KYBERNETIKA — VOLUME 37 (2001), NUMBER 4, PAGES 459–478

AN ASYMPTOTIC STATE OBSERVER FOR A CLASS OF NONLINEAR DELAY SYSTEMS

ALFREDO GERMANI, COSTANZO MANES AND PIERDOMENICO PEPE

The problem of state reconstruction from input and output measurements for nonlinear time delay systems is studied in this paper and a state observer is proposed that is easy to implement and, under suitable assumptions on the system and on the input function, gives exponential observation error decay. The proposed observer is itself a delay system and can be classified as an *identity observer*, in that it is such that if at a given time instant the system and observer states coincide, on a suitable Hilbert space, the observation error remains zero in all following time instants. The computation of the observer gain is straightforward. Computer simulations are reported that show the good performance of the observer.

1. INTRODUCTION

As well-known the state space of time delay systems has infinite dimension. This fact leads to difficulties not only in the system analysis and in the synthesis of controllers and/or observers, but also on their physical implementation. In the case of linear delay systems the control problem and the state observation problem, both in deterministic and stochastic settings, have been extensively studied in the past [1, 2, 6, 7, 15, 16, 18, 21, 22, 23, 24, 25, 26] and are still under investigation. In the case of nonlinear delay systems in recent years some papers on the approximation of dynamics and on control problems have appeared [8, 13, 19, 20, 23]. Difficulties arise in dealing with such systems due to the fact that the state space has infinite dimension and moreover the differential description is nonlinear.

In [8] a formalism has been introduced to overcome these difficulties for an interesting class of nonlinear delay systems. A feedback law for output control has been proposed there, that requires the knowledge of all system variables. Preliminary results on the problem of state reconstruction for nonlinear delay systems has been presented in [10, 11]. In [14] the problem of state reconstruction for nonlinear output-delay systems is considered.

In this paper a state observer for nonlinear delay systems is proposed and conditions for exponential observation error decay are given. As in [8], some concepts of standard nonlinear analysis [17] are extended to the case of delay systems and used to work out the observer equations and to prove convergence, following the same approach used in [3, 4, 12, 14]. The observer-gain computation is very straightforward and the implementation is easy. Although only single-input single-output systems are considered here, for simplicity, the same construction can be used to develop observers for multi-input multi-output systems.

The paper is organized as follows. Section 2 reports the necessary notations, definitions and preliminary results. In Section 3 the state observation problem is formulated, an observer is proposed and theoretical results are shown. An example of application is worked out and simulation results are reported in Section 4. Conclusions follow.

2. PRELIMINARIES

In this section some notations and definitions necessary for the analysis of the state observation problem and for the synthesis of the observer are presented in short. The formalism used has been introduced in [8].

The system under investigation is described by the following equation

$$\dot{x}(t) = f(x(t), x(t-\Delta)) + g(x(t), x(t-\Delta))\, u(t), \tag{2.1}$$

$$y(t) = h(x(t)), \qquad t \geq 0, \tag{2.2}$$

where $\Delta \geq 0$, $x(t) \in I\!R^n$, $u(t) \in I\!R$ and $y(t) \in I\!R$, the vector functions f and g are C^∞ with respect to both arguments, and h is a C^∞ scalar function. The model description is completed by the initial state in the space of C^1 functions in $[-\Delta, 0]$:

$$x(\tau) = \vartheta(\tau), \quad \tau \in [-\Delta, 0], \quad \vartheta \in C^1\left([-\Delta, 0], I\!R^n\right). \tag{2.3}$$

Throughout the paper, for a given function $q(t) \in I\!R^m$, the symbol $q_{i\Delta}(t)$, with i nonnegative integer, will denote its translation by $-i\Delta$, i.e. $q_{i\Delta}(t) = q(t - i\Delta)$. Some care must be put on the interval on which the translated function is defined. For instance, being $x(t)$ defined for $t \geq -\Delta$, the delayed function $x_{i\Delta}(t)$ is defined for $t \geq (i-1)\Delta$, while $u_{i\Delta}(t)$ is defined for $t \geq i\Delta$, being $u(t)$ defined for $t \geq 0$.

Also the following notation is needed in the paper: consider vectors $\chi_i \in I\!R^n$ and scalars v_i, with i integer. The symbols $\mathcal{X}_{i,j}$ and $\mathcal{V}_{i,j}$, with $i \leq j$, will denote the composed vectors

$$\mathcal{X}_{i,j} = \begin{bmatrix} \chi_i \\ \chi_{i+1} \\ \vdots \\ \chi_j \end{bmatrix} \in I\!R^{(j-i+1)n}, \qquad \mathcal{V}_{i,j} = \begin{bmatrix} v_i \\ v_{i+1} \\ \vdots \\ v_j \end{bmatrix} \in I\!R^{j-i+1}. \tag{2.4}$$

Here follows the definition of *observation delay relative degree* for nonlinear delay systems, a weaker version of the concept of delay relative degree introduced in [8].

Definition 2.1. System (2.1), (2.2) is said to have observation delay relative degree r in an open set $\Omega_r \in I\!R^{n(r+1)}$ if the following conditions are verified

$$\begin{aligned} &\forall\, \mathcal{X}_{0,r} \in \Omega_r, \quad L_G L_F^k H(\mathcal{X}_{0,r}) = 0, \qquad k = 0,1,\ldots,r-2, \\ &\exists\, \mathcal{X}_{0,r} \in \Omega_r : \quad L_G L_F^{r-1} H(\mathcal{X}_{0,r}) \neq 0, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} F(\mathcal{X}_{0,r}) &= \begin{bmatrix} f(\chi_0,\chi_1) \\ f(\chi_1,\chi_2) \\ \vdots \\ f(\chi_{r-1},\chi_r) \end{bmatrix}, \quad G(\mathcal{X}_{0,r}) = \text{diag}_{i=0}^{r-1}\{g(\chi_i,\chi_{i+1})\}, \\ H(\mathcal{X}_{0,r}) &= h(\chi_0), \end{aligned} \tag{2.6}$$

$$\begin{aligned} L_F^0 H(\mathcal{X}_{0,r}) &= H(\mathcal{X}_{0,r}), \\ L_F^k H(\mathcal{X}_{0,r}) &= \left(\tfrac{d}{d\mathcal{X}_{0,r-1}} L_F^{k-1} H\right) F(\mathcal{X}_{0,r}), \quad k \leq r \\ L_G L_F^k H(\mathcal{X}_{0,r}) &= \left(\tfrac{d}{d\mathcal{X}_{0,r-1}} L_F^{k} H\right) G(\mathcal{X}_{0,r}), \quad k \leq r-1. \end{aligned} \tag{2.7}$$

If $\Omega_r = I\!R^{n(r+1)}$, the system is said to have uniform observation delay relative degree equal to r.

Remark 2.2. Note that the term $L_F^k H(\mathcal{X}_{0,r})$, $k \leq r$, is actually a function of $\mathcal{X}_{0,k}$, and the term $L_G L_F^k H(\mathcal{X}_{0,r})$, $k \leq r-1$, is a function of $\mathcal{X}_{0,k+1}$.

Remark 2.3. The computation of the observation delay relative degree of a nonlinear delay system is made applying Definition 2.1 to integers $r = 1, 2, \ldots$, until the conditions (2.5) are verified.

At this point it is useful the definition of the *stack* operator.

Definition 2.4. Consider a function $q(t) \in I\!R^m$, defined for $t \in [t_1, t_2] \subseteq I\!R$. The symbol $Stack_{i,j}(q)$, with i, j such that $0 \leq j - i \leq (t_2 - t_1)/\Delta$, denotes the following function, defined for $t \in [t_1 + j\Delta, t_2 + i\Delta]$,

$$Stack_{i,j}(q)(t) = \begin{bmatrix} q_{i\Delta}(t) \\ q_{(i+1)\Delta}(t) \\ \vdots \\ q_{j\Delta}(t) \end{bmatrix} \in I\!R^{(j-i+1)m}. \tag{2.8}$$

Using the stack operator, the following vector functions can be defined:

$$X_{i,j}(t) = Stack_{i,j}(x)(t), \qquad U_{i,j}(t) = Stack_{i,j}(u)(t) \tag{2.9}$$

Using the previous definitions, the following differential equation can be derived, that holds for $t \geq (r-1)\Delta$,

$$\begin{aligned} \dot{X}_{0,r-1}(t) &= F(X_{0,r}(t)) + G(X_{0,r}(t))U_{0,r-1}(t), \\ y(t) &= H(X_{0,r}(t)), \end{aligned} \tag{2.10}$$

and the following map can be defined

$$z = \Phi(\mathcal{X}_{0,r-1}) = \begin{bmatrix} H(\mathcal{X}_{0,0}) \\ L_F H(\mathcal{X}_{0,1}) \\ \vdots \\ L_F^{r-1} H(\mathcal{X}_{0,r-1}) \end{bmatrix}. \tag{2.11}$$

For systems having observation delay relative degree r in Ω_r it is

$$\begin{aligned} y^{(k)}(t) &= L_F^k H\big(X_{0,k}(t)\big), \qquad k = 0,1,\dots r-1 \\ y^{(r)}(t) &= L_F^r H\big(X_{0,r}(t)\big) + L_G L_F^{r-1} H\big(X_{0,r}(t)\big) U_{0,r-1}(t), \end{aligned} \tag{2.12}$$

and therefore substitution of $X_{0,r-1}(t)$ in the map $\Phi(\cdot)$ provides the output derivatives up to order $(r-1)$

$$z(t) = \Phi\big(X_{0,r-1}(t)\big) = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(r-1)}(t) \end{bmatrix}. \tag{2.13}$$

Note that, being $x(t)$ defined for $t \geq -\Delta$, it follows that $X_{0,r-1}(t)$ and $z(t)$ are well defined for $t \geq (r-2)\Delta$.

In the next section an observer for nonlinear delay systems is presented and exponential state observation is proved under suitable assumptions.

Among the assumptions, in the case in which the input $u(t)$ is not identically zero, the following is needed:

Hp_0) system (2.1), (2.2) has uniform observation delay relative degree equal to n (the dimension of vector $x(t)$).

Note that under assumption Hp_0 the vector $z(t) \in I\!R^n$ is defined for $t \geq (n-2)\Delta$. Defining the Brunovsky triple

$$\begin{aligned} A_b &= \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{bmatrix}, \quad B_b = \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix}, \\ C_b &= \begin{bmatrix} 1 & 0_{1\times(n-1)} \end{bmatrix}, \end{aligned} \tag{2.14}$$

it can be verified that, thanks to (2.10), it is

$$\begin{aligned} \frac{d\Phi(\mathcal{X}_{0,n-1})}{d\mathcal{X}_{0,n-1}} F(\mathcal{X}_{0,n}) &= A_b \Phi(\mathcal{X}_{0,n-1}) + B_b L_F^n H(\mathcal{X}_{0,n}), \\ \frac{d\Phi(\mathcal{X}_{0,n-1})}{d\mathcal{X}_{0,n-1}} G(\mathcal{X}_{0,n}) &= B_b L_G L_F^{n-1}(\mathcal{X}_{0,n}), \\ H(\mathcal{X}_{0,n}) &= C_b \Phi(\mathcal{X}_{0,n-1}). \end{aligned} \tag{2.15}$$

From these, the following equation can be derived for the dynamics of the variable $z(t)$ defined in (2.13)

$$\begin{aligned} \dot{z}(t) &= A_b z(t) + B_b\Big(L_F^n H\big(X_{0,n}(t)\big) + L_G L_F^{n-1} H\big(X_{0,n}(t)\big)U_{0,n-1}(t)\Big), \\ y(t) &= C_b z(t), \qquad t \geq (n-1)\Delta. \end{aligned} \tag{2.16}$$

The pair A_b, C_b is observable, and it is an easy matter to assign eigenvalues to the matrix $A_b - KC_b$, that has the companion structure

$$A_b - KC_b = \begin{bmatrix} -k_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -k_{n-1} & 0 & \cdots & 1 \\ -k_n & 0 & \cdots & 0 \end{bmatrix}. \tag{2.17}$$

Let $K(\lambda)$ denote the gain vector that assigns eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$ to matrix $A_b - K(\lambda)C_b$ (the gain $K(\lambda)$ contains the coefficients of the monic polynomial that has the λ_j's as roots). If eigenvalues λ_j's are distinct, the matrix $A_b - K(\lambda)C_b$ is diagonalized by the Vandermonde matrix

$$V(\lambda) = \begin{bmatrix} \lambda_1^{n-1} & \cdots & \lambda_1 & 1 \\ \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ \lambda_n^{n-1} & \cdots & \lambda_n & 1 \end{bmatrix}, \tag{2.18}$$

so that

$$V(\lambda)\big(A_b - K(\lambda)C_b\big)V(\lambda)^{-1} = \mathrm{diag}\{\lambda\} = \Lambda. \tag{2.19}$$

Lemma 2.5. For any positive a, b, there exists $\lambda \in I\!R^n$ that satisfies $\lambda_n < \cdots < \lambda_1 < 0$, such that

$$b\|V^{-1}(\lambda)\| + \lambda_1 = -a. \tag{2.20}$$

Proof. In [3] it is shown that if the n reals λ_j are chosen as functions of a parameter $\rho > 0$ as follows: $\lambda_j(\rho) = -\rho^j$, for $j = 1, \ldots, n$, then

$$\lim_{\rho\to+\infty} \|V^{-1}(\lambda(\rho))\| = 1, \qquad \lim_{\rho\to 0^+} \|V^{-1}(\lambda(\rho))\| = +\infty. \tag{2.21}$$

It follows that the function $\sigma(\rho)$ defined for $\rho \in (0, +\infty)$ as

$$\sigma(\rho) = b\|V^{-1}(\lambda(\rho))\| - \rho, \tag{2.22}$$

is a continuous function such that

$$\lim_{\rho\to 0^+} \sigma(\rho) = +\infty, \qquad \lim_{\rho\to+\infty} \sigma(\rho) = -\infty. \tag{2.23}$$

This implies that there exists at least one solution $\bar{\rho}$ for the equation $\sigma(\rho) = -a$. Moreover, $\bar{\rho}$ is such that the vector $\lambda(\bar{\rho})$ solves equation (2.20). □

The following three lemmas are required in the proof of convergence of the proposed observer.

Lemma 2.6. Let c_0, c_1, c_2 and β be positive constants, and let $s(t)$ be a non negative function, defined for $t \in [t_0 - n\Delta, \infty)$, that for $t \geq t_0$ satisfies the following inequality

$$s(t) \leq c_0 e^{-\beta(t-t_0)} + c_1 \int_{t_0}^{t} e^{-\beta(t-\tau)} \sum_{i=1}^{n} s(\tau - i\Delta)\, \mathrm{d}\tau + c_2 \sum_{i=1}^{n-1} s(t - i\Delta). \qquad (2.24)$$

Let $s_0(t)$ be a function such that:

$$s_0(\theta) \geq s(\theta), \qquad \theta \in [t_0 - n\Delta, t_0], \qquad (2.25)$$

and for $t \geq t_0$

$$s_0(t) = c_0 e^{-\beta(t-t_0)} + c_1 \int_{t_0}^{t} e^{-\beta(t-\tau)} \sum_{i=1}^{n} s_0(\tau - i\Delta)\, \mathrm{d}\tau + c_2 \sum_{i=1}^{n-1} s_0(t - i\Delta). \quad (2.26)$$

Then

$$s_0(t) \geq s(t), \qquad t \in [t_0 - n\Delta, \infty). \qquad (2.27)$$

Proof. Consider the function $\delta(t) = s_0(t) - s(t)$. By assumption it is $\delta(t) \geq 0$ in $[t_0 - n\Delta, t_0]$. The theorem is proven by induction, by showing that if for a given non negative integer i it is $\delta(t) \geq 0$ in $[t_0 - n\Delta, t_0 + i\Delta]$, then it follows that $\delta \geq 0$ in $[t_0 - n\Delta, t_0 + (i+1)\Delta]$. This result is obtained by writing the inequality

$$\delta(t) \geq c_1 \int_{t_0}^{t} e^{-\beta(t-\tau)} \sum_{i=1}^{n} \delta(\tau - i\Delta)\, \mathrm{d}\tau + c_2 \sum_{i=1}^{n-1} \delta(t - i\Delta), \qquad (2.28)$$

that is obtained subtracting from both sides of equation (2.26) both sides of inequality (2.24). Being positive the constants c_1 and c_2, and being positive $\delta(t)$ for $t \in [t_0 - n\Delta, t_0 + i\Delta]$ by assumption, from (2.28) it follows that $\delta(t)$ is positive also for $t \in (t_0 + i\Delta, t_0 + (i+1)\Delta]$, and therefore in $[t_0 - n\Delta, t_0 + (i+1)\Delta]$. By finite induction it follows that for all $t \in [t_0 - n\Delta, \infty)$ it is $\delta(t) \geq 0$, that is the thesis. □

Lemma 2.7. Let c_0, c_1, c_2 and β be positive constants. If

$$\frac{c_1}{\beta} + c_2(n-1) < 1, \qquad (2.29)$$

then equation (2.26) admits the solution

$$s_0(t) = \bar{s}_0 e^{-\alpha(t-t_0)}, \quad \bar{s}_0 > 0,\ \alpha > 0, \qquad t \in [t_0 - n\Delta, +\infty) \qquad (2.30)$$

where the coefficient α is the unique solution in $[0, \beta)$ of equation

$$\frac{c_1}{\beta - \alpha} \sum_{i=1}^{n} (e^{\alpha\Delta})^i + c_2 \sum_{i=1}^{n-1} (e^{\alpha\Delta})^i = 1, \qquad (2.31)$$

and $\bar{s}_0$ is given by

$$\bar{s}_0 = \frac{c_0}{1 - c_2 \sum\limits_{i=1}^{n-1} (e^{\alpha\Delta})^i}. \tag{2.32}$$

Proof. First of all note that the function $\sigma : [0, \beta) \mapsto [0, \infty)$, defined as

$$\sigma(\alpha) = \frac{c_1}{\beta - \alpha} \sum_{i=1}^{n} (e^{\alpha\Delta})^i + c_2 \sum_{i=1}^{n-1} (e^{\alpha\Delta})^i, \tag{2.33}$$

is a continuous, monotonically increasing function such that

$$\sigma(0) = \frac{c_1}{\beta} + c_2(n-1) < 1, \qquad \lim_{\alpha \to \beta^-} \sigma(\alpha) = +\infty. \tag{2.34}$$

It follows that one and only one solution to equation $\sigma(\alpha) = 1$ exists in $(0, \beta)$. Now, by direct substitution, it can be readily verified that the expression (2.30) is a solution of equation (2.26) with α solution of (2.31) and $\bar{s}_0$ given by (2.32). Note that the denominator of (2.32) is positive because it is

$$1 - c_2 \sum_{i=1}^{n-1} (e^{\alpha\Delta})^i = \frac{c_1}{\beta - \alpha} \sum_{i=1}^{n} (e^{\alpha\Delta})^i, \tag{2.35}$$

where $c_1 > 0$ and $\beta > \alpha$. □

Lemma 2.8. Let c_0, c_1, c_2, β and s_M be positive constants, and let $s(t)$ be a non negative function defined for $t \in [t_0 - n\Delta, \infty)$, such that for $t \in [t_0 - n\Delta, t_0]$ it is $s(t) \leq s_M$ and for $t \geq t_0$ it satisfies inequality (2.24).

Then, if

$$\frac{c_1}{\beta} + c_2(n-1) < 1, \tag{2.36}$$

the following inequality holds

$$s(t) \leq \bar{s}_0 e^{-\alpha(t - t_0)}, \qquad t \geq t_0 - n\Delta, \tag{2.37}$$

where α is the unique solution of eq. (2.31) in $[0, \beta)$ and $\bar{s}_0$ is given by

$$\bar{s}_0 = \max \left\{ \frac{c_0}{1 - c_2 \sum\limits_{i=1}^{n-1} (e^{\alpha\Delta})^i}, s_M \right\}. \tag{2.38}$$

Proof. Lemma 2.7 ensures that the solution of (2.31) in $[0, \beta)$ exists unique, and that $1 - c_2 \sum\limits_{i=1}^{n-1} (e^{\alpha\Delta})^i$ is finite and positive, so that $\bar{s}_0$ is well defined. Define now

$$s_0(t) = \bar{s}_0 e^{-\alpha(t - t_0)} \tag{2.39}$$

and

$$\bar{c}_0 = \bar{s}_0\Big(1 - c_2\sum_{i=1}^{n-1}(e^{\alpha\Delta})^i\Big). \tag{2.40}$$

From (2.38) it is $\bar{c}_0 \geq c_0$. Moreover it is $s_0(t) \geq s(t)$, for $t \in [t_0 - n\Delta, t_0]$ and for $t \geq t_0$

$$s_0(t) = \bar{c}_0 e^{-\beta(t-t_0)} + c_1\int_{t_0}^{t} e^{-\beta(t-\tau)}\sum_{i=1}^{n} s_0(\tau - i\Delta)\,\mathrm{d}\tau + c_2\sum_{i=1}^{n-1} s_0(t - i\Delta), \tag{2.41}$$

as it can be checked by direct substitution. Note that, being $\bar{c}_0 \geq c_0$, in $[t_0 - n\Delta, t_0]$ it is also

$$s(t) \leq \bar{c}_0 e^{-\beta(t-t_0)} + c_1\int_{t_0}^{t} e^{-\beta(t-\tau)}\sum_{i=1}^{n} s(\tau - i\Delta)\,\mathrm{d}\tau + c_2\sum_{i=1}^{n-1} s(t - i\Delta), \tag{2.42}$$

and therefore the assumptions of Lemma 2.6 are satisfied, so that $s(t) \leq s_0(t)$, for all $t \in [t_0 - n\Delta, \infty)$. This proves the theorem. □

Let ϑ be a function in $C^1\big([-\Delta, 0], I\!R^n\big)$. In the following the symbol $x(t;\vartheta)$ will denote the state trajectory of the nonlinear delay system (2.1), (2.2) when the initial state is ϑ (note that for $\tau \in [-\Delta, 0]$ it is $x(\tau;\vartheta) = \vartheta(\tau)$). In the same way $z(t;\vartheta)$ will denote the vector of output derivatives defined in (2.13) (with $r = n$) when the initial state is ϑ.

Definition 2.9. A system of the form (2.1), (2.2) is said to be *globally delay drift-detectable* if for $u(t) \equiv 0$, and for any pair of initial states $\vartheta, \bar{\vartheta} \in C^1\big([-\Delta, 0], I\!R^n\big)$, the inequality

$$\|z(t;\vartheta) - z(t,\bar{\vartheta})\| \leq \nu e^{-\beta(t-t_0)}, \qquad t \geq t_0, \tag{2.43}$$

where t_0, ν and β are positive real, implies the inequality

$$\|x(t;\vartheta) - x(t,\bar{\vartheta})\| \leq \mu e^{-\alpha(t-t_0)}, \qquad t \geq t_0, \tag{2.44}$$

for suitable positive μ and α.

Remark 2.10. It can be shown that this definition, when applied to linear delay systems, implies the definition of $(-\alpha)$-*observability* given in [18], where inequality (2.43) is substituted by equality $\|z(t;\vartheta) - z(t,\bar{\vartheta})\| = 0,\ t \geq t_0$.

Throughout the paper it will be referred to the map $z = \Phi(\mathcal{X}_{0,n-1})$ as the *observability map* of system (2.1) – (2.1), because suitable assumptions on this map imply delay drift-detectability of the system and allow the construction of an observer.

The observability map can be seen as a square map from χ_0 to z, in which the sub-vector $\mathcal{X}_{1,n-1} \in I\!R^{n(n-1)}$ is considered as a vector of parameters. To stress this point of view, in the following the map Φ will be rewritten as follows

$$z = \Phi(\chi_0, \mathcal{X}_{1,n-1}). \tag{2.45}$$

Definition 2.11. The observability map associated to a system (2.1), (2.2) is said to be *globally partially invertible* if, for any $\mathcal{X}_{1,n-1} \in I\!\!R^{n(n-1)}$, the map (2.45) is a diffeomorphism in $I\!\!R^n$.

The inverse of the map (2.45) can be denoted as

$$\chi_0 = \Phi^{-1}(z, \mathcal{X}_{1,n-1}). \tag{2.46}$$

Substituting the trajectories $z(t)$ and $X_{1,n-1}(t)$ in the inverse map one has

$$x(t) = \Phi^{-1}\big(z(t), X_{1,n-1}(t)\big), \quad t \geq (n-2)\Delta. \tag{2.47}$$

The expression (2.47) can be substituted in the nonlinear perturbation term in expression (2.16) of system (2.1), (2.2), yielding

$$\begin{aligned} \dot{z}(t) &= A_b z(t) + B_b \widetilde{L}\big(z(t), X_{1,n}(t), U_{0,n-1}(t)\big), \\ y(t) &= C_b z(t), \qquad t \geq (n-1)\Delta, \end{aligned} \tag{2.48}$$

in which the function $\widetilde{L}(\cdot,\cdot,\cdot)$ is defined as

$$\begin{aligned} \widetilde{L}(z, \mathcal{X}_{1,n}, \mathcal{V}_{0,n-1}) &= L_F^n H\big(\Phi^{-1}(z, \mathcal{X}_{1,n-1}), \mathcal{X}_{1,n-1}\big) \\ &\quad + L_G L_F^{n-1} H\big(\Phi^{-1}(z, \mathcal{X}_{1,n-1}), \mathcal{X}_{1,n-1}\big)\mathcal{V}_{0,n-1}. \end{aligned} \tag{2.49}$$

The differential equation (2.48) can be used for the description of system (2.1), (2.2) completing it with (2.47) and by writing the following updating equation for $X_{1,n}(t)$ for $t \geq (n-1)\Delta$

$$X_{1,n}(t) = Stack_{1,n}(x)(t) = Stack_{1,n}\Big(\Phi^{-1}(z, X_{1,n-1})\Big)(t). \tag{2.50}$$

For a correct initialization of system (2.48) at the initial time $t_0 = (n-1)\Delta$ the vector $X_{1,n-1}(t)$ in the interval $[(n-2)\Delta, (n-1)\Delta]$ is needed. Since it is $X_{1,n-1}(t) = Stack_{1,n-1}(x)(t)$, the knowledge of $x(t)$ in $[-\Delta, (n-1)\Delta]$ is required.

An assumption that will be needed later in the paper is the following:

Hp_1) The observability map of system (2.1), (2.2) is such that there exist positive $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ such that

$$\|\Phi(\chi_0, \mathcal{X}_{1,n-1}) - \Phi(\hat{\chi}_0, \widehat{\mathcal{X}}_{1,n-1})\| + \tilde{\gamma}_1 \|\mathcal{X}_{1,n-1} - \widehat{\mathcal{X}}_{1,n-1}\| \geq \tilde{\gamma}_0 \|\chi_0 - \hat{\chi}_0\|,$$
$$\text{with} \quad \frac{\tilde{\gamma}_1}{\tilde{\gamma}_0}(n-1) < 1. \tag{2.51}$$

The following theorem can be given.

Theorem 2.12. If the observability map of system (2.1), (2.2) is globally partially invertible and if assumption Hp_1 holds, then the system is globally drift-detectable.

Proof. Inequality (2.51) of assumption Hp_1 can be rewritten as

$$\begin{aligned} \|\chi_0 - \hat{\chi}_0\| \;\le\;& \tfrac{1}{\tilde{\gamma}_0}\|\Phi(\chi_0, \mathcal{X}_{1,n-1}) - \Phi(\hat{\chi}_0, \widehat{\mathcal{X}}_{1,n-1})\| \\ &+ \tfrac{\tilde{\gamma}_1}{\tilde{\gamma}_0}\sum_{i=1}^{n-1} \|\chi_i - \hat{\chi}_i\|. \end{aligned} \tag{2.52}$$

From (2.47), for any pair $\vartheta, \bar{\vartheta} \in C^1([-\Delta, 0], I\!R^n)$ it is, for $t \ge (n-2)\Delta$,

$$\begin{aligned} z(t;\vartheta) &= \Phi\big(x(t;\vartheta), X_{1,n-1}(t;\vartheta)\big), \\ z(t;\bar{\vartheta}) &= \Phi\big(x(t;\bar{\vartheta}), X_{1,n-1}(t;\bar{\vartheta})\big). \end{aligned} \tag{2.53}$$

With substitutions $\chi_0 = x(t;\vartheta)$, $\hat{\chi}_0 = x(t;\bar{\vartheta})$, $\mathcal{X}_{1,n-1} = X_{1,n-1}(t;\vartheta)$, $\widehat{\mathcal{X}}_{1,n-1} = X_{1,n-1}(t;\bar{\vartheta})$, inequality (2.52) becomes

$$\|x(t;\vartheta) - \hat{x}(t,\bar{\vartheta})\| \le \gamma_0\|z(t;\vartheta) - z(t;\bar{\vartheta})\| + \gamma_1 \sum_{i=1}^{n-1} \|x_{i\Delta}(t;\vartheta) - \hat{x}_{i\Delta}(t;\bar{\vartheta})\|, \tag{2.54}$$

where $\gamma_0 = 1/\tilde{\gamma}_0$ and $\gamma_1 = \tilde{\gamma}_1/\tilde{\gamma}_0$.

Let $s(t) = \|x(t;\vartheta) - \hat{x}(t,\bar{\vartheta})\|$. If there exist positive ν and β such that inequality (2.43) holds, (t_0 necessarily must be greater than $(n-2)\Delta$), then from (2.54) the following is derived

$$s(t) \le \gamma_0 \nu e^{-\beta(t-t_0)} + \gamma_1 \sum_{i=1}^{n-1} s(t - i\Delta). \tag{2.55}$$

Since, by assumption Hp_1, it is $\gamma_1(n-1) < 1$, then a constant c_1 can be chosen small enough to satisfy

$$\frac{c_1}{\beta} + \gamma_1(n-1) < 1. \tag{2.56}$$

From (2.55) it is also

$$s(t) \le \gamma_0 \nu e^{-\beta(t-t_0)} + c_1 \int_{t_0}^{t} e^{\beta(t-\tau)} \gamma_1 \sum_{i=1}^{n} s(t - i\Delta) + \gamma_1 \sum_{i=1}^{n-1} s(t - i\Delta), \tag{2.57}$$

being positive the integral term. From Lemma 2.8, there exist positive μ and α such that $s(t) \le \mu e^{-\alpha(t-t_0)}$, $t \ge t_0$. Recalling the definition of $s(t)$, this inequality is precisely inequality (2.44), so that drift-detectability is proved. □

3. AN OBSERVER FOR NONLINEAR DELAY SYSTEMS

From Definition 2.11 the matrix function

$$Q_0(\mathcal{X}_{0,n-1}) = \frac{\partial \Phi(\chi_0, \mathcal{X}_{1,n-1})}{\partial \chi_0} \tag{3.1}$$

associated to a globally partially invertible observability map is nonsingular for all $\mathcal{X}_{0,n-1} \in I\!R^{n^2}$. Moreover it is such that

$$\frac{\partial \Phi^{-1}(z, \mathcal{X}_{1,n-1})}{\partial z}\bigg|_{z=\Phi(\mathcal{X}_{0,n-1})} = Q_0^{-1}(\mathcal{X}_{0,n-1}). \tag{3.2}$$

Also the following matrices can be defined

$$Q_i(\mathcal{X}_{0,n-1}) = \frac{\partial \Phi(\mathcal{X}_{0,n-1})}{\partial \chi_i}, \quad i = 1, \dots, n-1. \tag{3.3}$$

In many cases it will be preferred to split the argument of Q_i and of Q_0^{-1}, writing $Q_i(\chi_0, \mathcal{X}_{1,n-1})$ and $Q_0^{-1}(\chi_0, \mathcal{X}_{1,n-1})$.

The proposed observer for nonlinear delay systems that are globally delay drift-detectable is the following:

$$\begin{aligned} \dot{\hat{x}}(t) &= f(\hat{x}(t), \hat{x}_\Delta(t)) + g(\hat{x}(t), \hat{x}_\Delta(t))u(t) + w(t), \qquad t \geq 0 \\ w(t) &= Q_0^{-1}(\hat{x}(t), \widehat{X}_{1,n-1}(t)) \left(K\left(y(t) - h(\hat{x}(t))\right) - \sum_{i=1}^{n-1} Q_i\left(\hat{x}(t), \widehat{X}_{1,n-1}(t)\right) w_{i\Delta}(t) \right) \end{aligned} \tag{3.4}$$

with initial conditions

$$\begin{aligned} &\hat{x}(\tau) = \xi(\tau), \qquad \xi \in C^1([-n\Delta, 0]; I\!R^n), \\ &w(\tau) = \dot{\xi}(\tau) - f(\xi(\tau), \xi(\tau - \Delta)) - g(\xi(\tau), \xi(\tau - \Delta))\tilde{u}(\tau), \\ &\widehat{X}_{1,n-1}(t) = Stack_{1,n-1}(\xi)(t), \quad t \in [-\Delta, 0], \end{aligned} \tag{3.5}$$

in which $\tilde{u}(\tau)$ in $[-(n-1)\Delta, 0]$ is any bounded extension of the function $u(t)$ for negative times.

The gain vector $K \in I\!R^n$ is chosen such to assign the eigenvalues to the matrix $A_b - KC_b$. The function ξ that initializes the observer represents the a priori knowledge on the system state.

Lemma 3.1. Let system (2.1), (2.2) have observation delay relative degree equal to n, and let the observability map be globally partially invertible. Then, for $t \geq 0$, the observed state $\hat{x}(t)$ provided by the observer (3.4) can be obtained as follows

$$\hat{x}(t) = \Phi^{-1}\left(\hat{z}(t), \widehat{X}_{1,n-1}(t)\right), \tag{3.6}$$

where

$$\begin{aligned} \dot{\hat{z}}(t) &= A_b\hat{z}(t) + B_b\Big(L_F^n H\big(\widehat{X}_{0,n}(t)\big) + L_G L_F^{n-1} H\big(\widehat{X}_{0,n}(t)\big) U_{0,n-1}(t)\Big) \\ &\quad + K\big(y(t) - C_b\hat{z}(t)\big) \\ \hat{z}(0) &= \Phi(\widehat{X}_{0,n-1}(0)), \end{aligned} \tag{3.7}$$

and the initial values are chosen as in (3.5).

Proof. It is sufficient to verify that differentiation of (3.6) and substitution of (3.7) gives back the observer equation (3.4). In this proof the input $u(t)$ is extended in the interval $[-(n-1)\Delta, 0]$ by the function $\tilde{u}(t)$ used for the observer initialization in (3.5), so that $U_{0,n-1}(t)$ is defined for $t \geq 0$.

Consider the map (2.13) in the form

$$\hat{z}(t) = \Phi\big(\hat{x}(t), \widehat{X}_{1,n-1}(t)\big) = \Phi\big(\hat{x}(t), \hat{x}_\Delta(t), \ldots, \hat{x}_{(n-1)\Delta}(t)\big). \tag{3.8}$$

Differentiation of (3.6), recalling definitions (3.1) – (3.3) gives

$$\dot{\hat{x}}(t) = Q_0^{-1}\big(\widehat{X}_{0,n-1}(t)\big)\left(\dot{\hat{z}}(t) - \sum_{i=1}^{n-1} Q_i\big(\widehat{X}_{0,n-1}(t)\big)\dot{\hat{x}}_{i\Delta}(t)\right). \tag{3.9}$$

By (3.7), recalling also the definition (3.1), it is

$$\begin{aligned}\dot{\hat{x}}(t) \;&=\; Q_0^{-1}\big(\widehat{X}_{0,n-1}(t)\big)\Big(A_b\Phi\big(\widehat{X}_{0,n-1}(t)\big) + B_b\big(L_F^n H\big(\widehat{X}_{0,n}(t)\big)\\ &\quad + L_G L_F^{n-1} H\big(\widehat{X}_{0,n}(t)\big)U_{0,n-1}(t)\big) + K\big(y(t) - h(\hat{x}(t))\big)\Big)\\ &\quad - Q_0^{-1}\big(\widehat{X}_{0,n-1}(t)\big)\sum_{i=1}^{n-1} Q_i\big(\widehat{X}_{0,n-1}\big)\dot{\hat{x}}_{i\Delta}(t).\end{aligned} \tag{3.10}$$

Note that it is

$$\begin{aligned}\frac{d\Phi\big(\mathcal{X}_{0,n-1}\big)}{d\mathcal{X}_{0,n-1}}F(\mathcal{X}_{0,n}) \;&=\; \sum_{i=0}^{n-1} Q_i\big(\mathcal{X}_{0,n-1}\big)f(\chi_i, \chi_{i+1}),\\ \frac{d\Phi\big(\mathcal{X}_{0,n-1}\big)}{d\mathcal{X}_{0,n-1}}G(\mathcal{X}_{0,n})\mathcal{V}_{0,n-1} \;&=\; \sum_{i=0}^{n-1} Q_i\big(\mathcal{X}_{0,n-1}\big)g(\chi_i, \chi_{i+1})v_i.\end{aligned} \tag{3.11}$$

From these and from the first two equations of (2.15), with the substitution $\mathcal{X}_{0,n-1} = \widehat{X}_{0,n-1}(t)$ and $\mathcal{V}_{0,n-1} = U_{0,n-1}(t)$, it follows that for $t \geq 0$

$$A_b\Phi(\widehat{X}_{0,n-1}) + B_b L_F^n H(\widehat{X}_{0,n}) = \sum_{i=0}^{n-1} Q_i\big(\widehat{X}_{0,n-1}\big)f(\hat{x}_{i\Delta}, \hat{x}_{(i+1)\Delta}), \tag{3.12}$$

and

$$B_b L_G L_F^{n-1} H(\widehat{X}_{0,n})U_{0,n-1} = \sum_{i=0}^{n-1} Q_i\big(\widehat{X}_{0,n-1}\big)g(\hat{x}_{i\Delta}, \hat{x}_{(i+1)\Delta})u_{i\Delta}. \tag{3.13}$$

Substitution of these in (3.10) gives the following differential equation for $t \geq 0$

$$\begin{aligned}&\dot{\hat{x}} = f(\hat{x}, \hat{x}_\Delta) + g(\hat{x}, \hat{x}_\Delta)u + Q_0^{-1}\big(\widehat{X}_{0,n-1}\big)K\left(y - h(\hat{x})\right)\\ &-Q_0^{-1}\big(\widehat{X}_{0,n-1}\big)\sum_{i=1}^{n-1} Q_i\big(\widehat{X}_{0,n-1}\big)\left(\dot{\hat{x}}_{i\Delta} - f(\hat{x}_{i\Delta}, \hat{x}_{(i+1)\Delta}) - g(\hat{x}_{i\Delta}, \hat{x}_{(i+1)\Delta})u_{i\Delta}\right).\end{aligned} \tag{3.14}$$

Consider now that from the first of (3.4) for $t \geq 0$ it is

$$w(t) = \dot{\hat{x}}(t) - f(\hat{x}(t), \hat{x}(t-\Delta)) - g(\hat{x}(t), \hat{x}(t-\Delta))u(t), \tag{3.15}$$

and that (3.15) holds also for $t \in [-(n-1)\Delta, 0]$ thanks to the initialization (3.5). It follows that for $t \geq 0$ and $i = 0, 1, \ldots, n-1$ it is

$$w_{i\Delta}(t) = \dot{\hat{x}}_{i\Delta}(t) - f\big(\hat{x}_{i\Delta}(t), \hat{x}_{(i+1)\Delta}(t)\big) - g\big(\hat{x}_{i\Delta}(t), \hat{x}_{(i+1)\Delta}(t)\big)u_{i\Delta}(t). \tag{3.16}$$

Substitution of (3.16) in (3.14) gives back the observer equation (3.4). □

Remark 3.2. Using the definition (2.49) of the function $\widetilde{L}(\cdot,\cdot,\cdot)$ the observer (3.7) can be written as

$$\begin{aligned} \dot{\hat{z}}(t) &= A_b\hat{z}(t) + B_b\widetilde{L}\big(\hat{z}(t), \widehat{X}_{1,n}(t), U_{0,n-1}(t)\big) + K\big(y(t) - C_b\hat{z}(t)\big) \\ \hat{z}(0) &= \Phi(\widehat{X}_{0,n-1}(0)), \\ \hat{x}(t) &= \Phi^{-1}\big(\hat{z}(t), \widehat{X}_{1,n-1}(t)\big), \qquad t \geq 0, \end{aligned} \tag{3.17}$$

with the initial values chosen as in (3.5).

Remark 3.3. On the basis of expression (3.7) for the observer, it can be noted that if for a given $\bar{t}$ it is $\hat{x}(\tau) = x(\tau)$ for $\tau \in [\bar{t}-(n-1)\Delta, \bar{t}]$, it follows that $\hat{x}(t) = x(t)$ for all $t > \bar{t}$. This result follows from the fact that coincidence of x and $\hat{x}$ on the interval $\tau \in [\bar{t}-(n-1)\Delta, \bar{t}]$ implies that $w(\tau) = 0$ and $y(\tau) = h(\hat{x}(\tau))$ on the same interval, so that for $t \geq \bar{t}$ the feedback terms in (3.4) are identically zero.

Now it is possible to give the main result of the paper, that is the convergence theorem for the proposed observer (3.4).

Theorem 3.4. Consider system (2.1), (2.2) and assume the following assumptions:

H_1) the system has observation delay relative degree equal to n (Hp_0) and there exists a positive u_M such that $|u(t)| \leq u_M \; \forall\, t \geq 0$;

H_2) the observability map is globally partially invertible;

H_3) the observability map satisfies assumption Hp_1;

H_4) there exists a positive $\gamma_{\widetilde{L}}$ such that

$$\sup_{\mathcal{V}_{0,n-1}\in S} \|\widetilde{L}(z, \mathcal{X}_{1,n}, \mathcal{V}_{0,n-1}) - \widetilde{L}(\hat{z}, \widehat{\mathcal{X}}_{1,n}, \mathcal{V}_{0,n-1})\| \leq \gamma_{\widetilde{L}} \left\| \begin{matrix} z - \hat{z} \\ \mathcal{X}_{1,n} - \widehat{\mathcal{X}}_{1,n} \end{matrix} \right\|, \tag{3.18}$$

where $S = [-u_M, u_M]^n \subset I\!R^n$;

H_5) the observation error $\|x(t) - \hat{x}(t)\|$ is bounded in $[-\Delta, (n-1)\Delta]$.

Then, there exists a gain vector $K \in \mathbb{R}^n$ to be put in the observer (3.4) such that

$$\|x(t) - \hat{x}(t)\| \le \mu_0 e^{-\alpha t}, \tag{3.19}$$

for suitable positive μ_0 and α.

Proof. Let λ be a n-ple of real eigenvalues, with $\lambda_n < \ldots < \lambda_2 < \lambda_1 < 0$. Let $K(\lambda)$ be the gain vector that assigns such eigenvalues to matrix $A_b - K(\lambda)C_b$. Consider the expression (2.48) of system (2.1) and the expression (3.17) of the observer. For $t \ge (n-1)\Delta$, the dynamics of the error in z-coordinates $e_z = z - \hat{z}$ can be written as

$$\dot{e}_z = (A_b - K(\lambda)C_b)e_z + B_b\Big(\widetilde{L}(z, X_{1,n}, U_{0,n-1}) - \widetilde{L}(\hat{z}, \widehat{X}_{1,n}, U_{0,n-1})\Big). \tag{3.20}$$

As stated in the introductory section, the Vandermonde matrix $V(\lambda)$ defined in (2.18) diagonalizes $A_b - K(\lambda)C_b$. Let $\xi(t) = V(\lambda)e_z(t)$ and let $\Lambda = \mathrm{diag}\{\lambda\}$, so that

$$\dot{\xi}(t) = \Lambda\xi(t) + V(\lambda)B_b\Big(\widetilde{L}(z(t), X_{1,n}(t), U_{0,n-1}(t)) - \widetilde{L}(\hat{z}(t), \widehat{X}_{1,n}(t), U_{0,n-1}(t))\Big). \tag{3.21}$$

Note that assumption H_4 implies that for $t \ge (n-1)\Delta$

$$\|\widetilde{L}(z, X_{1,n}, U_{0,n-1}) - \widetilde{L}(\hat{z}, \widehat{X}_{1,n}, U_{0,n-1})\| \le \gamma_{\widetilde{L}}\Big(\|e_z\| + \sum_{i=1}^{n} \|x_{i\Delta} - \hat{x}_{i\Delta}(\tau)\|\Big), \tag{3.22}$$

By integration of (3.20), taking also into account that $\|V(\lambda)B_b\| = \sqrt{n}$ and that $\|e_z\| \le \|V^{-1}(\lambda)\| \cdot \|\xi\|$, it follows

$$\begin{aligned}\|\xi(t)\| \le{} & e^{\lambda_1(t-t_0)}\|\xi(t_0)\| \\ & + \int_{t_0}^{t} e^{\lambda_1(t-\tau)}\sqrt{n}\gamma_{\widetilde{L}}\left(\|V^{-1}(\lambda)\| \cdot \|\xi(\tau)\| + \sum_{i=1}^{n} \|x_{i\Delta}(\tau) - \hat{x}_{i\Delta}(\tau)\|\right) d\tau,\end{aligned} \tag{3.23}$$

where $t_0 = (n-1)\Delta$. Rewriting (3.23) in terms of the variable $e^{-\lambda_1(t-t_0)}\|\xi(t)\|$, applying the Gronwall inequality and returning to $\|\xi(t)\|$, yields

$$\begin{aligned}\|\xi(t)\| \le{} & e^{\left(\sqrt{n}\gamma_{\widetilde{L}}\|V^{-1}(\lambda)\| + \lambda_1\right)(t-t_0)}\|\xi(t_0)\| \\ & + \int_{t_0}^{t} e^{\left(\sqrt{n}\gamma_{\widetilde{L}}\|V^{-1}(\lambda)\| + \lambda_1\right)(t-\tau)} \cdot \sqrt{n}\gamma_{\widetilde{L}} \sum_{i=1}^{n} \|x_{i\Delta}(\tau) - \hat{x}_{i\Delta}(\tau)\| \, d\tau.\end{aligned} \tag{3.24}$$

Being assumption Hp_1 satisfied, and being

$$\begin{aligned} z(t) &= \Phi\big(x(t), X_{1,n-1}(t)\big), \\ \hat{z}(t) &= \Phi\big(\hat{x}(t), \widehat{X}_{1,n-1}(t)\big), \end{aligned} \tag{3.25}$$

it is, from (2.54),

$$\|x(t) - \hat{x}(t)\| \le \gamma_0\|e_z(t)\| + \gamma_1 \sum_{i=1}^{n-1} \|x_{i\Delta}(t) - \hat{x}_{i\Delta}(t)\|, \tag{3.26}$$

where $\gamma_0 = 1/\tilde{\gamma}_0$ and $\gamma_1 = \tilde{\gamma}_1/\tilde{\gamma}_0$. Since it is $\|e_z(t)\| \leq \|V^{-1}(\lambda)\| \cdot \|\xi(t)\|$, and $\|\xi(t_0)\| \leq \|V(\lambda)\| \cdot \|e_z(t_0)\|$, it follows

$$\begin{aligned} \|x(t) - \hat{x}(t)\| &\leq \gamma_0 \|V^{-1}(\lambda)\| e^{\left(\sqrt{n}\gamma_{\tilde{L}} \|V^{-1}(\lambda)\| + \lambda_1\right)(t-t_0)} \|V(\lambda)\| \cdot \|e_z(t_0)\| \\ &+ \gamma_0 \|V^{-1}(\lambda)\| \int_{t_0}^{t} e^{\left(\sqrt{n}\gamma_{\tilde{L}} \|V^{-1}(\lambda)\| + \lambda_1\right)(t-\tau)} \cdot \sqrt{n}\gamma_{\tilde{L}} \sum_{i=1}^{n} \|x_{i\Delta}(\tau) - \hat{x}_{i\Delta}(\tau)\| \, \mathrm{d}\tau \\ &+ \gamma_1 \sum_{i=1}^{n-1} \|x_{i\Delta}(t) - \hat{x}_{i\Delta}(t)\|. \end{aligned} \tag{3.27}$$

Note that, setting $s(t) = \|x(t) - \hat{x}(t)\|$, inequality (3.27) has the same structure of inequality (2.24) considered in Lemmas from 2.6 to 2.8, with

$$\begin{aligned} \beta &= \sqrt{n}\gamma_{\tilde{L}} \|V^{-1}(\lambda)\| + \lambda_1, \\ c_0 &= \gamma_0 \|V^{-1}(\lambda)\| \cdot \|V(\lambda)\| \cdot \|e_z(t_0)\|, \\ c_1 &= \gamma_0 \|V^{-1}(\lambda)\| \sqrt{n}\gamma_{\tilde{L}}, \\ c_2 &= \gamma_1. \end{aligned} \tag{3.28}$$

Moreover, by assumption Hp_1, it is $c_2(n-1) < 1$. It follows that a sufficiently large positive constant β can be chosen such that

$$\frac{c_1}{\beta} + c_2(n-1) < 1. \tag{3.29}$$

By Lemma 2.5 it is always possible to choose a set of eigenvalues λ such to ensure

$$\sqrt{n}\gamma_{\tilde{L}} \|V^{-1}(\lambda)\| + \lambda_1 = \beta. \tag{3.30}$$

Then, all the assumptions of Lemma 2.8 are satisfied by inequality (3.27), with $t_0 = (n-1)\Delta$. It follows that there exist positive μ_0 and α such to satisfy (3.19), and this proves the theorem. □

Remark 3.5. It must be stressed that only assumption H_2 of Theorem 3.4 is *necessary* for the observer implementation. The other conditions are only *sufficient* to ensure exponential convergence of the observation error to zero. Indeed, in computer simulation, the observer performed well also on many systems that did not satisfy such conditions.

Remark 3.6. An interesting class of systems that satisfy hypotheses of Theorem 3.4 is the one described by the following nth order differential equation

$$\begin{aligned} x^{(n)}(t) &= \varphi(x(t), x(t-\Delta), x^{(1)}(t), x^{(1)}(t-\Delta), \ldots, x^{(n-1)}(t), x^{(n-1)}(t-\Delta)) \\ &\quad + \psi(x(t), x(t-\Delta), x^{(1)}(t), x^{(1)}(t-\Delta), \ldots, x^{(n-1)}(t), x^{(n-1)}(t-\Delta)) u(t), \\ y(t) &= x(t), \end{aligned} \tag{3.31}$$

where $x^{(i)}(t)$ denotes the ith derivative of the scalar function $x(t)$, for any given C^∞ Lipschitz functions φ and ψ.

4. EXAMPLE

In this section let ${}^i x_{j\Delta}(t)$, $i = 1, 2$, denote the ith component of a vector $x \in I\!R^2$ delayed of $j\Delta$, that is ${}^i x_{j\Delta}(t) = {}^i x(t - j\Delta)$, $j = 0, 1, 2, \ldots$.

Consider the following nonlinear delay system:

$$\begin{aligned} {}^1\dot{x}(t) &= -3\,{}^2x(t) + 0.5\,{}^1x_\Delta(t)\,{}^2x_\Delta(t), \\ {}^2\dot{x}(t) &= -({}^1x_\Delta)^2(t)\,{}^2x_\Delta(t) + u(t), \\ y(t) &= {}^1x(t). \end{aligned} \tag{4.1}$$

The observation delay relative degree can be computed exploiting Definition 2.1, obtaining $r = n = 2$. Computations give

$$\begin{aligned} H(X_{0,2}) &= {}^1x, \\ F(X_{0,2}) &= \begin{bmatrix} -3\,{}^2x + 0.5\,{}^1x_\Delta\,{}^2x_\Delta \\ -({}^1x_\Delta)^2\,{}^2x_\Delta \\ -3\,{}^2x_\Delta + 0.5\,{}^1x_{2\Delta}\,{}^2x_{2\Delta} \\ -({}^1x_{2\Delta})^2\,{}^2x_{2\Delta} \end{bmatrix} \quad G(X_{0,2}) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \\ L_F H(X_{0,2}) &= [1 \quad 0 \quad 0 \quad 0]\, F(X_{0,2}) = -3\,{}^2x + 0.5\,{}^1x_\Delta\,{}^2x_\Delta \\ L_F^2 H(X_{0,2}) &= [0 \quad -3 \quad 0.5\,{}^2x_\Delta \quad 0.5\,{}^1x_\Delta]\, F(X_{0,2}) \\ &= 3({}^1x_\Delta)^2\,{}^2x_\Delta + 0.5\,{}^2x_\Delta(-3\,{}^2x_\Delta + 0.5\,{}^1x_{2\Delta}\,{}^2x_{2\Delta}) \\ &\quad +0.5\,{}^1x_\Delta(-({}^1x_{2\Delta})^2\,{}^2x_{2\Delta}) \\ L_G H(X_{0,2}) &= [1 \quad 0 \quad 0 \quad 0]\, G(X_{0,2}) = [0 \quad 0] \\ L_G L_F H(X_{0,2}) &= [0 \quad -3 \quad 0.5\,{}^2x_\Delta \quad 0.5\,{}^1x_\Delta]\, G(X_{0,2}) = [-3 \quad 0.5\,{}^1x_\Delta]. \end{aligned}$$

The map Φ is as follows

$$z = \Phi(x, x_\Delta) = \begin{bmatrix} {}^1x \\ -3\,{}^2x + 0.5\,{}^1x_\Delta\,{}^2x_\Delta \end{bmatrix},$$

and

$$Q_0^{-1}(x, x_\Delta) = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}.$$

In this case, the observer has the following equations:

$$\begin{aligned} {}^1\dot{\hat{x}} &= -3\,{}^2\hat{x} + 0.5\,{}^1\hat{x}_\Delta\,{}^2\hat{x}_\Delta + {}^1w, \\ {}^2\dot{\hat{x}} &= -\,{}^1\hat{x}_\Delta^2\,{}^2\hat{x}_\Delta + u + {}^2w, \\ w &= \begin{bmatrix} {}^1w \\ {}^2w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} K(y - {}^1\hat{x}) - \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0.5\,{}^2\hat{x}_\Delta & 0.5\,{}^1\hat{x}_\Delta \end{bmatrix} w_\Delta, \end{aligned} \tag{4.2}$$

where, as usual, $w_\Delta(t) = w(t - \Delta)$. In the simulations here reported it has been taken $\Delta = 0.1$. The initial state of the system has been chosen constant over the interval$[-\Delta, 0]$

$$x(\tau) = \begin{bmatrix} \bar{x} \\ \bar{x} \end{bmatrix}, \qquad \tau \in [-0.1, 0]. \tag{4.3}$$

Two simulations are reported here, one with $\bar{x} = 1$ and one with $\bar{x} = -1$. In both cases the observer has been initialized with

$$\hat{x}(\tau) = 0, \quad w(\tau) = 0, \quad \tau \in [-0.1, 0]. \tag{4.4}$$

The vector K has been chosen such to assign eigenvalues $\lambda = (-1, -2)$ for the matrix $A - KC$ in (2.17). The input applied is $u(t) = \sin 4t$. In Figures 1–2 the two components of the true and estimated state are plotted, in the interval $[-0.1, 5]$, in the case of $\bar{x} = 1$. Figures 3–4 report simulation results for $\bar{x} = -1$.

Many simulations on different systems have been carried out, and in most cases they showed good performance, also when hypotheses of Theorem 3.4 were not satisfied. The observer has been tested successfully also with respect to robustness to disturbance on output measures.

5. CONCLUSIONS

An observer for a class of nonlinear delay systems has been proposed in this paper. The observer is very easy to implement, and sufficient conditions for the convergence of the estimated state to the true one are provided. Global and delay independent results are presented in this paper. Computer simulations have shown the good performance of the proposed observer.

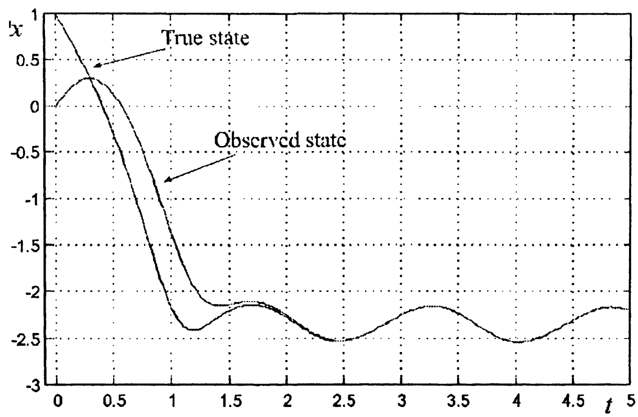

Fig. 1. True and observed variable ${}^1x$ in the case of $\bar{x} = 1$.

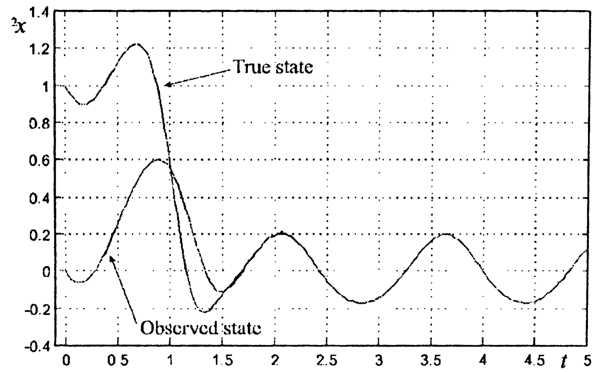

Fig. 2. True and observed variable ^2x in the case of $\bar{x} = 1$.

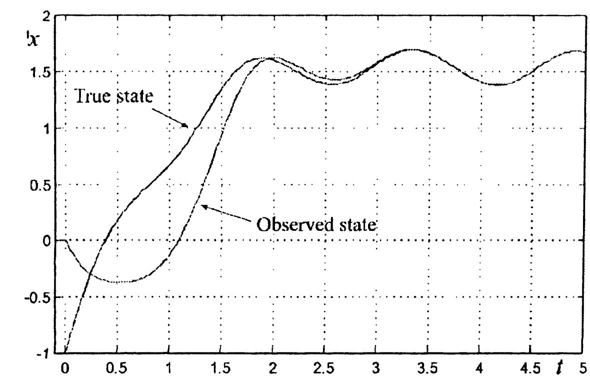

Fig. 3. True and observed variable ^1x in the case of $\bar{x} = -1$.

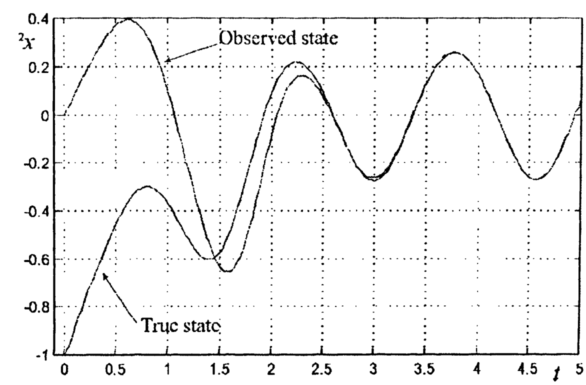

Fig. 4. True and observed variable ^2x in the case of $\bar{x} = -1$.

ACKNOWLEDGEMENT

This work is supported by MURST (*Ministry for University and Technological Research*) and by ASI (*Italian Aerospace Agency*).

(Received November 22, 2000.)

REFERENCES

[1] H. T. Banks and F. Kappel: Spline approximations for functional differential equations. J. Differential Equations *34* (1979), 496–522.
[2] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter: Representation and control of Infinite Dimensional Systems. Birkhauser, Boston 1992.
[3] G. Ciccarella, M. Dalla Mora, and A. Germani: A Luenberger-like observer for nonlinear systems. Internat. J. Control *57* (1993), 3, 537–556.
[4] M. Dalla Mora, A. Germani, and C. Manes: Design of state observers from a drift-observability property. IEEE Trans. Automat. Control *45* (2000), 6, 1536–1540.
[5] M. Dambrine, A. Goubet, and J. P. Richard: New results on constrained stabilizing control of time-delay systems. In: Proc. 34th IEEE Conference on Decision and Control, Vol. 2, New Orleans 1995, pp. 2052–2057.
[6] F. W. Fairman and A. Kumar: Delayless observers for systems with delay. IEEE Trans. Automat. Control *AC-31* (1986), 3, 258–259.
[7] A. Fattouh, O. Sename, and J. M. Dion: Robust observer design for time-delay sysems: a Riccati equation approach. Kybernetika *35* (1999), 6, 753–764.
[8] A. Germani, C. Manes, and P. Pepe: Linearization of input-output mapping for nonlinear delay systems via static state feedback. In: Proc. of IMACS Multiconference on Computational Engineering in Systems Applications, Vol. 1, Lille 1996, pp. 599–602.
[9] A. Germani, C. Manes, and P. Pepe: Linearization and Decoupling of nonlinear delay systems. In: Proc. IEEE 1998 American Control Conference (ACC'98), Philadelphia 1998.
[10] A. Germani, C. Manes, and P. Pepe: A state observer for nonlinear delay systems. In: Proc. 37th IEEE Conference on Decision and Control (CDC'98), Tampa 1998, Vol. 1, pp. 355–360.
[11] A. Germani, C. Manes, and P. Pepe: An observer for M.I.M.O. nonlinear delay systems. In: IFAC World Congress 99, Beijing 1999, Vol. E, pp. 243–248.
[12] A. Germani and C. Manes: State observers for nonlinear systems with Smooth/Bounded Input. Kybernetika *35* (1999), 4, 393–413.
[13] A. Germani, C. Manes, and P. Pepe: Local asymptotic stability for nonlinear state feedback delay systems. Kybernetika *36* (2000), 1, 31–42.
[14] A. Germani, C. Manes, and P. Pepe: State observation of nonlinear systems with delayed Output Measurements. In: IFAC Workshop on Time Delay Systems (LTDS2000), Ancona 2000.
[15] A. Germani, C. Manes, and P. Pepe: A twofold spline approximation for finite horizon LQG control of hereditary systems. SIAM J. Control Optim. *39* (2000), 4, 1233–1295.
[16] J. S. Gibson: Linear quadratic optimal control of hereditary differential systems: infinite-dimensional Riccati equations and numerical approximations. SIAM J. Control Optim. *31* (1983), 95–139.
[17] A. Isidori: Nonlinear Control Systems. Third edition. Springer–Verlag, Berlin 1995.
[18] E. B. Lee and A. W. Olbrot: Observability and related structural results for linear hereditary systems. Internat. J. Control *34* (1981), 6, 1061–1078.
[19] B. Lehman, J. Bentsman, S. V. Lunel, and E. I. Verriest: Vibrational control of nonlinear time lag systems with bounded delay: averaging theory, stabilizability, and transient behavior. IEEE Trans. Automat. Control *5* (1994), 898–912.

[20] C. H. Moog, R. Castro, and M. Velasco: The disturbance decoupling problem for nonlinear systems with multiple time-delays: static state feedback solutions. In: Proc. IMACS Multiconference on Computational Engineering in Systems Applications, Lille 1996.
[21] A. W. Olbrot: Observability and observers for a class of Linear systems with delays. IEEE Trans. Automat. Control *AC-26* (1981), 2, 513–517.
[22] A. E. Pearson and Y. A. Fiagbedzi: An observer for time lag systems. IEEE Trans. Automat. Control *34* (1989), 7, 775–777.
[23] I. G. Rosen: Difference equation state approximations for nonlinear hereditary control problems. SIAM J. Control Optim. *2* (1984), 302–326.
[24] D. Salamon: Observers and duality between observation and state feedback for time delay systems. IEEE Trans. Automat. Control *AC-25* (1980), 6, 1187–1192.
[25] K. Watanabe: Finite spectrum assignment and observer for multivariable systems with commensurate delays. IEEE Trans. Automat. Control *AC-31* (1986), 6, 543–550.
[26] Y. X. Yao, Y. M. Zhang, and R. Kovacevic: Functional observer and state feedback for input time-delay systems. Internat. J. Control *66* (1997), 4, 603–617.

Prof. Dr. Alfredo Germani, Dr. Costanzo Manes and Dr. Pierdomenico Pepe, Dipartimento di Ingegneria Elettrica, Università degli Studi dell'Aquila, Monteluco di Roio, 67040 L'Aquila. Italy.
e-mails: germani,manes,pepe@ing.univaq.it