaussian martingales with $U_i(0) = U(0) = 0$, and

$$\mathrm{Cov}\,(U_i(s_1), U_i(s_2)) = \frac{1}{l_i}\frac{1 - S_i(s_1 \wedge s_2)}{S_i(s_1 \wedge s_2)} := \sigma_i^2(s_1 \wedge s_2),$$

$$\mathrm{Cov}\,(U(s_1), U(s_2)) = \frac{1}{l_0}\frac{1 - S(s_1 \wedge s_2)}{S(s_1 \wedge s_2)} := \sigma^2(s_1 \wedge s_2), \tag{10}$$

with $S_i = \exp\{-A_i\}$, $S = \exp\{-A\}$.

Let us consider the limit distribution of the stochastic process $T_n(t), t \in [0, t_m]$. Note that

$$\frac{K(v)}{\sqrt{n}} \xrightarrow{\mathsf{P}} k(v) = \frac{l_0 l_i S(v) S_i(v)}{l_0 S(v) + l_i S_i(v)}\, g\left(l_0 S(v) + l_i S_i(v)\right),\ v \in [t_{i-1}, t_i).$$

The convergence is uniform on $[0, t_m]$.

Proposition 1. Under Assumptions A

$$T_n(t) \xrightarrow{\mathcal{D}} V_k(t) = \int_0^t k(v)\,\mathrm{d}U(v) - \mathbf{1}\{i \geq 2\}\sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j} k(v)\,\mathrm{d}U_j(v) - \int_{t_{i-1}}^{t} k(v)\,\mathrm{d}U_i(v),$$

$$t \in [t_{i-1}, t_i),\ i = 1, \cdots, m,\ t_0 = 0 \tag{11}$$

on $D[0, t_m]$.

Proof. For $t \in [t_{i-1}, t_i)$, $i = 1, \cdots, m$, write the statistic (7) in the form

$$T_n(t) = \int_0^t K(v)\,\mathrm{d}\{\hat{A}^{(1)}(t) - A(t)\} - \int_0^t K(v)\,\mathrm{d}\{\hat{A}^{(2)}(t) - A(t)\}$$

$$= \int_0^t J(v)K(v)\,\frac{\mathrm{d}M(v)}{Y(v)} - \mathbf{1}\{i \geq 2\} \sum_{j=1}^{i-1} \int_0^t J_j(v)K(v)\,\frac{\mathrm{d}M_j(t)}{Y_j(t)} - \int_{t_{i-1}}^t J_i(v)K(v)\,\frac{\mathrm{d}M_i(t)}{Y_i(t)},$$

where

$$M(t) = N(t) - \int_0^t Y(u)\,\mathrm{d}A(u), \quad M_i(t) = N_i(t) - \int_0^t Y_i(u)\,\mathrm{d}A_i(u),$$

$$J(t) = \mathbf{1}_{\{Y(t)>0\}}, \quad J_i(t) = \mathbf{1}_{\{Y_i(t)>0\}}.$$

Note that

$$\left\langle \int_0^t J(v)K(v)\,\frac{\mathrm{d}M(v)}{Y(v)} \right\rangle = \int_0^t J(v)K^2(v)\,\frac{\mathrm{d}A(v)}{Y(v)} \xrightarrow{\mathsf{P}} \int_0^t k^2(v)\,\frac{\mathrm{d}A(v)}{l_0 S(v)}$$

and for any $\varepsilon > 0$:

$$\left\langle \int_0^t J(v)\frac{K(v)}{Y(v)}\,\mathbf{1}_{\{|\frac{K(v)}{Y(v)}| \geq \varepsilon\}}\,\mathrm{d}M(v) \right\rangle = \int_0^t J(v)\frac{K^2(v)}{Y(v)}\,\mathbf{1}_{\{|\frac{K(v)}{Y(v)}| \geq \varepsilon\}}\,\mathrm{d}A(v) \xrightarrow{\mathsf{P}} 0$$

on $D[0, t_m]$. The Rebolledo's theorem (see Andersen et al [1]) implies that

$$\int_0^t J(v)K(v)\,\frac{\mathrm{d}M(v)}{Y(v)} \xrightarrow{\mathcal{D}} \int_0^t k(v) \left(\frac{\alpha(v)}{l_0 S(v)}\right)^{1/2} \mathrm{d}W(v),$$

on $D[0, t_m]$; here W is the standard Wiener process. The limit process has the same variance-covariance structure as the Gaussian process

$$\int_0^t k(v)\,\mathrm{d}U(v).$$

So

$$\int_0^t J(v)K(v)\,\frac{\mathrm{d}M(v)}{Y(v)} \xrightarrow{\mathcal{D}} \int_0^t k(v)\,\mathrm{d}U(v).$$

Analogously it is obtained that

$$\int_0^t J_i(v)K(v)\,\frac{\mathrm{d}M_i(v)}{Y_i(v)} \xrightarrow{\mathcal{D}} \int_0^t k(v)\,\mathrm{d}U_i(v)$$

on $D[0, t_m]$. □

Corollary 1. Under the assumptions of the theorem

$$\mathsf{Cov}(V_k(s), V_k(t)) = \sigma^2_{V_k}(s \wedge t)$$

where

$$\sigma^2_{V_k}(t) = \int_0^t k^2(v)\,\mathrm{d}\,\sigma^2(v) + \mathbf{1}\{i \geq 2\} \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} k^2(t)\,\mathrm{d}\sigma_j^2(v) + \int_{t_{i-1}}^{t} k^2(t)\,\mathrm{d}\sigma_i^2(v)$$

$$= \int_0^t \frac{k^2(v)}{l_0 S(v)}\,\mathrm{d}A(v) + \mathbf{1}\{i \geq 2\} \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} \frac{k^2(t)}{l_j S_j(v)}\,\mathrm{d}A_j(v)\mathbf{1}\{i \geq 2\}$$

$$+ \int_{t_{i-1}}^{t} \frac{k^2(t)}{l_i S_i(v)}\,\mathrm{d}A_i(v), \quad t \in [t_{i-1}, t_i),\ i = 1, \cdots, m,\ t_0 = 0,$$

and

$$T_n \xrightarrow{\mathcal{D}} N(0, \sigma^2_{V_k}(t_m)).$$

Proposition 2. The variance $\sigma^2_{V_k}(t_m)$ can be consistently estimated by the statistic

$$\hat{\sigma}^2_{V_k}(t_m) = \int_0^{t_m} K^2(v)\frac{\mathrm{d}N(v)}{Y^2(v)} + \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} K^2(v)\frac{\mathrm{d}N_i(v)}{Y_i^2(v)}.$$

P r o o f. Let us consider the difference

$$\int_0^{t_m} K^2(v)\frac{\mathrm{d}N(v)}{Y^2(v)} - \int_0^{t_m} k^2(v)\frac{\mathrm{d}A(v)}{l_0 S(v)}$$

$$= \int_0^{t_m} J(v) K^2(v)\frac{Y(v)\,\mathrm{d}A(v) + \mathrm{d}M(v)}{Y^2(v)} - \int_0^{t_m} k^2(v)\frac{\mathrm{d}A(v)}{l_0 S(v)}$$

$$= \int_0^{t_m} J(v)\left(\frac{K^2(v)/n}{(n_0/n)Y(v)} - \frac{k^2(v)}{l_0 S(v)}\right)\mathrm{d}A(v)$$

$$+ \int_0^{t_m} J(v) K^2(v)\frac{\mathrm{d}M(v)}{Y^2(v)} + \int_0^{t_m} (1 - J(v))\frac{\mathrm{d}A(v)}{l_0 S(v)}$$

$$= B_1 + B_2 + B_3.$$

We have

$$|B_1| \leq \sup_{[0,t_m]} \left|\frac{K^2(v)/n}{(n_0/n)Y(v)} - \frac{k^2(v)}{l_0 S(v)}\right| A(t_m) \xrightarrow{\mathsf{P}} 0,$$

and

$$\langle B_2 \rangle = \left\langle \int_0^{t_m} J(v) K^2(v)\frac{\mathrm{d}M(v)}{Y^2(v)} \right\rangle$$

$$= \int_0^{t_m} J(v) K^4(v)\frac{\mathrm{d}A(v)}{Y^3(v)} \leq \frac{1}{n} \sup_{[0,t_m]} \frac{(K(v)/\sqrt{n})^4}{(Y(v)/n)^3} A(t_m) \xrightarrow{\mathsf{P}} 0,$$

which imply that $B_i \xrightarrow{\mathsf{P}} 0$ $(i = 1, 2)$. Convergence $B_3 \xrightarrow{\mathsf{P}} 0$ is evident. □

4. LOGRANK-TYPE TEST

The hypothesis

$$H_0 : \alpha_{x(\cdot)}(t) = \alpha_{x_i}(t), t \in [t_{i-1}, t_i) \; (i = 1, \cdots, m)$$

(or the AM model) is rejected with the approximative significance level α, if

$$\left(\frac{T_n}{\hat{\sigma}_{V_k}(t_m)}\right)^2 > \chi^2_{1-\alpha}(1),$$

where $\chi^2_{1-\alpha}(1)$ is the $(1-\alpha)$-quantile of the chi-square distribution with one degree of freedom.

5. CONSISTENCY AND THE POWER OF THE TEST AGAINST THE APPROACHING ALTERNATIVES

Let us find the power of the test against the following alternatives:

H_1 : *GS model with specified non-exponential time-to-failure distributions under constant stresses.*

Under H_1

$$\hat{A}^{(1)}(v) \xrightarrow{P} A^{(1)}_*(v) = A_i(v - t_{i-1} + t^*_{i-1}), \quad v \in [t_{i-1}, t_i) \; (i = 1, \cdots, m),$$

where t^*_i can be found by solving the equations

$$A_1(t_1) = A_2(t_1{}^*), \cdots, A_i(t_i - t_{i-1} + t^*_{i-1}) = A_{i+1}(t^*_i) \quad (i = 1, \cdots, m-1),$$

$$\hat{A}^{(2)}(v) \xrightarrow{P} A^{(2)}(v) = A_i(t) - A_i(t_{i-1}) + \mathbf{1}_{\{i \ge 2\}} \sum_{j=1}^{i-1} (A_j(t_j) - A_j(t_{j-1})), \; v \in [t_{i-1}, t_i) \quad (i = 1, \ldots, m),$$

and

$$\frac{1}{\sqrt{n}} K(v) \xrightarrow{P} k_*(v), \quad Y(v)/n_0 \xrightarrow{P} S^{(1)}_*(v)$$

where $S^{(1)}_*(v) = \exp\{-A^{(1)}_*(v)\}$,

$$k_*(v) = \frac{l_0 l_i S^{(1)}_*(v) S_i(v)}{l_0 S^{(1)}_*(v) + l_i S_i(v)} g\left(l_0 S^{(1)}_*(v) + l_i S_i(v)\right)$$

$$= \frac{l_0 l_i S_i(v) S_i(v - t_{i-1} + t^*_{i-1})}{l_0 S_i(v - t_{i-1} + t^*_{i-1}) + l_i S_i(v)} g\left(l_0 S_i(v - t_{i-1} + t^*_{i-1}) + l_i S_i(v)\right), v \int [t_{i-1}, t_i).$$

Convergence is uniform on $[0, t_m]$.

Proposition 3. Suppose that Assumptions A hold and

$$\Delta^* = \Delta^*(t_m) \neq 0,$$

where

$$\Delta^*(t) = \mathbf{1}_{\{i\geq 2\}} \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} k_*(v) \{\alpha_j(v - t_{j-1} + t_{j-1}^*) - \alpha_j(v)\} \, \mathrm{d}v$$

$$+ \int_{t_{i-1}}^{t} k_*(v) \{\alpha_i(v - t_{i-1} + t_{i-1}^*) - \alpha_i(v)\} \, \mathrm{d}v.$$

Then the test is consistent against H_1.

Proof. Write the test statistic in the form

$$T_n = \int_0^{t_m} K(v) \, \mathrm{d}\{\hat{A}^{(1)}(v) - A_*^{(1)}(v)\} - \int_0^{t_m} K(v) \, \mathrm{d}\{\hat{A}^{(2)}(v) - A^{(2)}(v)\}$$

$$+ \int_0^{t_m} K(v) \, \mathrm{d}\{A_*^{(1)}(v) - A^{(2)}(v)\} = T_{1n} + T_{2n} + T_{3n}. \tag{12}$$

Analogously as in the case when seeking the limit distribution of the statistic T_n under the hypothesis H_0, we obtain that under H_1

$$T_{1n} + T_{2n} \xrightarrow{\mathcal{D}} N(0, {\sigma_{V_k}^*}^2(t_m)),$$

where ${\sigma_{V_k}^*}^2(t)$ has the same form (11) with only difference that $k(v)$ is replaced by $k_*(v)$ and $\sigma^2(v)$ is replaced by

$$(\sigma^{(1)})^2(v) = \frac{1}{l_0} \left(\frac{1}{S_*^{(1)}(v)} - 1 \right),$$

i. e.

$${\sigma_{V_k}^*}^2(t) = \int_0^t \frac{k_*^2(v)}{l_0 S_*(v)} \, \mathrm{d}A_*(v) + \mathbf{1}\{i \geq 2\} \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} \frac{k_*^2(t)}{l_j S_j(v)} \, \mathrm{d}A_j(v)$$

$$+ \int_{t_{i-1}}^{t} \frac{k^2(t)}{l_i S_i(v)} \, \mathrm{d}A_i(v), \quad t \in [t_{i-1}, t_i), \; i = 1, \cdots, m, \; t_0 = 0. \tag{13}$$

Under H_1 we have

$$\hat{\sigma}_{V_k}^2(t) \xrightarrow{\mathsf{P}} {\sigma_{V_k}^*}^2(t) \tag{14}$$

uniformly on $D[0, t_m]$, and

$$\frac{T_{1n} + T_{2n}}{\sigma_{V_k}^*(t_m)} \xrightarrow{\mathcal{D}} N(0, 1). \tag{15}$$

The third member in (12) can be written in the form

$$T_{3n} = \sum_{i=1}^{m} \int_{t_i-1}^{t_i} K(v)\,\{\alpha_i(v - t_{i-1} + t_{i-1}^*) - \alpha_i(v)\}\,dv. \tag{16}$$

The assumptions of the proposition and the equalities (13) – (16) imply that under H_1

$$\frac{1}{\sqrt{n}} T_{3n} \xrightarrow{\mathsf{P}} \Delta^*, \quad \frac{T_n}{\hat{\sigma}_{V_k}(t_m)} \xrightarrow{\mathsf{P}} \infty.$$

Thus under H_1

$$\mathsf{P}\left\{ \left(\frac{T_n}{\hat{\sigma}_{V_k}(t_m)} \right)^2 > \chi^2_{1-\alpha}(1) \right\} \to 1.$$

□

Proposition 4. If α_i are increasing (decreasing) then the test is consistent against H_1.

Proof. We shall show now by recurrence that $t_i > t_i^*$ for all i. Really, the inequalities $x_1 < \cdots < x_m$ imply that

$$S_1(t_1{}^*) > S_2(t_1{}^*) = S_1(t),$$

which give $t_1 > t_1{}^*$. If we assume that $t_{i-1} > t_{i-1}^*$ then

$$S_{i+1}(t_i^*) = S_i(t_i - t_{i-1} + t_{i-1}^*) > S_i(t_i - t_{i-1} + t_{i-1}) = S_i(t_i) > S_{i+1}(t_i),$$

which imply $t_i > t_i^*$. If α_i are increasing (decreasing) then $\Delta^* > 0$ ($\Delta^* < 0$) under H_1. The proposition implies the consistency of the test. □

Let us consider the sequence of the approaching alternatives

$$H_n : \; GS \text{ with } \alpha_i(t) = \left(\frac{t}{\theta_i} \right)^{\frac{\varepsilon_i}{\sqrt{n}}} \tag{17}$$

with fixed $\varepsilon_i > 0$ ($i = 1, \cdots, m$). Then

$$T_{3n} \xrightarrow{\mathsf{P}} \mu = \sum_{i=1}^{m} \varepsilon_i \int_{t_i-1}^{t_i} k_*(v)\, \ln\left(1 + \frac{t_{i-1}^* - t_{i-1}}{v}\right) dv < 0,$$

and

$$\frac{T_n}{\hat{\sigma}_{V_k}(t_m)} \xrightarrow{\mathcal{D}} N(a,1), \quad \left(\frac{T}{\hat{\sigma}_{V_k}(t_m)} \right)^2 \xrightarrow{\mathcal{D}} \chi^2(1,a),$$

where $a = -\mu/\sigma_T^*$, and $\chi^2(1,a)$ denotes the chi-square distribution with one degree of freedom and the non-centrality parameter a (or the random variable having such distribution).

The power function of the test is approximated by

$$\beta = \lim_{n\to\infty} \mathsf{P}\left\{ \left(\frac{T}{\hat{\sigma}_{V_k}(t_m)} \right)^2 > \chi^2_{1-\alpha}(1) | H_n \right\} = \mathsf{P}\left\{ \chi^2(1,a) > \chi^2_{1-\alpha}(1) \right\}. \tag{18}$$

6. KOLMOGOROV–TYPE TEST

Let us reject now the condition $x_1 < \cdots < x_m$. Logrank-type tests may be not powerful in such situations with non-monotone step-stresses. In such cases Kolmogorov-type tests could be useful. Such tests are constructed using the following considerations.

The limit process $V_k(t)$ obtained in Proposition 1 is a zero mean Gaussian martingale with the covariance function

$$\mathsf{Cov}(V_k(s), V_k(t)) = \sigma^2_{V_k}(s \wedge t).$$

It implies that $V_k(t) = W(\sigma^2_{V_k}(t))$, where W is the standard Wiener process. We have

$$\frac{1}{\sigma_{V_k}(t_m)} \sup_{0 \le t \le t_m} |V_k(t)| = \sup_{0 \le t \le t_m} \left| W\left(\frac{\sigma^2_{V_k}(t)}{\sigma^2_{V_k}(t_m)} \right) \right| = \sup_{0 \le u \le 1} |W(u)|. \tag{19}$$

The variance $\sigma^2_{V_k}(t)$ is consistently estimated by the statistic

$$\hat\sigma^2_{V_k}(t) = \int_0^t \frac{K^2(v)}{Y^2(v)} \, \mathrm{d}N(v) + \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} \frac{K^2(t)}{Y_j^2(v)} \, \mathrm{d}N_j(v) \mathbf{1}\{i \ge 2\}$$

$$+ \int_{t_{i-1}}^t \frac{K^2(t)}{Y_i^2(v)} \, \mathrm{d}A_i(v), \quad t \in [t_{i-1}, t_i),\ i = 1, \cdots, m.$$

So the test statistic is

$$Z_K = \frac{1}{\hat\sigma_{V_k}(t_m)} \sup_{t \in [0, t_m]} \left| \int_0^t K(v) \, \mathrm{d}\{\hat A^{(1)}(v) - \hat A^{(2)}(v)\} \right|. \tag{20}$$

If $n \to \infty$ then (10) and (18) imply

$$Z_K \xrightarrow{\mathcal{D}} \sup_{0 \le u \le 1} |W(u)|.$$

Denote by $W_{1-\alpha}$ the $(1-\alpha)$-quantile of the supremum of the Wiener process on the interval $[0,1]$. The hypothesis H_0 is rejected with approximative significance level α if $Z_K > W_{1-\alpha}$.

Consistence of the first test against H_1 implies consistence of this test against H_1 because the convergence $T_n \xrightarrow{\mathsf{P}} \infty$ implies the convergence $Z_K \xrightarrow{\mathsf{P}} \infty$.

Let us consider the sequence of the approaching alternatives (18). Similarly as in the case of the hypothesis H_0 we have

$$T_n(t) = \int_0^t K(v) \, \mathrm{d}\{\hat A^{(1)}(v) - A_*^{(1)}(v)\} - \int_0^t K(v) \, \mathrm{d}\{\hat A^{(2)}(v) - A^{(2)}(v)\}$$

$$+ \int_0^t K(v) \, \mathrm{d}\{A_*^{(1)}(v) - A^{(2)}(v)\} = T_{1n}(t) + T_{2n}(t) + T_{3n}(t) \xrightarrow{\mathcal{D}} V_k{}^*(t) + \Delta^*(t)$$

on $D[0, t_m]$, where

$$V_k^{*}(t) = \int_0^t k_*(v)\,\mathrm{d}U^*(v) - \sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j} k_*(v)\,\mathrm{d}U_j(v)\mathbf{1}\{i \ge 2\} - \int_{t_{i-1}}^{t} k(v)\,\mathrm{d}U_i(v)$$

$$t \in [t_{i-1}, t_i),\ i = 1, \cdots, m,\ t_0 = 0, \tag{21}$$

U^* is a Gaussian martingale with $U^*(0) = 0$, and

$$\mathrm{Cov}\,(U^*(s_1), U^*(s_2)) = \frac{1}{l_0}\frac{1 - S_*(s_1 \wedge s_2)}{S_*(s_1 \wedge s_2)},$$

with $S_* = \exp\{-A_*\}$,

$$\Delta^*(t) = \mathbf{1}_{\{i \ge 2\}} \sum_{j=1}^{i-1} \varepsilon_j \int_{t_j-1}^{t_j} k_*(v)\,\ln(1 + \frac{t^*_{j-1} - t_{j-1}}{v})\,\mathrm{d}v$$

$$+\varepsilon_i \int_{t_i-1}^{t} k_*(v)\,\ln\left(1 + \frac{t^*_{i-1} - t_{i-1}}{v}\right)\mathrm{d}v < 0,\ t \in [t_{i-1}, t_i).$$

Analogously as in the case of the hypothesis H_0,

$$V_k^{*}(t) = W(\sigma^2_{V_k}(t)),$$

and

$$Z_K = \frac{1}{\hat{\sigma}_{V_k}(t_m)} \sup_{t \in [0, t_m]} |T_n(t)| \xrightarrow{\mathcal{D}} \sup_{t \in [0, t_m]} \left| W\left(\frac{\sigma^2_{V_k}(t)}{\sigma^2_{V_k}(t_m)}\right) + \frac{\Delta^*(t)}{\sigma_{V_k}(t_m)} \right|$$

$$= \sup_{0 \le u \le 1} \left| W(u) + \frac{1}{c}\Delta^*(h(cu)) \right|,$$

where $h(s)$ is the function inverse to $\sigma_{V_k}(t)$, and $c = \sigma_{V_k}(t_m)$.

7. SIMULATION RESULTS

Suppose that a group of n_0 units is tested the time t_m under the step-stress (1) and $T_{01}, \cdots, T_{i\tilde{n}_0}$ are observed failure times, where $\tilde{n}_0$ is the number of elements, failed until time t_m. Let m groups of $n_1, \cdots, n_m$ units be tested the time t_m under constant in time stresses $x_1, \cdots, x_m$, respectively, and $T_{i1}, \cdots, T_{i\tilde{n}_i}$, $i = 1 \ldots m$ be the observed failure times of the ith group until the time t_m; here T_{ij} is the jth failure in the jth group, $\tilde{n}_i$ is the number of elements failed until t_m.

We simulate the failure moments when

$$\alpha_{x_i}(t) = \frac{\gamma}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\gamma-1},\quad S_{x_i}(t) = \exp\left(-\left(\frac{t}{\theta_i}\right)^{\gamma}\right),\quad i = 1 \ldots m,\quad \theta_i = \theta(x_i), \tag{22}$$

i. e. the Weibull distribution of the failure times under the constant stresses is supposed.

Set

$$\alpha_i = \alpha_{x_i},\quad \alpha = \alpha_{x(\cdot)},\quad S_i = S_{x_i},\quad S = S_{x(\cdot)},\quad i = 1, \ldots, m.$$

7.1. Tables of the simulation results

Denote N — the number of runs, m – the number of groups, tested under constant in time stresses, $\theta = (\theta_1, \cdots, \theta_m)$ – the stresses, $t = (t_1, \cdots, t_m)$ – the partition of the interval $[0,\ t_m]$, α – the significance level of test, β – the power of test, $n = \sum_{i=0}^{m} n_i$, γ – the parameter of the Weibull distribution.

Suppose $N = 5000$, $m = 3$. Let $\theta = (5000,\ 100,\ 10)$, when $\gamma < 1$ and $\theta = (15,\ 10,\ 8)$, when $\gamma \geq 1$.

Table 1. The values of $t = (t_1, t_2, t_3)$ and $t = (t_1{}^*, t_2{}^*, t_3{}^*)$, calculated from conditions

$$S_{x(.)}(t_1) = S(t_1) = 0.9, \quad S_{x(.)}(t_2) = S(t_2) = 0.5, \quad S_{x(.)}(t_3) = S(t_3) = 0.1$$

under the alternative (i. e. the level of censoring is the same for various γ values; we need that for the correct comparison of the test power under various γ values). Thus

$$t_1 = \theta_1\ (-\ln 0.9)^{\frac{1}{\gamma}}, \quad t_1{}^* = \frac{\theta_2}{\theta_1}\, t_1, \quad t_2 = t_1 - t_1{}^* + \theta_2\ (-\ln 0.5)^{\frac{1}{\gamma}},$$

$$t_2{}^* = \frac{\theta_3}{\theta_2}\ (t_2 - t_1 + t_1{}^*), \quad t_3 = t_2 - t_2{}^* + \theta_3\ (-\ln 0.1)^{\frac{1}{\gamma}}.$$

The seventh column: the values of σ^2, calculated by numerical methods using the formula:

$$\sigma^2_{V_k}(t_m) = \sigma^2 = \sum_{i=1}^{3} \int_{t_{i-1}}^{t_i} \frac{k^2(t)\alpha_i(t)\,(S(t) + S_i(t))}{l_0\ S(t)S_i(t)}\, \mathrm{d}t,$$

where $S(t)$ defined by (4), S_i, α_i – the survival function and the hazard rate of Weibull distribution and

$$k(t) = \frac{l_0\ l_i\ S(t)S_i(t)}{l_0\ S(t) + l_i\ S_i(t)}, \quad t \in [t_{i-1},\ t_i), \quad i = 1,\ 2,\ 3, \quad l_i = l_0 = \frac{n_0}{n} = \frac{n_i}{n} = \frac{1}{4}.$$

The eight column: the values of $\sigma_*^2 = \sigma_{V_k}^{*\ 2}(t_m)$, calculated by numerical methods using the formula:

$$\sigma_*^2 = \sum_{i=1}^{3} \int_{t_{i-1}}^{t_i} \frac{\gamma}{4\,\theta_i}\ \frac{e^{-\left(\frac{t}{\theta_i}\right)^\gamma} e^{-\left(\frac{t-t_{i-1}+t^*_{i-1}}{\theta_i}\right)^\gamma}}{\left(e^{-\left(\frac{t}{\theta_i}\right)^\gamma} e^{-\left(\frac{t-t_{i-1}+t^*_{i-1}}{\theta_i}\right)^\gamma}\right)^2} \times$$

$$\times \left(e^{-\left(\frac{t}{\theta_i}\right)^\gamma} \left(\frac{t - t_{i-1} + t^*_{i-1}}{\theta_i}\right)^{\gamma-1} + e^{-\left(\frac{t-t_{i-1}+t^*_{i-1}}{\theta_i}\right)^\gamma} \left(\frac{t}{\theta_i}\right)^{\gamma-1}\right)\ dt.$$

The ninth column: the values of Δ^*:

$$\Delta^* = \sum_{i=1}^{3} \int_{t_{i-1}}^{t_i} \frac{\gamma}{4\,\theta_i} \frac{e^{-\left(\frac{t}{\theta_i}\right)^\gamma} e^{-\left(\frac{t-t_{i-1}+t^*_{i-1}}{\theta_i}\right)^\gamma}}{\left(e^{-\left(\frac{t}{\theta_i}\right)^\gamma} e^{-\left(\frac{t-t_{i-1}+t^*_{i-1}}{\theta_i}\right)^\gamma}\right)^2} \left(\left(\frac{t-t_{i-1}+t^*_{i-1}}{\theta_i}\right)^{\gamma-1} - \left(\frac{t}{\theta_i}\right)^{\gamma-1}\right) dt.$$

Table 2. $\hat{\alpha_L}$ – the estimator of significance level for the Logrank-type test ($\alpha = 0.1$, $\chi^2_{1-\alpha}(1) = 2.706$), $\hat{\alpha_K}$ – the estimator of significance level for Kolmogorov-type test ($\alpha = 0.1$, $\chi^2_{1-\alpha}(1) = 1.96$).

Remark. The approximate confidence interval of estimator of significance level (for Logrank-type test) with confidence level $Q = 0.95$ is $\alpha \pm 1.96\sqrt{\frac{\alpha\,(1-\alpha)}{N}} =$ [0.0917, 0.1083].

$a = \frac{\sqrt{n}\Delta^*}{\sigma^*}$ – the non-centrality parameter for Logrank-type test.

β_T – the theoretical power of test for Logrank-type test:

$$\begin{aligned}\beta_T = \beta(\Delta^*) &= \mathsf{P}\left\{\left(\frac{T_n}{\sigma^*}\right)^2 > \chi^2_\alpha(1)|H_1\right\} \\ &= \mathsf{P}\left\{\frac{T_n}{\sigma^*} - \frac{\Delta^*\sqrt{n}}{\sigma^*} < -\sqrt{\chi^2_\alpha(1)} - \frac{\Delta^*\sqrt{n}}{\sigma^*}|H_1\right\} \\ &\quad + \mathsf{P}\left\{\frac{T_n}{\sigma^*} - \frac{\Delta^*\sqrt{n}}{\sigma^*} > \sqrt{\chi^2_\alpha(1)} - \frac{\Delta^*\sqrt{n}}{\sigma^*}|H_1\right\} \\ &= \Phi\left(-\sqrt{\chi^2_\alpha(1)} - \frac{\Delta^*\sqrt{n}}{\sigma^*}\right) + 1 - \Phi\left(\sqrt{\chi^2_\alpha(1)} - \frac{\Delta^*\sqrt{n}}{\sigma^*}\right)\end{aligned}$$

$\hat{\beta_L}$ – the estimator of the power of test for Logrank-type test.

$\hat{\beta_K}$ – the estimator of the power of test for Kolmogorov-type test.

The first number in the cell is calculated when $n = 400$, the second number – when $n = 800$. (The significance level $\alpha = 0.1$.)

Table 3. The values of estimators of power of test ($\hat{\beta_L}$ – for Logrank-type test, $\hat{\beta_K}$ – for Kolmogorov-type test) for non-monotone stresses. Also are given values of these estimators for the same stresses but in monotonous order. ($\alpha = 0.1$, $N = 5000$, $n = 400$.)

Table 1.

γ	t_1	t_2	t_3	t_1^*	t_2^*	σ^2	σ_*^2	Δ^*
0.40	18.0	57.6	134.0	0.4	4.0	0.05124	0.06078	0.03814
0.55	83.6	133.0	173.6	1.7	5.1	0.03057	0.03629	0.02241
0.60	117.0	169.0	204.0	2.4	5.4	0.02738	0.03162	0.01754
0.70	200.8	256.0	283.0	4.0	5.9	0.02300	0.02466	0.00999
0.75	248.0	305.0	329.0	4.9	6.0	0.02086	0.02167	0.00671
1.00	1.6	7.5	20.3	1.1	5.6	0.10520	0.10520	0.00000
1.80	4.0	9.5	15.7	2.8	6.5	0.10100	0.09883	-0.01803
2.30	5.6	10.4	15.0	3.8	6.8	0.09677	0.09316	-0.02932
2.60	6.0	10.8	14.8	4.0	6.9	0.09366	0.08905	-0.03452
3.00	7.0	11.0	14.7	4.7	7.0	0.08948	0.08361	-0.03973
3.50	7.9	11.6	14.5	5.0	7.0	0.08686	0.08037	-0.04297

Table 2.

γ	$\hat{\alpha_L}$	$\hat{\alpha_K}$	a	β_T	$\hat{\beta_L}$	$\hat{\beta_K}$
0.40	0.0938	0.0848	3.0943	0.9264	0.9276	0.9150
	0.1038	0.1006	4.3761	0.9968	0.9948	0.9948
0.55	0.0988	0.0858	2.3523	0.7604	0.7454	0.7076
	0.1022	0.0902	3.3267	0.9537	0.9446	0.9270
0.60	0.1058	0.0908	1.9729	0.6286	0.6118	0.5698
	0.1044	0.0934	2.7900	0.8739	0.8658	0.8318
0.70	0.0990	0.0830	1.2731	0.3568	0.3700	0.3272
	0.0982	0.0836	1.8005	0.5621	0.5656	0.5098
0.75	0.1024	0.0854	0.9117	0.2369	0.2570	0.2204
	0.0968	0.0900	1.2893	0.3627	0.3812	0.3384
1.00	0.0972	0.0910	0.0000	0.0999	0.0962	0.0910
	0.0990	0.0862	0.0000	0.0999	0.1002	0.0952
1.80	0.0964	0.0876	-1.1468	0.3118	0.2942	0.2558
	0.1020	0.0928	-1.6218	0.4913	0.4846	0.4326
2.30	0.1068	0.0972	-1.9212	0.6089	0.6210	0.5460
	0.0966	0.0908	-2.7169	0.8581	0.8652	0.8174
2.60	0.1040	0.0878	-2.3138	0.7482	0.7614	0.6840
	0.1036	0.0940	-3.2721	0.9481	0.9594	0.9342
3.00	0.1038	0.0968	-2.7479	0.8649	0.8914	0.8326
	0.0988	0.0928	-3.8861	0.9875	0.9930	0.9858
3.50	0.0956	0.0886	-3.0318	0.9173	0.9624	0.9386
	0.0932	0.0944	-4.2877	0.9959	0.9996	0.9986

Table 3.

γ	θ	$\hat{\beta_L}$	$\hat{\beta_K}$
0.40	(5000, 100, 10)	0.9276	0.9150
0.40	(5000, 10, 100)	0.3496	0.5434
1.00	(15, 8, 20)	0.0958	0.0868
2.00	(15, 8, 20)	0.0790	0.2100
2.20	(15, 8, 20)	0.0724	0.3222
2.50	(20, 15, 8)	0.3612	0.2984
2.50	(15, 20, 8)	0.2310	0.2292
2.50	(15, 8, 20)	0.0530	0.5328
3.00	(20, 15, 8)	0.5440	0.4884
3.00	(20, 8, 15)	0.5226	0.7190
3.00	(15, 8, 20)	0.0502	0.8110
3.00	(15, 20, 8)	0.3998	0.3652
3.20	(15, 8, 20)	0.0422	0.8734
3.50	(15, 8, 20)	0.0352	0.9222

7.2. Conclusions

The results from Table 2 imply that under monotone step-stresses the power of both tests increases when n increases or the parameter γ goes away from 1.

If $n = 400$ then the tests separate the hypothesis H_0 from the alternative H_1 sufficiently well for $\gamma \leq 0.6$ or $\gamma \geq 2.3$. If $n = 800$ then the tests separate the hypothesis H_0 from the alternative H_1 sufficiently well for $\gamma \leq 0.7$ or $\gamma \geq 1.8$ (see Figure 1).

The simulated values of the power of the Logrank-type test are close to the values of the theoretical power calculated by numerical methods (see Figure 2).

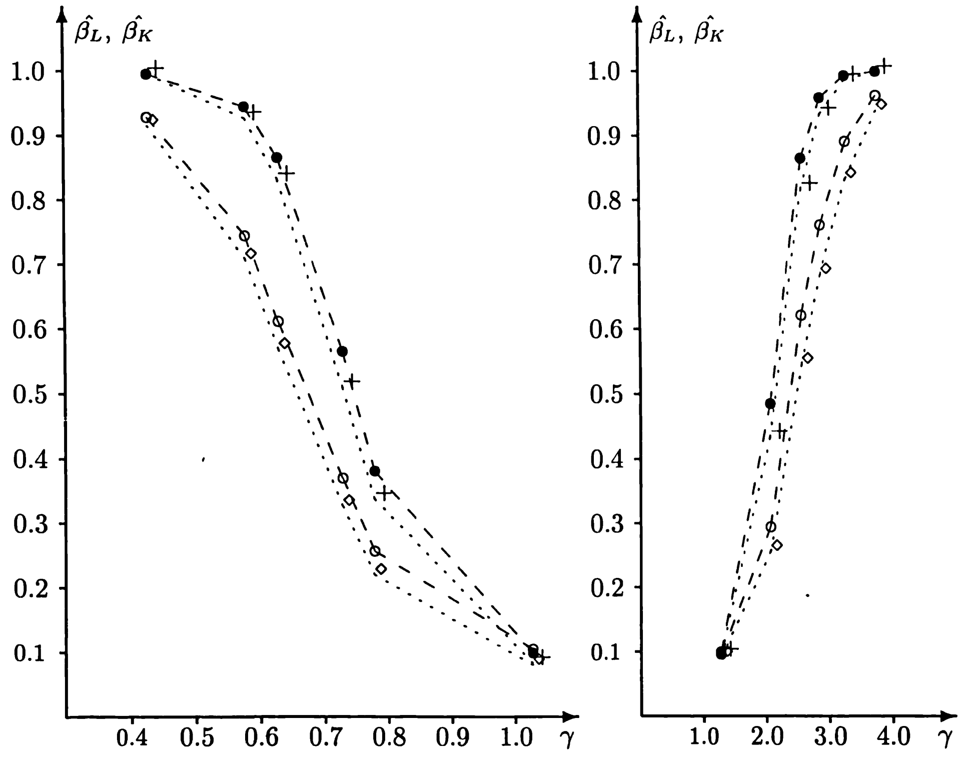

Fig. 1. The dependence of $\hat{\beta}_L$ and $\hat{\beta}_K$ on γ.
∘ – $\hat{\beta}_L$, $n = 400$; ⋄ – $\hat{\beta}_K$, $n = 400$
• – $\hat{\beta}_L$, $n = 800$; + – $\hat{\beta}_K$, $n = 800$

The logrank-type test was constructed for the monotone step-stresses. The results from Table 3 show that it is possible to find such plan of experiment with non-monotone stresses that Logrank-type test does not distinguish the hypothesis H_0 from the alternative and is even biased. The Kolmogorov-type test can be used and for such stresses. It distinguishes well the hypothesis and the alternative. The estimator of the power of the test increases when the parameter γ goes away from 1 (see Figure 3).

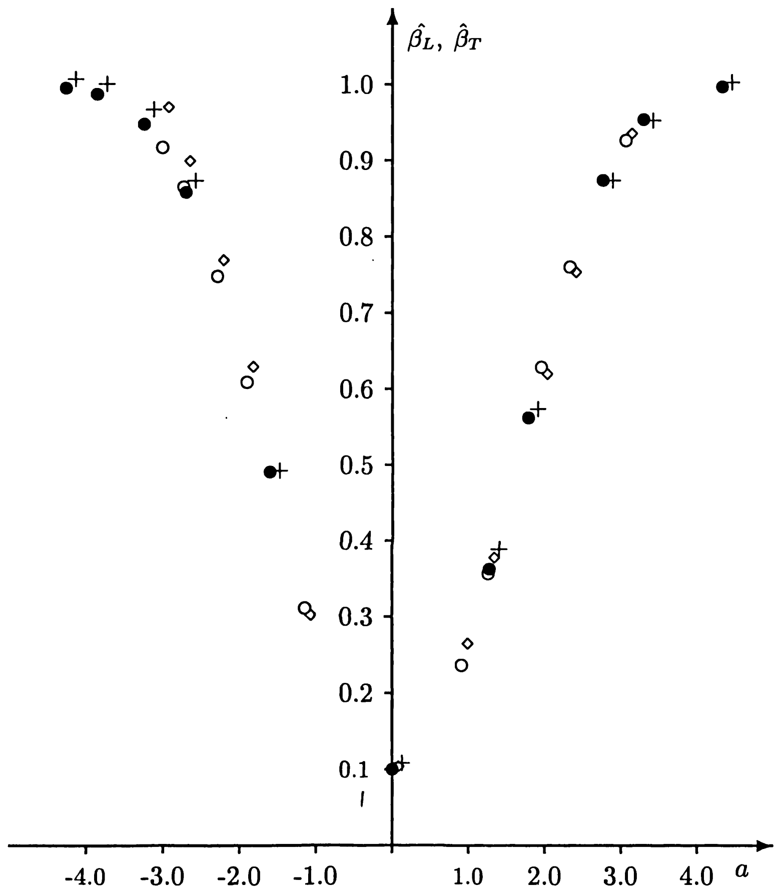

Fig. 2. The dependence of $\hat\beta_L$ and $\hat\beta_T$ on a.

○ – β_T, $n = 400$; ◇ – $\hat\beta_L$, $n = 400$

● – β_T, $n = 800$; + – $\hat\beta_L$, $n = 800$

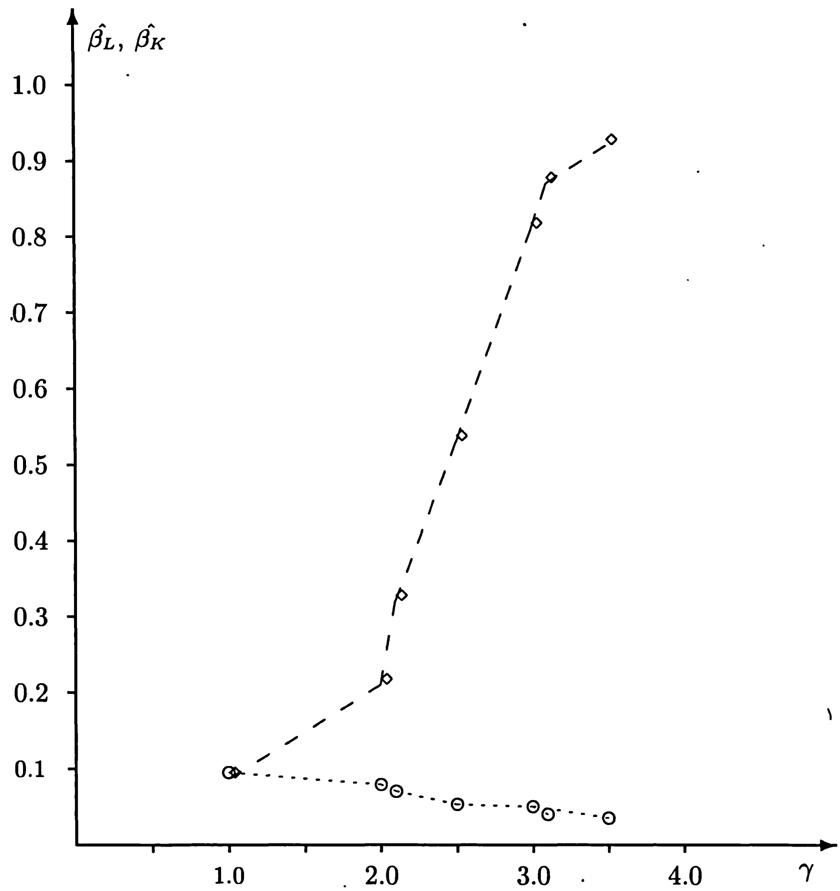

Fig. 3. The dependence of $\hat{\beta}_L$ and $\hat{\beta}_K$ on γ for non-monotone stresses.

$\circ$ – $\hat{\beta}_L$, $\diamond$ – $\hat{\beta}_K$, $n = 400$

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