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# DISCRETIZATION SCHEMES FOR LYAPUNOV–KRASOVSKII FUNCTIONALS IN TIME–DELAY SYSTEMS

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This article gives an overview of discretized Lyapunov functional methods for time-delay systems. Quadratic Lyapunov–Krasovskii functionals are discretized by choosing the kernel to be piecewise linear. As a result, the stability conditions may be written in the form of linear matrix inequalities. Conservatism may be reduced by choosing a finer mesh. Simplification techniques, including elimination of variables and using integral inequalities are also discussed. Systems with multiple delays and distributed delays are also treated. Finally, the treatment of uncertainties and input-output performance requirements are discussed.

## LIST OF NOTATIONS

| | |
|---|---|
| $W^T$ | Transpose of matrix $W$ |
| $\dot{W}$ | Derivative of $W$ with respect to time $t$ |
| $\dot{W}(\alpha)$ | Derivative of $W$ with respect to the argument and evaluated at $\alpha$ |
| $\partial_1 W(\alpha,\beta)$ | Partial derivative of $W$ with respect to first argument $\alpha$ |
| $\partial_{1+2} W(\alpha,\beta)$ | $\partial_1 W(\alpha,\beta) + \partial_2 W(\alpha,\beta)$ |
| $\mathcal{R}$, $\mathcal{R}^k$, $\mathcal{R}^{k\times m}$ | The set of real numbers, $k$-vectors and $k$ by $m$ matrices |
| $r$ | Time-delay |
| $n$ | Number of states |
| $\mathcal{C}$ | The set of continuous $\mathcal{R}^n$ valued function on $[-r,0]$ |
| $N$ | Number of divisions of the interval $[-r,0]$ in discretization |
| $h$ | Length of each division $= \frac{r}{N}$ |
| $I$ | Identity matrix of appropriate dimensions |
| $O_{k\times m}$ | $kn$ by $mn$ dimensional zero matrix |
| $I_r$ | Matrix $[O_{N\times 1}, I]$ |
| $I_l$ | Matrix $[I, O_{N\times 1}]$ |
| $I_d$ | $= I_l - I_r$ |

## 1. INTRODUCTION

Time-delay systems are frequently encountered in engineering, biology, economy, and other areas [18]. In the wake of intensive research on the robust stability and çontrol theory, the stability and control of time-delay systems has received renewed interests. The development of efficient computational algorithm for nonsmooth convex optimization problem [29], which made it possible to efficiently solve Linear Matrix Inequalities (LMI) [1, 5], inspired intensive activities to formulate such problems in an LMI form. I will only mention some more recent activities in the time-domain approaches using Lyapunov–Krasovskii and Razumikhin methods. For a more comprehensive survey, see [25, 32] and [23].

For systems with small coefficient matrix for the delayed term, a delay-independent stability criterion, based on a rather simple Lyapunov–Krasovskii functional argument, is often sufficient in practice. This formulation considers the delayed term of the system as always detrimental to stability. For many practical systems, the delayed inputs are often needed to stabilize the system. Obviously, the delay-independent stability would be inappropriate in such situations.

For systems with small delay, a model transformation technique is often used to transform a system with a point-wise delay into one with a distributed delay, and a rather simple Lyapunov–Krasovskii or Razumikhin stability criterion is used to the resulting system. There are still many recent publications in this approach, see, for example, [28] and [3]. The resulting stability criteria explicitly depend on the delay (delay-dependent), and often reflect the reality better. For relatively large delay, however, this method can be rather conservative. One source of conservatism is due to the application of Razumikhin theory or the type of Lyapunov–Krasovskii functionals used. The other conservatism is due to the fact that the model transformation may introduce additional poles which are not present in the original system, and one of these additional poles may cross the imaginary axis before any of the poles of the original system do as the delay increases from zero [16, 24]. Another shortcoming of the above mentioned results is that the corresponding system without delay needs to be stable. It is well known that there are systems which are stable with some nonzero delay, but is unstable without delay. However, for systems where the time-delay may be fast time-varying, Razumikhin approach remains the only method available.

To resolve these problems, a *discretized Lyapunov functional method* turns out to be very effective. In this article, I will give an overview. The basic ideas of discretized Lyapunov functional method, initially proposed in [6] for systems with single constant delay, is to choose a piecewise linear kernel for a quadratic Lyapunov–Krasovskii functional. A series of stability criteria in the form of linear matrix inequalities (LMIs) [1] can be generated depending on the grid size of the discretization. Computational experience shows that with a coarse grid, significant reduction of conservatism is achieved with similar computation requirement as compared to most of the existing methods using Lyapunov–Krasovskii methods. As grid size decreases, and computational requirement increases, true stability limit can be approached for systems without uncertainty. A more general setup was proposed in [10] showing the possibility of improved accuracy at a cost of more computation.

A technique to simplify LMIs by eliminating some variables was developed in [9]. This technique made it possible to allow the relaxation parameters to depend on the uncertainties since they can be subsequently eliminated. Reduction of conservatism may be achieved for uncertain systems with at least three states. Another technique to simplify the LMIs is to utilize a quadratic integral inequality [11]. It turns out that a combination of the two techniques may dramatically improves the results [12].

The applicability of discretized Lyapunov functional method can be extended to more complicated systems as well. Systems with multiple delays was discussed in [8]. Due to the possibility of incommensurate delays, a particular challenge is the necessity to use nonuniform grid. Systems with distributed delays and piecewise constant coefficients were investigated in [15] and [14]. Systems with time-varying delay with known derivative bound of the delay with respect to time were treated in [13] and [20].

An important issue investigated in, for example, [28] and [3] is the system uncertainty. For norm bounded uncertainty, it was shown in [19] that a significant reduction of conservatism can be achieved as well using discretized Lyapunov functional method as compared to previous methods. For more general setting, similar to systems without time-delays, it is often possible to "pull out uncertainties" and write an uncertain time-delay system as a time-delay system without uncertainty with a block-diagonal uncertainty in the feedback path [2, 4, 33, 34]. The performance specification of limiting the ratio between output norm and disturbance input norm is equivalent to one additional uncertain block. The stability of such systems with block-diagonal uncertainty was treated in [7]. Although only systems with single delay was treated, there is no additional conceptual difficulty in extending the formulation to systems with multiple delays or distributed delays with piecewise constant coefficients. With this in mind, the discretized Lyapunov functional method can be used in a wider class of systems. For example, for a general distributed delay system, one may approximate the system by a distributed system with piecewise constant coefficients, with the errors modeled as uncertainty. Similarly, such a system may also be approximated by a system with multiple delays with uncertainty to account for the errors.

In this article, I will give a brief overview of the above mentioned results, trying to bring out the essential ideas. The readers may refer to the original articles listed in the reference section for technical details.

## 2. QUADRATIC LYAPUNOV–KRASOVSKII FUNCTIONAL

Let us start by considering the stability of the linear time-invariant system with single time-delay

$$\dot{x}(t) = Ax(t) + A_d x(t-r) \tag{1}$$

with initial condition

$$x(t) = \phi(t), \qquad -r \le t \le 0. \tag{2}$$

With experience on the Lyapunov function method in studying systems without time-delay, it is tempting to use a Lyapunov function $V(x(t))$ as a quadratic function of $x(t)$, the state $x$ at the current time $t$. Due to the fundamental work of Krasovskii

[27] on a more general type of systems, it becomes clear that such a Lyapunov function is not sufficient. One needs to use a Lyapunov function $V$ which depends on all the values of the state $x$ in the time interval $[t-r, t]$, i.e., a Lyapunov–Krasovskii functional $V(x_t)$, where the function $x_t : [-r, 0] \to R^n$ is defined as

$$x_t(\theta) = x(t+\theta).$$

This is not surprising in view of the fact that $x_t$ is the minimum information needed to predict the future evolution of the time-delay system (1) after time $t$ while $x(t)$ is sufficient to predict future trajectory of a system without time-delay.

If system (1) is stable, Infante and Castelan [22] proposed an explicit procedure to construct a quadratic Lyapunov–Krasovskii functional. Huang [21] extended the procedure to a more general time-delay system. A consequence of these results is that the existence of a quadratic Lyapunov–Krasovskii functional $V(x_t)$ of the form

$$\begin{aligned} V(\phi) &= \frac{1}{2}\phi^T(0)P\phi(0) + \phi^T(0)\int_{-r}^{0} Q(\xi)\,\phi(\xi)\,\mathrm{d}\xi \\ &\quad + \frac{1}{2}\int_{-r}^{0}\mathrm{d}\xi\int_{-r}^{0}\phi^T(\xi)\,R(\xi,\eta)\,\phi(\eta)\,\mathrm{d}\eta + \frac{1}{2}\int_{-r}^{0}\phi^T(\xi)S(\xi)\,\phi(\xi)\,\mathrm{d}\xi, \end{aligned} \tag{3}$$

satisfying $V(x_t) > 0$ and $\dot{V}(x_t) < 0$ whenever $x(t) \neq 0$ is a necessary and sufficient condition for system (1) to be stable. In the above,

$$P \in \mathcal{R}^{n\times n}, \quad P = P^T > 0, \tag{4}$$

$$Q : [-r, 0] \to \mathcal{R}^{n\times n}, \tag{5}$$

$$S : [-r, 0] \to \mathcal{R}^{n\times n}, \quad S^T(\xi) = S(\xi) > 0, \tag{6}$$

$$\begin{aligned} R &: [-r, 0]\times[-r, 0] \to \mathcal{R}^{n\times n}, \\ & R(\eta,\xi) = R^T(\xi,\eta). \end{aligned} \tag{7}$$

Theoretically, we may choose $S = 0$ and $R(\xi,\eta) = R(\xi - \eta)$ (i.e., $R$ depends only on the difference of $\xi$ and $\eta$) according to [21]. However, we choose to use this more general form, and will comment on the effects of restricting to a more special form in view of our objective in arriving at a practically computable criteria, as well as applying to a more general type of systems.

The systems considered here may be expressed as

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,x(t-r), \tag{8}$$

where $A(t) \in \mathcal{R}^{n\times n}$ and $B(t) \in \mathcal{R}^{n\times n}$ are uncertain matrices, which are unknown and possibly time-varying, but are known to be bounded by some compact set $\Omega$, i.e.,

$$(A(t), B(t)) \in \Omega \subset R^{n\times 2n}, \quad \text{for all } t \in (0,\infty). \tag{9}$$

When the set $\Omega$ is a singleton, then the system expressed by equations (8) and (9) reduces to a system without uncertainty expressed in (1). We will often suppress the explicit dependence of $A$ and $B$ on time $t$ in the following discussions for the sake of convenience. The derivative of $V(x_t)$ along the system trajectory, $\dot{V}(t,x_t)$, can also be expressed in a quadratic form, after integration by parts and consolidating terms, as follows:

$$\begin{aligned}\dot{V}(t,\phi) &= -\frac{1}{2}\phi^T(0)[-PA-A^TP-Q(0)-Q^T(0)-S(0)]\phi(0)\\ &\quad -\frac{1}{2}\phi^T(-r)S(-r)\,\phi(-r)-\frac{1}{2}\int_{-r}^{0}\phi^T(\xi)\dot{S}(\xi)\,\phi(\xi)\,\mathrm{d}\xi\\ &\quad -\frac{1}{2}\int_{-r}^{0}\mathrm{d}\xi\int_{-r}^{0}\phi^T(\xi)\partial_{1+2}R(\xi,\eta)\,\phi(\eta)\,\mathrm{d}\eta+\phi^T(0)[PB-Q(-r)]\phi(-r)\\ &\quad +\phi^T(0)\int_{-r}^{0}[A^TQ(\xi)-\dot{Q}(\xi)+R^T(\xi,0)]\phi(\xi)\,\mathrm{d}\xi\\ &\quad +\phi^T(-r)\int_{-r}^{0}[B^TQ(\xi)-R^T(\xi,-r)]\phi(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{10}$$

The explicit dependence of $t$ by $\dot{V}$ arises due to the possibility that $A$ and $B$ may be time-varying. One may use Lyapunov–Krasovskii functional to study the stability of a time-delay system based on the following theorem.

**Theorem 1.** The system (8) is asymptotically stable if there exists a quadratic Lyapunov–Krasovskii functional $V(x_t)$ of the form (3) such that for some $\epsilon > 0$, it satisfies

$$V(\phi) \geq \epsilon\phi^T(0)\,\phi(0), \tag{11}$$

and its derivative along the solution of (8) satisfies

$$\dot{V}(t,\phi) \leq -\epsilon\phi^T(0)\,\phi(0). \tag{12}$$

The theorem is a special case of Theorem 2.1 in Chapter 5 of [18] by restricting $V(x_t)$ to the form (3) (notice, the additional condition $V(\phi) \leq K|\phi|^2$ for some sufficiently large $K > 0$ required by [18] is automatically satisfied since this quadratic Lyapunov functional is clearly a bounded quadratic form). The theorem roughly says that a system is assured of asymptotic stability if we can find a quadratic Lyapunov–Krasovskii functional $V(x_t)$, such that $V(x_t) > 0$ and its derivative along the system trajectory satisfies $\dot{V}(t,x_t) < 0$ whenever $x(t) = x_t(0) \neq 0$.

It is important to note that the above criterion can detect stability for a system which is unstable if the delay $r$ is set to zero.

Unfortunately, the search for such a Lyapunov–Krasovskii functional is not easy in practice. Even if $V$ is given, there is no systematic way of determining whether it satisfies conditions (11) and (12). Therefore, a discretization process is proposed.

## 3. DISCRETIZATION

The discretization discussed here is mainly from [10]. The simplified version is from [6]. Let $h = r/N$ and $\theta_i = -r + ih$.This divides the interval

$$\mathcal{I} = [-r, 0]$$

into $N$ small intervals,

$$\mathcal{I}_i = [\theta_{i-1}, \theta_i], \quad i = 1, 2, \ldots, N.$$

It also divides the square region

$$\mathcal{S} = [-r, 0] \times [-r, 0]$$

into $N \times N$ small square regions

$$\mathcal{S}_{ij} = [\theta_{i-1}, \theta_i] \times [\theta_{j-1}, \theta_j], \quad i, j = 1, 2, \ldots, N.$$

Each small square is further divided into two triangular regions

$$\begin{aligned} \mathcal{T}_{ij}^u &= \left\{ (\theta_{i-1} + \alpha h, \theta_{j-1} + \beta h) \;\middle|\; \begin{array}{l} 0 \le \beta \le 1, \\ 0 \le \alpha \le \beta \end{array} \right\}, \\ \mathcal{T}_{ij}^l &= \left\{ (\theta_{i-1} + \alpha h, \theta_{j-1} + \beta h) \;\middle|\; \begin{array}{l} 0 \le \alpha \le 1, \\ 0 \le \beta \le \alpha \end{array} \right\}. \end{aligned}$$

Choose $Q$ and $S$ to be continuous in $\mathcal{I}$ and linear within each $\mathcal{I}_i$, and $R$ to be continuous in $\mathcal{S}$ and linear within each $\mathcal{T}_{ij}^u$ and $\mathcal{T}_{ij}^l$. Then, the quadratic Lyapunov–Krasovskii functional (3) is completely determined by the matrices $P$, $Q_i = Q(\theta_i)$, $S_i = S(\theta_i)$, and $R_{ij} = R(\theta_i, \theta_j)$, $i, j = 0, 1, 2, \ldots, N$, since, for example, the value of $R(\xi, \eta)$ within $\mathcal{T}_{ij}^u$ may be calculated by linear interpolation $R(\theta_{i-1}+\alpha h, \theta_{j-1}+\beta h) = (1-\beta)R_{i-1,j-1} + \alpha R_{ij} + (\beta - \alpha)R_{i-1,j}$. Introduce notations

$$\begin{aligned} \phi^i(\alpha) &= \phi(\theta_{i-1} + \alpha h), \\ \psi^i(\alpha) &= \int_0^\alpha \phi^i(\tau)\,\mathrm{d}\tau, \\ \tilde{\phi}(\alpha) &= (\phi^{1T}(\alpha), \phi^{2T}(\alpha), \ldots, \phi^{NT}(\alpha))^T, \\ \tilde{\psi}(\alpha) &= (\psi^{1T}(\alpha), \psi^{2T}(\alpha), \ldots, \psi^{NT}(\alpha))^T. \end{aligned}$$

Then, the Lyapunov–Krasovskii functional may be written as

$$\begin{aligned} V(\phi) = {} & \frac{1}{2} \int_0^1 (\phi^T(0), h[\tilde{\psi}^T(1) I_r - \tilde{\psi}^T(\alpha) I_d]) \\ & \begin{pmatrix} U & \tilde{W} \\ \tilde{W}^T & \tilde{R} \end{pmatrix} \begin{pmatrix} \phi(0) \\ h[I_r^T \tilde{\psi}(1) - I_d^T \tilde{\psi}(\alpha)] \end{pmatrix} \mathrm{d}\alpha + \frac{1}{2} \int_{-r}^0 (\phi^T(0), \phi^T(\xi)) \\ & \begin{pmatrix} \frac{1}{r}[P - U] + Z(\xi) & Q(\xi) - W(\xi) \\ [Q(\xi) - W(\xi)]^T & S(\xi) \end{pmatrix} \\ & \begin{pmatrix} \phi(0) \\ \phi(\xi) \end{pmatrix} \mathrm{d}\xi \end{aligned} \tag{13}$$

where $W(\xi)$ and $Z(\xi)$ are arbitrary continuous piecewise linear matrix functions determined by $W_i$ and $Z_i$ of the same structure and symmetry as $Q(\xi)$ and $S(\xi)$, respectively; and $Z(\xi)$ is further required to satisfy

$$\int_{-r}^0 Z(\xi)\,\mathrm{d}\xi = \frac{h}{2}(Z_0 + Z_N) + h \sum_{i=1}^{N-1} Z_i = 0. \tag{14}$$

It is therefore obvious that condition (11) is satisfied if the two matrices in the integration in (13) are positive definite. Based on this observation, after some additional derivation, we can arrive at the following theorem.

**Theorem 2.** With the continuous piecewise linear $Q$, $S$ and $R$ chosen, the Lyapunov functional satisfies (11) if there exist $n \times n$ matrices $W_i$, $i = 0, 1, \dots, N$ and $U = U^T$ such that

$$\begin{pmatrix} U & \tilde{W} \\ \tilde{W}^T & \tilde{R} \end{pmatrix} \geq 0, \tag{15}$$

and

$$\begin{pmatrix} (P-U)/h & \tilde{Q}-\tilde{W} \\ (\tilde{Q}-\tilde{W})^T & \hat{S} \end{pmatrix} > 0, \tag{16}$$

where

$$\tilde{R} = \begin{pmatrix} R_{00} & R_{01} & \dots & R_{0N} \\ R_{10} & R_{11} & \dots & R_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N0} & R_{N1} & \dots & R_{NN} \end{pmatrix}, \tag{17}$$

$$\tilde{W} = (W_0, W_1, \dots, W_N), \tag{18}$$

$$\tilde{Q} = (Q_0, Q_1, \dots, Q_N), \tag{19}$$

$$\hat{S} = \mathrm{diag}(2S_0, S_1, S_2, \dots, S_{N-1}, 2S_N). \tag{20}$$

Similarly, the $\dot{V}(t,\phi)$ may also be expressed as

$$\begin{aligned} \dot{V}(t,\phi) = &-\frac{1}{2}\int_0^1 (\phi^T(0), \phi^T(-r), h\tilde{\phi}^T(\alpha)) \\ &\begin{pmatrix} \Delta - T + (1-2\alpha)Y & D(\alpha) - X \\ (D(\alpha)-X)^T & \frac{1}{h}S_d \end{pmatrix} \\ &\begin{pmatrix} \phi(0) \\ \phi(-r) \\ h\tilde{\phi}(\alpha) \end{pmatrix} \mathrm{d}\alpha - \frac{1}{2}\left(\phi^T(0), \phi^T(-r), h\int_0^1 \tilde{\phi}^T(\alpha)\,\mathrm{d}\alpha\right) \\ &\begin{pmatrix} T & X \\ X^T & R_d \end{pmatrix} \begin{pmatrix} \phi(0) \\ \phi(-r) \\ h\int_0^1 \tilde{\phi}(\alpha)\,\mathrm{d}\alpha \end{pmatrix} \end{aligned}$$

for arbitrary matrices $T_{ij}$, $Y_{ij}$ and $X_i$, $i,j = 1,2$, where

$$\begin{aligned} \Delta_{11} &= -PA - A^TP - Q_N - Q_N^T - S_N, \qquad (21) \\ \Delta_{12} &= PB - Q_0, \\ \Delta_{22} &= S_0, \\ \Delta &= \begin{pmatrix} \Delta_{11} & -\Delta_{12} \\ -\Delta_{12}^T & \Delta_{22} \end{pmatrix}, \end{aligned}$$

$$
\begin{aligned}
T &= \begin{pmatrix} T_{11} & T_{12} \\ T_{12}^T & T_{22} \end{pmatrix}, \\
Y &= \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{pmatrix}, \\
S_d &= \mathrm{diag}(S_{d1}, S_{d2}, \ldots, S_{dN}), \\
S_{di} &= \frac{1}{h}(S_i - S_{i-1}), \\
R_d &= \begin{pmatrix} R_{d11} & R_{d12} & \ldots & R_{d1N} \\ R_{d21} & R_{d22} & \ldots & R_{d2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{dN1} & R_{dN2} & \ldots & R_{dNN} \end{pmatrix}, \\
R_{dij} &= \frac{1}{h}(R_{ij} - R_{i-1,j-1}), \\
D_{1i}^k &= A^T Q_{i-1+k} - \frac{1}{h}(Q_i - Q_{i-1}) + R_{i-1+k,N}^T, \\
D_{2i}^k &= B^T Q_{i-1+k} - R_{i-1+k,0}^T, \\
D_j^k &= \begin{pmatrix} D_{j1}^k & D_{j2}^k & \ldots & D_{jN}^k \end{pmatrix}, \\
D^i &= \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix}, \\
D &= -(1-\alpha)D^0 - \alpha D^1, \\
X &= \begin{pmatrix} X_{11} & X_{12} & \ldots & X_{1N} \\ X_{21} & X_{22} & \ldots & X_{2N} \end{pmatrix}.
\end{aligned} \tag{22}
$$

It is again obvious that condition (12) is satisfied if the two matrices in the above expression are positive definite, leading to the following theorem after some manipulations.

**Theorem 3.** With the piecewise linear $Q$, $S$ and $R$ chosen, (12) is satisfied if there exist $n \times n$ matrices

$$X_{ij}, \quad i = 1,2; \quad j = 1,2,\ldots,N,$$

and

$$T_{ij}^T = T_{ji}, Y_{ij}^T = Y_{ji}, \quad i,j = 1,2,$$

such that

$$\begin{pmatrix} T & X \\ X^T & R_d \end{pmatrix} \geq 0, \tag{23}$$

$$\begin{pmatrix} \Delta - T + Y & -D^0 - X \\ (-D^0 - X)^T & \frac{1}{h}S_d \end{pmatrix} > 0, \tag{24}$$

$$\begin{pmatrix} \Delta - T + Y & -D^1 - X \\ (-D^1 - X)^T & \frac{1}{h}S_d \end{pmatrix} > 0, \tag{25}$$

for all $(A, B) \in \Omega$.

When $\Omega$ is polytopic, one only needs to check the condition (23) to (25) at the vertices of $\Omega$. The above two theorems indicate that system (8) is stable if the LMIs (15), (16) and (23) to (25) have a solution. As $N$ increases, the conservatism decreases, eventually approaching the true stability limit if the system does not have uncertainty.

It is possible to further simplify the LMIs by introducing some constraints, and still guarantee the convergence to the true stability limit for uncertainty-free system as is discussed in [10]. This reduces the formulation to the version discussed in [6]. This results in less computational effort required to solve LMIs with the same $N$. The convergence rate, of course, will be slower, and the result may be more conservative even for $N \to \infty$ for systems with uncertainty.

**Example 4.** This example was studied in [28] and [30]. This is a linear time-invariant time-delay system without uncertainty

$$\dot{x}(t) = \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} x(t-r). \tag{26}$$

The maximum time-delay for stability, $r_{\max}$, can be calculated analytically (find $r$ such that there is a pole on the imaginary axis, the true stability limits of the other examples are obtained similarly) as 6.17. The complete version consisting of LMIs (15), (16) and (23) to (25), and the simplified version discussed in [6], are used to estimate the maximum delay, $r_{\max}$, for the system to retain stability with different $N$. The results are listed in the following.

| $N$ | 1 | 2 | 5 | 10 | 20 |
|---|---|---|---|---|---|
| Complete | 5.30 | 5.74 | 6.07 | 6.14 | 6.16 |
| Simplified | 5.30 | 5.74 | 5.99 | 6.06 | 6.10 |

As expected, the complete version converges to the true stability limit much faster.

The next example illustrates the application of the approach to a polytopic uncertain system.

**Example 5.** Consider the following uncertain system,

$$\dot{x}(t) = \begin{pmatrix} -2+\rho(t) & \rho(t) \\ \rho(t) & -0.9+\rho(t) \end{pmatrix} x(t) + \begin{pmatrix} -1+\rho(t) & 0 \\ -1 & -1-\rho(t) \end{pmatrix} x(t-r),$$

where

$$|\rho(t)| \leq 0.1.$$

It can be modeled as a system with polytopic uncertainty set $\Omega$ with 2 vertices $(A_i, B_i)$, $i = 1, 2$, where

$$\begin{aligned} A_1 &= \begin{pmatrix} -2.1 & -0.1 \\ -0.1 & -1.0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -1.1 & 0 \\ -1 & -0.9 \end{pmatrix}, \\ A_2 &= \begin{pmatrix} -1.9 & 0.1 \\ 0.1 & -0.8 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -0.9 & 0 \\ -1 & -1.1 \end{pmatrix}. \end{aligned}$$

The maximum delay for stability $r_{\max}$ are estimated using the two methods mentioned above, and are summarized in the following table. Again, the complete version shows better results.

| $N$ | 1 | 3 | 5 | 10 |
|---|---|---|---|---|
| Complete | 2.37 | 2.58 | 2.62 | 2.64 |
| Simplified | 2.36 | 2.55 | 2.59 | 2.61 |

## 4. IMPROVEMENTS

While the methods discussed in the last section can theoretically produce very accurate results, it requires rather extensive calculation. Indeed, in large $N$, it often requires hours of calculation in a personal computer to produce the above numerical example. It is therefore clearly of interest to improve the above formulations.

### 4.1. Variable elimination in LMIs

The first method of improving the above results is the elimination of matrix variables in LMI proposed in [7]. In addition to the possibility of simplifying the LMIs by eliminating some relaxation parameters such as $W(\xi)$, it also makes it possible to allow some relaxation parameters such as $X$ in the LMIs to vary with the uncertainty $(A, B) \in \Omega$, resulting in less conservative stability criteria for uncertain systems. Although the resulting matrix inequalities involving, say $X(A, B)$, are difficult to solve directly, they can be eliminated from the LMIs analytically. The basis of the simplifications are the following three facts:

**Proposition 6.** (Elimination of an off-diagonal independent variable) There exists a matrix $X$ such that

$$\begin{pmatrix} P & Q & X \\ Q^T & R & V \\ X^T & V^T & S \end{pmatrix} > 0 \tag{27}$$

if and only if

$$\begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix} > 0 \tag{28}$$

$$\begin{pmatrix} R & V \\ V^T & S \end{pmatrix} > 0 \tag{29}$$

**Corollary 7.** (Elimination of related off-diagonal variable) There exists a matrix $X$ such that

$$\begin{pmatrix} P & Q + XE & X \\ (Q + XE)^T & R & V \\ X^T & V^T & S \end{pmatrix} > 0 \tag{30}$$

if and only if

$$\begin{pmatrix} P & Q \\ Q^T & R - VE - E^T V^T + E^T SE \end{pmatrix} > 0 \qquad \begin{pmatrix} R & V \\ V^T & S \end{pmatrix} > 0.$$

**Proposition 8.** (Elimination of a common matrix variable in a diagonal entry of two LMIs) There exists a symmetric matrix $X$ such that

$$\begin{pmatrix} P_1+X & Q_1 \\ Q_1^T & R_1 \end{pmatrix} > 0, \tag{31}$$

$$\begin{pmatrix} P_2-X & Q_2 \\ Q_2 & R_2 \end{pmatrix} > 0 \tag{32}$$

if and only if

$$\begin{pmatrix} P_1+P_2 & Q_1 & Q_2 \\ Q_1^T & R_1 & 0 \\ Q_2^T & 0 & R_2 \end{pmatrix} > 0. \tag{33}$$

The above three results are still valid for a continuum of LMIs. For example, Proposition 6 may be extended as follows: Let $P$, $Q$, $R$, $S$, $V$ are continuous matrix functions of a parameter $\theta$, and $\Theta$ is a compact set. There exists a continuous matrix function $X$ of $\theta$ to satisfy (27) for all $\theta \in \Theta$ if and only if (28) and (29) are satisfied for all $\theta \in \Theta$.

Using the above to eliminate relaxation parameters $U$ and $\tilde{W}$ in (15) and (16) results in an equivalent set of LMIs:

$$\tilde{R} > 0, \tag{34}$$

$$\hat{S} > 0, \tag{35}$$

$$\begin{pmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \frac{1}{h}\hat{S}+\tilde{R} \end{pmatrix} > 0. \tag{36}$$

For the conditions on the Lyapunov derivative, it turns out that (23) to (25) are still sufficient condition for (12), even though the parameters $X_i$, $T_{ij}$ and $Y_{ij}$ are arbitrary continuous functions of uncertain matrices $A$ and $B$. Obviously, in general, this will be less conservative than (23) to (25) with constant parameters. Eliminating these parameters using the above three facts results in two LMIs:

$$\begin{pmatrix} \Delta & \frac{1}{2}(D^1+D^0) & \frac{1}{2}(D^1-D^0) \\ & \frac{1}{h}S_d+R_d & 0 \\ \text{symmetric} & & \frac{1}{h}S_d \end{pmatrix} > 0 \tag{37}$$

$$R_d > 0. \tag{38}$$

To summarize the above discussion, a system is stable if the LMIs (34) to (38) (which we will refer to as the modified criterion) have a solution. Of course, the solution should satisfy these LMIs for all possible system matrices in the uncertainty set $(A, B) \in \Omega$.

Numerical calculation shows that the simplified version after variable elimination may give less conservative results for uncertain systems. It is interesting to note that such improvements of accuracy is not found in the calculation tried by the author for systems with two states (i. e., $x$ has two components) [12].

### 4.2. Quadratic integral inequality

A very useful quadratic integral inequality was proposed in [11]. It may be used to give less conservative result, especially if it is used in combination with the variable elimination technique. The results in this subsection is from [11] and [12]. The following quadratic integral inequality plays a central role.

**Lemma 9.** For any symmetric positive definite constant matrix $M \in \mathcal{R}^{m\times m}$, scalar $\gamma > 0$, vector function $\omega : [0,\gamma] \to \mathcal{R}^m$ such that the integrations in the following are well defined, then

$$\gamma \int_0^\gamma \omega^T(\beta) M \omega(\beta)\,\mathrm{d}\beta \geq \left(\int_0^\gamma \omega(\beta)\,\mathrm{d}\beta\right)^T M \left(\int_0^\gamma \omega(\beta)\,\mathrm{d}\beta\right). \tag{39}$$

Indeed, the above is a special case of Jensen's inequality since $x^T M x$ is a convex function of $x$.

Using (39), it can be shown that

$$\begin{aligned} V_S(\phi) &= \frac{1}{2}\int_{-r}^{0} \phi^T(\xi) S(\xi)\,\phi(\xi)\,\mathrm{d}\xi \\ &\geq \frac{h^2}{2}\int_0^1 [\tilde{\psi}^T(1) I_r - \tilde{\psi}^T(\alpha) I_d]\,\tilde{S}[I_r^T \tilde{\psi}(1) - I_d^T \tilde{\psi}(\alpha)]\,\mathrm{d}\alpha \end{aligned} \tag{40}$$

where

$$\tilde{S} = \mathrm{diag}(S_0, S_1, \ldots, S_N).$$

As a result, it is possible to prove that a sufficient condition for (11), is

$$\tilde{S} > 0 \tag{41}$$

and

$$\begin{pmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \tilde{R} + \frac{1}{h}\tilde{S} \end{pmatrix} > 0. \tag{42}$$

It is interesting to compare the above conditions with (34) to (36), which is equivalent to (15) and (16). The condition $\tilde{R} > 0$ is no longer needed, which saves computation, and has an effect of making the criterion less conservative. On the other hand, the first and last entries of $\tilde{S}$ no longer have a coefficient 2 as compared to $\hat{S}$, which tends to make the criterion more conservative. An alternative condition which is more complicated than the above conditions, but can be theoretically shown to be less conservative than (34) to (36), is also proposed in [12].

For Lyapunov derivative condition, some manipulation using a combination of the above quadratic integral inequality and variable elimination yields a substantially improved result similar idea in combination with the variable elimination technique are used. The resulting sufficient condition for (12) is

$$\begin{pmatrix} \Delta & \frac{1}{2}(D^1 + D^0) & \frac{1}{2}(D^1 - D^0) \\ & \frac{1}{h}S_d + R_d & 0 \\ \text{symmetric} & & \frac{3}{h}S_d \end{pmatrix} > 0. \tag{43}$$

Notice, not only the condition (38) is no longer needed, but the coefficient in the (3, 3) entry above is three times as large as compared to (37), making this criterion substantially less conservative. Indeed, a few seconds of computation using the above formulation often gives results of similar or higher accuracy which requires hours of calculation using the results described earlier.

**Example 10.** Consider

$$\dot{x}(t) = \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} x(t-r). \tag{44}$$

Use the criterion discussed in this subsection, the resulting $r_{\max}$ are listed in the following

| $N$ | 1 | 2 | 3 |
|---|---|---|---|
| $r_{\max}$ | 6.059 | 6.165 | 6.171 |

It is clear that the convergence to the analytical solution is greatly accelerated as compared to the results presented in the last section.

**Example 11.** Consider the following uncertain system,

$$\dot{x}(t) = \begin{pmatrix} -2+\rho(t) & \rho(t) \\ \rho(t) & -0.9+\rho(t) \end{pmatrix} x(t) + \begin{pmatrix} -1+\rho(t) & 0 \\ -1 & -1-\rho(t) \end{pmatrix} x(t-r),$$

where

$$|\rho(t)| \leq 0.1.$$

Using the stability criterion discussed in this subsection, the estimated $r_{\max}$ is listed in the following table:

| $N$ | 1 | 2 | 3 |
|---|---|---|---|
| $r_{\max}$ | 2.628 | 2.653 | 2.654 |

The dramatic improvements of accuracy with identical $N$ is again easily seen.

**Example 12.** This example has not been published before, and is included here to illustrate the fact that discretized Lyapunov functional can be used to detect the stability of a time-delay system which is unstable without delay. Consider the system

$$\ddot{x}(t) - 0.1\dot{x}(t) + 2x(t) - x(t-r) = 0.$$

This system is clearly unstable for $r = 0$. The system is stable for $r \in (r_{\min}, r_{\max})$. The stability criterion in the subsection with different $N$ is used to estimate $r_{\min}$

and $r_{\max}$. The computation consists of sweeping through a range of $r$ with relatively large step size, and a bisection process near the lower limit $r_{\min}$ and the upper limit $r_{\max}$. The result is listed in the following table along with the true stability limits

| $N$ | 1 | 2 | 3 | true limits |
|---|---|---|---|---|
| $r_{\min}$ | 0.1006 | 0.1003 | 0.1003 | 0.1002 |
| $r_{\max}$ | 1.4272 | 1.6921 | 1.7161 | 1.7178 |

## 5. SYSTEMS WITH MULTIPLE DELAYS

In this section, we will consider the uncertain system with multiple time-delays

$$\dot{x}(t) = \sum_{i=0}^{K} A^i(t)\, x(t - r^i), \tag{45}$$

where

$$0 = r^0 < r^1 < \ldots < r^K = r \tag{46}$$

are time-delays, and $A^i(t) \in \mathcal{R}^{n\times n}$, $i = 0, 1, \ldots, K$ are the uncertain matrices. The exact values of $A^i(t)$ at any given $t$ are unknown and possibly time-varying, except that they are bounded by a known set

$$\left(A^0(t), A^1(t), \ldots, A^K(t)\right) \in \Omega, \qquad \text{for any } t \geq 0. \tag{47}$$

Due to the complexity of the formulation, we will only mention the main features of the formulation. For details, see [8]. The Lyapunov functional chosen is

$$\begin{aligned} V(\phi) &= \frac{1}{2}\phi^T(0)P\phi(0) + \sum_{i=1}^{K} \phi^T(0) \int_{-r^i}^{0} Q^i(\xi)\, \phi(\xi)\, \mathrm{d}\xi \\ &+\frac{1}{2}\sum_{i=1}^{K} \int_{-r^i}^{0} \phi^T(\xi) S^i(\xi)\, \phi(\xi)\, \mathrm{d}\xi \\ &+\frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K} \int_{-r^i}^{0} \mathrm{d}\xi \int_{-r^j}^{0} \phi^T(\xi) R^{ij}(\xi,\eta)\, \phi(\eta)\, \mathrm{d}\eta, \end{aligned} \tag{48}$$

with $P = P^T$, $Q^i(\xi)$, $S^i(\xi) = S^{iT}(\xi)$, $R^{ij}(\xi,\eta) = R^{jiT}(\eta,\xi)$ continuous. It is sufficient to restrict to

$$\text{for } i = 1, 2, \ldots, K-1 \qquad \text{and } j = 1, 2, \ldots, K-1,$$

$$Q^i(\xi) = Q^i = \text{constant}, \tag{49}$$

$$S^i(\xi) = S^i = \text{constant}, \tag{50}$$

$$R^{ij}(\xi,\eta) = R^{ij} = \text{constant}, \tag{51}$$

$$\begin{aligned} R^{iK}(\xi,\eta) &= R^{iK}(\eta) = R^{KiT}(\eta) \\ &= \text{independent of } \xi. \end{aligned} \tag{52}$$

without loss of generality. A tedious calculation shows that the derivative of Lyapunov functional is again a quadratic expression

$$\begin{aligned}\dot{V}(t,\phi) &= -\frac{1}{2}\sum_{i=0}^{K}\sum_{j=0}^{K}\phi^T(-r^i)\Delta^{ij}\phi\left(-r^j\right)\\&\quad+\sum_{i=0}^{K}\sum_{j=1}^{K-1}\phi^T(-r^i)\Pi^{ij}\int_{-r^j}^{0}\phi(\xi)\,\mathrm{d}\xi+\sum_{i=0}^{K}\phi^T(-r^i)\int_{-r}^{0}\Pi^{iK}(\xi)\,\phi(\xi)\,\mathrm{d}\xi\\&\quad-\frac{1}{2}\int_{-r}^{0}\phi^T(\xi)\dot{S}^K(\xi)\,\phi(\xi)\,\mathrm{d}\xi-\sum_{i=1}^{K-1}\int_{-r^i}^{0}\phi^T(\xi)\,\mathrm{d}\xi\int_{-r}^{0}\dot{R}^{iK}(\eta)\,\phi(\eta)\,\mathrm{d}\eta\\&\quad-\frac{1}{2}\int_{-r}^{0}\phi^T(\xi)\,\mathrm{d}\xi\int_{-r}^{0}\left[\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)R^{KK}(\xi,\eta)\right]\phi(\eta)\,\mathrm{d}\eta,\end{aligned}\tag{53}$$

where the matrices $\Delta^{ij}$ and $\Pi^{ij}$ depend linearly on Lyapunov functional parameters $P$, $Q^i$, $S^i$ and $R^{ij}$, and also depend linearly on system parameters $A^i$.

In discretization, the mesh needs to be compatible with the delays in the sense that each delay needs to be exactly at a dividing point of the mesh. This may mandate a non-uniform mesh due to the possibility of incommensurate delays (i. e., the ratio between two delays is irrational). Even in the case of commensurate delays, a uniform mesh compatible with such delays may require too many dividing points for efficient computation. Therefore, the formulation here will be based on nonuniform mesh. Divide the interval $[-r,0]$ into $N$ segments of length $h_p$, $p=1,2,\dots,N$. Then the dividing points can be calculated as

$$\theta_p=-\sum_{q=1}^{p}h_q,\quad p=0,1,2,\dots,N,\tag{54}$$

and the $p$th segment starting from the origin is $[\theta_p,\theta_{p-1}]$. Notice, this convention of index direction is opposite to the single delay case discussed in Section 3. The division is made in such a way that

$$\begin{aligned}r^i &= -\theta_{N^i}=\sum_{q=1}^{N^i}h_q,\quad i=0,1,\dots,K,&(55)\\0&=N^0<N^1<\cdots<N^K=N.&(56)\end{aligned}$$

It is useful to define, for $p=0,1,\dots,N$,

$$M_p=\min\left\{i\mid N^i\geq p\right\}.\tag{57}$$

In other words, $M_p$ can take $K+1$ distinct values

$$M_p=\begin{cases}i, & \text{if}\quad N^{i-1}<p\leq N^i \text{ for some } 0<i\leq K,\\0, & \text{if}\quad p=0.\end{cases}$$

Then, the discretized Lyapunov–Krasovskii functional can be expressed as the following quadratic expression:

$$\begin{aligned}
V(\phi) &= \frac{1}{2}\int_0^1 \begin{pmatrix} \phi^T(0) & \Phi^T & [I_r^T\tilde{\psi}(1) - I_d^T\tilde{\psi}(\alpha)]^T \end{pmatrix} \\
&\begin{pmatrix} P-U & \bar{Q}-\bar{V} & \hat{Q}^K - \hat{V}^K \\ \bar{Q}^T - \bar{V}^T & \bar{R} & \check{R}^{\cdot K} \\ \hat{Q}^{KT} - \hat{V}^{KT} & \check{R}^{\cdot KT} & \hat{R}^{KK} \end{pmatrix} \\
&\begin{pmatrix} \phi(0) \\ \Phi \\ I_r^T\tilde{\psi}(1) - I_d^T\tilde{\psi}(\alpha) \end{pmatrix} \mathrm{d}\alpha + \frac{1}{2}\sum_{p=1}^{N}\int_{\theta_p}^{\theta_{p-1}} \begin{pmatrix} \phi^T(0) & \phi^T(\xi) \end{pmatrix} \\
&\left[\begin{pmatrix} U^K(\xi) & V^K(\xi) \\ V^{KT}(\xi) & S^K(\xi) \end{pmatrix} + \sum_{i=M_p}^{K-1} \begin{pmatrix} U^i & V^i \\ V^{iT} & S^i \end{pmatrix}\right] \begin{pmatrix} \phi(0) \\ \phi(\xi) \end{pmatrix} \mathrm{d}\xi
\end{aligned} \tag{58}$$

and its derivative can be expressed in another quadratic expression:

$$\begin{aligned}
\dot{V}(t,\phi) &= -\frac{1}{2}\int_0^1 \begin{pmatrix} \hat{\phi}^T & \tilde{\phi}^T(\alpha) \end{pmatrix} \\
&\begin{pmatrix} \Delta + (1-2\alpha)Y - T & X - \Pi(\alpha) \\ X^T - \Pi^T(\alpha) & S'^K - \alpha W - (1-\alpha)Z \end{pmatrix} \\
&\begin{pmatrix} \hat{\phi} \\ \tilde{\phi}(\alpha) \end{pmatrix} \mathrm{d}\alpha - \frac{1}{2}\begin{pmatrix} \hat{\phi} & \int_0^1 \tilde{\phi}^T(\alpha)\,\mathrm{d}\alpha \end{pmatrix} \\
&\begin{pmatrix} T & -X \\ -X^T & R_s'^{KK} + R_s'^{\cdot K} \end{pmatrix} \begin{pmatrix} \hat{\phi} \\ \int_0^1 \tilde{\phi}(\alpha)\,\mathrm{d}\alpha \end{pmatrix} \\
&-\frac{1}{2}\int_0^1 \mathrm{d}\alpha \int_0^\alpha \begin{pmatrix} \tilde{\phi}^T(\alpha) & \tilde{\phi}^T(\beta) \end{pmatrix} \\
&\begin{pmatrix} W & R_a'^{KK} \\ -R_a'^{KK} & Z \end{pmatrix} \begin{pmatrix} \tilde{\phi}(\alpha) \\ \tilde{\phi}(\beta) \end{pmatrix} \mathrm{d}\beta.
\end{aligned} \tag{59}$$

The notations in the matrices expressions can be found in [8]. The last term of the above expression is needed only for the case of non-uniform mesh. For uniform mesh, $R_a'^{KK} = 0$, and therefore, we may set the relaxation parameters $W = Z = 0$. Again, the stability conditions can be obtained from the positive definiteness of the matrices in the quadratic expressions. Due to space constraint, the final LMIs for stability will not be repeated here. Two numerical examples are presented in the following to illustrate the effectiveness of the method.

**Example 13.** Consider the following uncertainty-free time-delay system (there was a printing error in [8] in the $(2,2)$ entry of the first term below)

$$\dot{x}(t) = \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} [0.05x(t-0.5r) + 0.95x(t-r)].$$

It is not difficult to find that the maximum time-delay $r$ for retaining stability is

$$r_{\max} = 8.59.$$

Divide $[-0.5r, 0]$ into $N_{d1}$ uniform segments

$$h_p = \frac{0.5r}{N_{d1}}, \ p = 1, 2, \ldots, N_{d1},$$

and $[-r, -0.5r]$ into $N_{d2}$ uniform segments

$$h_p = \frac{0.5r}{N_{d2}}, \ p = N_{d1} + 1, N_{d2} + 2, \ldots, N_{d1} + N_{d2}.$$

The estimated maximum delay $r_{\max}$ for guaranteed stability is presented in the following table.

| | $N_{d2}$ | | | |
|---|---|---|---|---|
| $N_{d1}$ | 1 | 2 | 3 | 4 |
| 1 | 8.25 | 8.27 | 8.27 | 8.27 |
| 2 | 8.44 | 8.47 | 8.48 | 8.48 |
| 3 | 8.50 | 8.53 | 8.53 | 8.54 |
| 4 | 8.52 | 8.55 | 8.56 | 8.56 |

It is clear that good results can be obtained by even very coarse mesh. The convergent trend to analytical solution is also clear. It should also be noticed from the above table that a judicious choice on the distribution of segment size can produce better results with fewer dividing segments.

**Example 14.** This example considers the uncertain system

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -2+\rho(t) & \rho(t) \\ \rho(t) & -0.9+\rho(t) \end{pmatrix} x(t) \\ &\quad +0.15 \begin{pmatrix} -1+\rho(t) & 0 \\ -1 & -1-\rho(t) \end{pmatrix} x\left(t - \frac{r}{\sqrt{3}}\right) \\ &\quad +0.85 \begin{pmatrix} -1-\rho(t) & \rho(t) \\ -1-\rho(t) & -1-\rho(t) \end{pmatrix} x(t-r), \end{aligned}$$

where $\rho$ is a time-varying parameter satisfying (there was a printing error in the following uncertainty bound in [8])

$$|\rho(t)| \leq 0.05, \text{ for all } t.$$

Notice, also, that the ratio of the two delays is an irrational number, which is known to present difficulty in spectrum type of stability computation. This does not present any special difficulty for the scheme proposed here. Similar to the last example, divide $[-r/\sqrt{3}, 0]$ into $N_{d1} = 3$ segments, and $[-r, -r/\sqrt{3}]$ into $N_{d2} = 2$ segments. The computation gives a maximum delay

$$r_{\max} \geq 5.75$$

for guaranteed stability.

Of course, one may also use the variable elimination and integral inequality to improve the stability criterion for multiple delay case. The process is much more tedious, but the extent of improvement is similar according to the investigation by the author, and will be described in a future work.

## 6. DISTRIBUTED DELAY WITH PIECEWISE CONSTANT COEFFICIENTS

In this section, we will consider dynamical systems with *distributed* delays:

$$\dot{x}(t) = \int_{-r}^{0} A(\theta)\, x(t+\theta)\, \mathrm{d}\theta. \tag{60}$$

The coefficient matrix $A(\theta)$ is assumed to be piecewise constant. Similar to the multiple delay case discussed in the last section, it is necessary to include all the discontinuous points of $A(\theta)$ in the dividing points of the mesh in discretization process. This will insure that the coefficient matrix be constant within each segment. Therefore, non-uniform mesh is again usually necessary. The materials in this section are mainly from [14] and [15].

Use the quadratic Lyapunov–Krasovskii functional (3), its derivative along the system trajectory can be calculated as

$$\begin{aligned}
&\dot{V}(t,\phi) = -\frac{1}{2} \begin{pmatrix} \phi^T(0) & \phi^T(-r) \end{pmatrix} \\
&\begin{pmatrix} -Q(0) - Q^T(0) - S(0) & Q(-r) \\ Q^T(-r) & S(-r) \end{pmatrix} \begin{pmatrix} \phi(0) \\ \phi(-r) \end{pmatrix} \\
&- \begin{pmatrix} \phi^T(0) & \phi^T(-r) \end{pmatrix} \int_{-r}^{0} \begin{pmatrix} \Gamma^0(\xi) \\ \Gamma^1(\xi) \end{pmatrix} \phi(\xi)\, \mathrm{d}\xi - \frac{1}{2}\int_{-r}^{0} \phi^T(\xi)\dot{S}(\xi)\, \phi(\xi)\, \mathrm{d}\xi \\
&-\frac{1}{2}\int_{-r}^{0} \mathrm{d}\xi \int_{-r}^{0} \phi^T(\xi)[\partial_{1+2}R(\xi,\eta) - A^T(\xi)Q(\eta) - Q^T(\xi)A(\eta)]\phi(\eta)\, \mathrm{d}\eta.
\end{aligned} \tag{61}$$

The discretization is similar to the last section. Divide the interval $[-r, 0]$ into $N$ segments of length $h_p$, $p = 1, 2, \ldots, N$. Then the dividing points $\theta_p$, which should include all the discontinuities of $A(\xi)$, can be calculated by (54), and the $p$th segment is $[\theta_p, \theta_{p-1}]$. Within $p$th segment, $A(\xi)$ is a constant, which will be denoted as $A_p$. With $Q$, $S$, $R$ continuous piecewise linear, it turns out that the expression for the discretized Lyapunov–Krasovskii functional is very similar to the case with uniform mesh:

$$\begin{aligned}
V(\phi) \;=\; & \frac{1}{2}\int_{0}^{1} \begin{pmatrix} \phi^T(0) & [I_r^T\tilde{\psi}(1) - I_d^T\tilde{\psi}(\alpha)]^T \end{pmatrix} \\
& \begin{pmatrix} P - U & \hat{Q} - \hat{V} \\ \hat{Q}^T - \hat{V}^T & \hat{R} \end{pmatrix} \begin{pmatrix} \phi(0) \\ I_r^T\tilde{\psi}(1) - I_d^T\tilde{\psi}(\alpha) \end{pmatrix} \mathrm{d}\alpha \\
& + \frac{1}{2}\sum_{p=1}^{N} \int_{\theta_p}^{\theta_{p-1}} \begin{pmatrix} \phi^T(0) & \phi^T(\xi) \end{pmatrix} \begin{pmatrix} U(\xi) & V(\xi) \\ V^T(\xi) & S(\xi) \end{pmatrix} \begin{pmatrix} \phi(0) \\ \phi(\xi) \end{pmatrix} \mathrm{d}\xi.
\end{aligned}$$

Therefore, the sufficient conditions for (11) has similar form as single point delay case. The following is another equivalent form of sufficient condition for (11):

$$\begin{pmatrix} P-U & \hat{Q}-\hat{V} \\ \hat{Q}^T-\hat{V}^T & \hat{R} \end{pmatrix} > 0, \qquad \begin{pmatrix} U_p & V_p \\ V_p^T & S_p \end{pmatrix} > 0, \quad p=0,1,\ldots,N$$

where

$$U=\sum_{p=1}^{N}\frac{h_p}{2}(U_{p-1}+U_p).$$

The expression for the derivative $\dot{V}$ is more complicated:

$$\dot{V}(t,\phi)=-\frac{1}{2}\int_0^1 \begin{pmatrix} \phi^T(0) & \phi^T(-r) & \tilde{\phi}^T(\alpha)\end{pmatrix} \left[\begin{pmatrix} -Q_0-Q_0^T-S_0 & Q_N & \Pi^0(\alpha) \\ & S_N & \Pi^1(\alpha) \\ \text{sym} & & S' \end{pmatrix}\right.$$

$$\left.+\begin{pmatrix} -T+(1-2\alpha)Y & X \\ X^T & \Theta(\alpha)\end{pmatrix}\right]\begin{pmatrix}\phi(0)\\ \phi(-r)\\ \tilde{\phi}(\alpha)\end{pmatrix}\mathrm{d}\alpha$$

$$-\frac{1}{2}\begin{pmatrix}\phi^T(0) & \phi^T(-r) & \int_0^1\tilde{\phi}^T(\alpha)\,\mathrm{d}\alpha\end{pmatrix}\begin{pmatrix} T & -X \\ -X^T & R_s'-\tilde{A}^T\bar{Q}-\bar{Q}^T\tilde{A}\end{pmatrix}\begin{pmatrix}\phi(0)\\ \phi(-r)\\ \int_0^1\tilde{\phi}(\alpha)\,\mathrm{d}\alpha\end{pmatrix}$$

$$-\frac{1}{2}\int_0^1\mathrm{d}\alpha\int_0^\alpha\begin{pmatrix}\tilde{\phi}^T(\alpha) & \tilde{\phi}^T(\beta)\end{pmatrix}\begin{pmatrix} W^R & R_a' \\ R_a'^T & Z^R\end{pmatrix}\begin{pmatrix}\tilde{\phi}(\alpha)\\ \tilde{\phi}(\beta)\end{pmatrix}\mathrm{d}\beta$$

$$-\frac{1}{2}\int_0^1\mathrm{d}\alpha\int_0^1\begin{pmatrix}\tilde{\phi}^T(\alpha) & \tilde{\phi}^T(\beta)\end{pmatrix}\begin{pmatrix} W^Q(\beta) & \tilde{A}^T[\bar{Q}-\tilde{Q}(\beta)] \\ \text{sym} & Z^Q\end{pmatrix}\begin{pmatrix}\tilde{\phi}(\alpha)\\ \tilde{\phi}(\beta)\end{pmatrix}\mathrm{d}\beta$$

where,

$$\Theta(\alpha)=-\alpha W^R-(1-\alpha)Z^R-W^{Q0}-Z^Q$$

and the condition for (12) is obtained by requiring the matrices in each term of the above expression to be positive definite. See [14] for details.

Let's again look at some numerical examples:

**Example 15.** This example considers a scalar constant distributed delay

$$\dot{x}(t)=-a\int_{-r}^{0}x(t+\theta)\,\mathrm{d}\theta$$

where $a>0$, $r>0$. In [26], it was concluded that the system is stable for

$$r<r_{\max}^{\text{kr}}=\sqrt{\frac{2}{a}}.$$

The true stability limit can also be analytically calculated as

$$r_{\max}^{\text{analytical}}=\frac{\pi}{2}\sqrt{\frac{2}{a}}.$$

Let $a = 1$, with a uniformly distributed grid ($h_p = h, p = 1, 2, \ldots, N$), the discretized Lyapunov functional method is applied. The resulting stability limits obtained are listed in the following table, along with the analytical limit and the limit obtained in [26]. It is easily seen that discretized Lyapunov functional works very well, and the trend to converge to analytical solution as $N$ increases is also clear, although the rate of convergence as compared to the case of pointed delay is not as fast.

| $r_{\max}^{\text{analytical}}$ | $r_{\max}^{\text{kr}}$ | $r_{\max}^{N=1}$ | $r_{\max}^{N=2}$ | $r_{\max}^{N=5}$ | $r_{\max}^{N=10}$ |
|---|---|---|---|---|---|
| 2.22 | 1.41 | 1.18 | 1.43 | 1.87 | 2.19 |

**Example 16.** This example considers a two state case

$$\dot{x}(t) = A \int_{-r}^{0} x(t+\theta)\,\mathrm{d}\theta$$

where

$$A = \begin{pmatrix} -4 & 1 \\ 0 & -3 \end{pmatrix}.$$

The analytical stability limit can also be calculated. The numerical results are again listed along with the true stability limit in the following table. The convergence to the true stability limit is again obvious.

| $r_{\max}^{\text{true}}$ | $r_{\max}^{N=1}$ | $r_{\max}^{N=2}$ | $r_{\max}^{N=5}$ | $r_{\max}^{N=10}$ |
|---|---|---|---|---|
| 1.11 | 0.59 | 0.71 | 0.94 | 1.10 |

**Example 17.** In this example, the coefficient matrix is piecewise linear

$$\dot{x}(t) = \int_{-r_1}^{0} A_1 x(t+\theta)\,\mathrm{d}\theta + \int_{-r}^{-r_1} A_2 x(t+\theta)\,\mathrm{d}\theta$$

where

$$r_1 = \frac{r}{\sqrt{3}}$$

$$A_1 = \begin{pmatrix} -2.2 & 1 \\ 0 & -1.15 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -2.4 & 1 \\ 0 & -1.4 \end{pmatrix}.$$

Since the ratio $r_1/(r - r_1)$ is irrational, the grid has to be nonuniform. Let the interval $[-r_1, 0]$ be uniformly divided into $N_1$ segments of length $h_{(1)} = r_1/N_1$, and $[-r, -r_1]$ into $N_2$ segments of length $h_{(2)} = (r - r_1)/N_2$. In other words, $h_p = h_{(1)}$, $p = 1, 2, \ldots, N_1$; $h_p = h_{(2)}, p = N_1+1, N_1+2, \ldots, N_1+N_2$. The estimated maximum delay $r_{\max}$ for different $N_1$ and $N_2$ are listed in the following table.

| | $N_1 = 1$ | $N_1 = 2$ | $N_1 = 3$ |
|---|---|---|---|
| $N_2 = 1$ | 0.92 | 1.05 | 1.13 |
| $N_2 = 2$ | 1.01 | 1.15 | 1.22 |
| $N_2 = 3$ | 1.06 | 1.19 | 1.27 |

Again, the variable elimination and integral inequality may be used to improve the above results, and will be described in a future work.

## 7. TIME–VARYING DELAY

In this section, we will discuss the system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,x(t - r(t)), \tag{62}$$

where the time-delay and its derivative have a known bounds

$$r_m \le r(t) \le r_M, \quad \dot{r}(t) \le \beta \tag{63}$$

where $r_m > 0$, $0 \le \beta < 1$.

The materials discussed in this section are mainly from [13] and [20]. We will use a simplified version of (3). Use (62), we can write

$$\begin{aligned} x(t - r(t)) &= x(t - r_m) - \int_{-r(t)}^{r_m} \dot{x}(t+\theta)\,\mathrm{d}\theta \\ &= x(t - r_m) - \int_{-r(t)}^{r_m} [A(t+\theta)\,x(t+\theta) + B(t+\theta)\,x(t+\theta - r(t+\theta))]\,\mathrm{d}\theta \end{aligned}$$

to re-write system (62) as

$$\begin{aligned} \dot{x}(t) &= A(t)\,x(t) + B(t)\,x(t - r_m) \\ &\quad - B(t)\int_{-r(t)}^{r_m} [A(t+\theta)\,x(t+\theta) + B(t+\theta)\,x(t+\theta - r(t+\theta))]\,\mathrm{d}\theta. \end{aligned} \tag{64}$$

It should be pointed out that systems (64) and (62) are not equivalent unless additional constraints on initial conditions are imposed on (64). The stability of (64) implies that of (62) similar to the case discussed in [31]. The fact that the reverse is not true on this type of transformation was made clear in [16], with additional cases discussed in [17]. In this section, we will discuss the conditions for the stability of (64).

Let $x_t : [-r_M, 0] \to R^n$ be defined as

$$x_t(\theta) = x(t+\theta)$$

and choose the Lyapunov–Krasovskii functional $V(x_t)$ in the following form

$$\begin{aligned} V(\phi) &= \frac{1}{2}\phi^T(0)P\phi(0) + \phi^T(0)\int_{-r_m}^{0} Q(\xi)\,\phi(\xi)\,\mathrm{d}\xi \\ &+ \frac{1}{2}\int_{-r_m}^{0}\mathrm{d}\xi\int_{-r_m}^{0}\phi^T(\xi)R(\xi-\eta)\,\phi(\eta)\,\mathrm{d}\eta + \frac{1}{2}\int_{-r_m}^{0}\phi^T(\xi)S(\xi)\,\phi(\xi)\,\mathrm{d}\xi \\ &+ \frac{1}{2}\int_{-r_M}^{-r_m}\int_{\theta}^{0}\phi^T(\xi)K_1\phi(\xi)\,\mathrm{d}\xi\mathrm{d}\theta + \frac{1}{2}\int_{-r_M}^{-r_m}\int_{\theta-r(t+\theta)}^{0}\phi^T(\xi)K_2\phi(\xi)\,\mathrm{d}\xi\mathrm{d}\theta. \end{aligned} \tag{65}$$

The derivative $\dot{V}(x_t)$ satisfies

$$\dot{V}(t,\phi) \le -\frac{1}{2}\int_{-r_m}^{0}\omega^T(\xi)[\dot{S}(\xi) + H^2]^{-1}\omega(\xi)\,\mathrm{d}\xi$$

$$-\frac{1}{2}\left(\begin{array}{cc}\phi^T(0) & \phi^T(-r_m)\end{array}\right)\Delta\left(\begin{array}{c}\phi(0)\\ \phi(-r_m)\end{array}\right)$$

$$-\frac{1}{2r_m}\int_{-r_m}^{0}\int_{-r(t)}^{-r_m} q^T(t,\xi,\theta,r)$$

$$\left(\begin{array}{cccc}E_1 & E_{12} & \Delta^5(\theta) & \Delta^6(\theta)\\ E_{12}^T & E_2 & \Delta^7(\xi,\theta) & \Delta^8(\xi,\theta)\\ \Delta^{5T}(\theta) & \Delta^{7T}(\xi,\theta) & K_1 & 0\\ \Delta^{6T}(\theta) & \Delta^{8T}(\xi,\theta) & 0 & (1-\beta)K_2\end{array}\right) q(t,\xi,\theta,r)\,\mathrm{d}\theta\mathrm{d}\xi$$

where $\omega(\xi)$ is a linear combination of $\phi(0)$, $\phi(-r_m)$ and $\phi(\xi)$, and

$$q(t,\xi,\theta,r) = \left(\begin{array}{cccc}\phi^T(0) & \phi^T(\xi) & \phi^T(\theta) & \phi^T(\theta - r(t+\theta))\end{array}\right)^T$$

All the matrices entries are linear with respect to parameters. A discretization process leads to a set of LMIs. In the following, two numerical examples will be presented

**Example 18.** Consider the system

$$\dot{x}(t) = \left(\begin{array}{cc}-2 & 0\\ 0 & -0.9\end{array}\right)x(t) + \left(\begin{array}{cc}-1 & 0\\ -1 & -1\end{array}\right)x(t - r(t)). \tag{66}$$

with $r(t)$ satisfying (63). This is a slight modification of Example 4. Let $\delta = r_m/r_M$. The stability limit of $r_M$ for some different combinations of $\beta$, $\delta$ and the number of segments $N$ are listed in the following table.

| $\beta$ | $\delta$ | $N = 1$ | $N = 2$ | $N = 5$ | $N = 10$ |
|---|---|---|---|---|---|
| 0.1 | 0.9 | 2.140 | 2.143 | 2.167 | 2.195 |
| 0.9 | 0.9 | 1.554 | 1.558 | 1.576 | 1.591 |
| 0.1 | 0.5 | 1.626 | 1.627 | 1.631 | 1.635 |
| 0.9 | 0.5 | 1.000 | 1.000 | 1.000 | 1.000 |

It is also interesting to see whether the stability limit converges to the time-invariant result as the time-delay interval size approaches zero ($\delta \to 1$) without the derivative bound approaching 0 ($\beta \to 0$). The following table shows the results for $N = 1$ and $\beta = 0.1$.

| $\delta$ | 0.9 | 0.99 | 0.999 | 0.9999 | 0.99999 |
|---|---|---|---|---|---|
| $r_M$ | 2.14 | 3.851 | 5.032 | 5.270 | 5.297 |

The corresponding limit for time-invariant delay is 5.30. The convergent trend is clear.

**Example 19.** Consider the following uncertain system,

$$\dot{x}(t) = \left(\begin{array}{cc}-2+\rho(t) & \rho(t)\\ \rho(t) & -0.9+\rho(t)\end{array}\right)x(t) + \left(\begin{array}{cc}-1+\rho(t) & 0\\ -1 & -1-\rho(t)\end{array}\right)x(t - r(t)),$$

where

$$|\rho(t)| \le 0.1.$$

The results for $\beta = 0.1$ and $\delta = 0.9$ and different $N$ are listed in the following table

| $N$ | 1 | 2 | 5 | 10 |
|---|---|---|---|---|
| $r_M$ | 1.397 | 1.455 | 1.495 | 1.517 |

For $N = 1$ and $\beta = 0.1$, the following table shows that the calculated stability results as $\delta \to 1$. It approaches the constant time-delay case in Example 5.

| $\delta$ | 0.9 | 0.99 | 0.999 | 0.9999 |
|---|---|---|---|---|
| $r_M$ | 1.397 | 2.104 | 2.333 | 2.363 |

## 8. BLOCK-DIAGONAL UNCERTAINTY

In this section, we consider the system with feedback uncertainty,

$$\dot{x}(t) = Ax(t) + A_d x(t-r) + Bu(t), \tag{67}$$

$$y(t) = Cx(t) + C_d x(t-r) + Du(t), \tag{68}$$

where $A, A_d \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times p}$, $C, C_d \in \mathcal{R}^{q\times n}$ and $D \in \mathcal{R}^{q\times p}$ are constant system matrices. The system is subject to uncertain feedback

$$u = \Delta y, \tag{69}$$

where $\Delta$ is a possibly nonlinear and dynamic uncertainty. Most of the materials presented in this section is from [7]. It is assumed that the uncertainty characteristics is such that

$$V_1(u,y) = \frac{1}{2}\int_0^t y^T(t)K_t y(t)\,\mathrm{d}t - \frac{1}{2}\int_0^t u^T(t)K_u u(t)\,\mathrm{d}t \ge 0 \tag{70}$$

for any $t > 0$, and

$$(K_u, K_y) \in \mathcal{K} \subset \mathcal{R}^{q\times q} \times \mathcal{R}^{p\times p}. \tag{71}$$

The types of uncertainty which may be characterized by such an expression have been discussed extensively in the literature, see, for example, [2, 4, 7, 33, 34].

The uncertainty is of block-diagonal structure. In other words, the inputs and outputs can be partitioned into $m$ parts, and the $i$th input $u_i$ depends only on the $i$th output $y_i$ in the feedback:

$$u = (u_1^T, u_2^T, \dots, u_m^T)^T, \qquad y = (y_1^T, y_2^T, \dots, y_m^T)^T,$$

where

$$u_i(t) \in \mathcal{R}^{p_i}, \quad y_i(t) \in \mathcal{R}^{q_i},$$

and

$$u_i = \Delta_i y_i.$$

We will assume that $\Delta_i$ all have unit gain. Otherwise, a standard scaling procedure can be used. Correspondingly, $\mathcal{K}$ consists of all the diagonal matrices

$$(\mathrm{diag}(K_{u1}, K_{u2}, \ldots, K_{um}),\ \mathrm{diag}(K_{y1}, K_{y2}, \ldots, K_{ym}))$$

where

$$(K_{ui}, K_{yi}) \in \mathcal{K}_i \subset \mathcal{R}^{p_i \times p_i} \times \mathcal{R}^{q_i \times q_i}.$$

Correspondingly,

$$V_1 = \sum_{i=1}^{m} V_{1i}.$$

To study the stability of systems with block-diagonal uncertainty, choose a quadratic Lyapunov functional

$$V : \mathcal{C} \times \mathcal{R}_+ \mapsto \mathcal{R}, \qquad V(x_t, t) = V_0(x_t) + V_1(u, y), \tag{72}$$

where $V_1$ is defined in (70), and

$$\begin{aligned} V_0(\phi) &= \frac{1}{2}\phi^T(0)P\phi(0) + \phi^T(0)\int_{-r}^{0} Q(\xi)\,\phi(\xi)\,\mathrm{d}\xi \\ &+\frac{1}{2}\int_{-r}^{0}\mathrm{d}\xi\int_{-r}^{0}\phi^T(\xi)R(\xi,\eta)\,\phi(\eta)\,\mathrm{d}\eta + \frac{1}{2}\int_{-r}^{0}\phi^T(\xi)\,S(\xi)\,\phi(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{73}$$

Due to the uncertainty characteristics (70), $V \geq V_0$. Therefore, it is sufficient for $V_0$ instead of $V$ to satisfy (11), conditions of which have already been discussed in earlier sections. The derivative can be calculated as:

$$\begin{aligned} &\dot{V}_0(t, u, \phi) = \phi^T(0)\,PBu + u^T B^T \int_{-r}^{0} Q(\xi)\,\phi(\xi)\,\mathrm{d}\xi \\ &-\frac{1}{2}\phi^T(0)[-PA - A^T P - Q(0) - Q^T(0) - S(0)]\phi(0) \\ &-\frac{1}{2}\phi^T(-r)S(-r)\,\phi(-r) - \frac{1}{2}\int_{-r}^{0}\phi^T(\xi)\dot{S}(\xi)\,\phi(\xi)\,\mathrm{d}\xi \\ &-\frac{1}{2}\int_{-r}^{0}\mathrm{d}\xi\int_{-r}^{0}\phi^T(\xi)\partial_{1+2}R(\xi,\eta)\,\phi(\eta)\,\mathrm{d}\eta + \phi^T(0)[PA_d - Q(-r)]\phi(-r) \\ &+\phi^T(0)\int_{-r}^{0}[A^T Q(\xi) - \dot{Q}(\xi) + R^T(\xi, 0)]\phi(\xi)\,\mathrm{d}\xi \\ &+\phi^T(0)\int_{-r}^{0}[A_d^T Q(\xi) - R^T(\xi, -r)]\phi(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{74}$$

Using (68) and (70), it can be obtained

$$\dot{V}_1(t, u, \phi) = \frac{1}{2}[C\phi(0) + C_d\phi(-r) + Du(t)]^T K_y [C\phi(0) + C_d\phi(-r) + Du(t)] - \frac{1}{2}u^T K_u u.$$

Therefore, $\dot{V} = \dot{V}_0 + \dot{V}_1$ can be written in a quadratic form of $\phi(0)$, $\phi(-r)$, $\phi(\xi)$, $u$ and $y$. Discretization process can be used to find condition for (12).

The idea of block-diagonal feedback uncertainty formulation, of course, can be extended to systems with multiple delays or distributed delays.

## 9. CONCLUSIONS

The discretized Lyapunov functional method is an effective method for developing computable stability criteria for time-delay systems. For time-invariant systems without uncertainty, the analytical limit may be approached as the grid size approaches zero. Various versions of formulations are possible due to the eliminations of parameters and utilizations of quadratic integral inequalities. Extensions to systems with multiple delays, distributed delays with piecewise constant coefficients as well as time-varying delays are possible. The modeling errors may often be modeled as a block-diagonal uncertainty with effective computational algorithms.

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