KYBERNETIKA — VOLUME 37 (2001), NUMBER 3, PAGES 345–353

STABILIZATION OF FRACTIONAL EXPONENTIAL SYSTEMS INCLUDING DELAYS

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This paper analyzes the BIBO stability of fractional exponential delay systems which are of retarded or neutral type. Conditions ensuring stability are given first. As is the case for the classical class of delay systems these conditions can be expressed in terms of the location of the poles of the system. Then, in view of constructing robust BIBO stabilizing controllers, explicit expressions of coprime and Bézout factors of these systems are determined. Moreover, nuclearity is analyzed in a particular case.

1. INTRODUCTION

Systems with scalar transfer functions which involve polynomials and exponentials of fractional powers of s combined with delays are considered in this paper. Many examples of fractional differential systems can be found in the literature. Simple examples such as $G(s) = \frac{\exp(-a\sqrt{s})}{s}$ with $a > 0$ arising in the theory of transmission lines are given in [12]. Several examples linked to the heat equation which leads to transfer functions such as $G(s) = \frac{\cosh(\sqrt{s}x)}{\sqrt{s}\sinh(\sqrt{s})}$ (with $0 \le x \le 1$) or $G(s) = \frac{2e^{-a\sqrt{s}}}{b(1-e^{-2a\sqrt{s}})}$ can be found in [4] and [9] for example. Also, we can find in [8] a fractional delay system with transfer function of the type $G(s) = \frac{e^{-s}}{a\sqrt{s}+b+(c\sqrt{s}+d)e^{-2s}}$ that arises in the study of pressure signal transmission in a tube with viscous perturbation presented in [10].

The stability properties of such fractional systems have been studied in [8, 9], [3] and [2]. Reference [9] and [3] consider transfer functions with polynomials and exponentials in fractional powers of s (reference [9] considering the particular case of the heat equation) as reference [8] and [2] analyze transfer functions with polynomials in fractional powers of s and delay terms (reference [7] considering the particular case of transfer functions of the type $G(s) = \frac{1}{as^\alpha+b+(cs^\alpha+d)e^{-hs}}$, a, b, c, d, h real).

We consider here a larger class of fractional systems, that is systems with transfer function which contain polynomials and exponentials of fractional powers of s as well as delay terms. These systems may model for example the behaviour of a delayed heat equation. The main result of this paper is the characterization of their

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BIBO stability through a condition given in terms of the location of their poles. This condition is necessary and sufficient in the case of retarded systems and only sufficient in the case of neutral systems. The analysis of retarded systems continues with the determination of explicit expressions of their coprime factorization and associated Bézout factors which are of use when constructing robust BIBO stabilizing controllers. Finally, we give a necessary and sufficient condition for nuclearity in a particular case.

2. PRELIMINARIES AND DEFINITIONS

For $x \in \mathbb{R}$, $[x]$ denotes the integer part of x and $\{x\}$ the fractional part, so $x = [x] + \{x\}$.

$\mathbb{R}^-$ denotes the negative real axis $\{x \in \mathbb{R} : x \leq 0\}$.

L^∞ denotes the complex-valued measurable functions on the nonnegative real axis such that $\operatorname{ess\,sup}_{t \in \mathbb{R}_+} |f(t)| < \infty$.

$L^1(\mathbb{R}^+)$ or L^1 denotes the complex-valued measurable functions on the nonnegative real axis such that $\int_0^\infty |f(t)|\, dt < \infty$, and $L^1(\mathbb{R})$ denotes the complex-valued measurable functions on the real axis such that $\int_{-\infty}^\infty |f(t)|\, dt < \infty$.

$\mathcal{A}$ denotes the space of distributions of the form $h(t) = h_a(t) + \sum_{i=0}^\infty h_i \delta(t - t_i)$ where $t_i \in [0, \infty)$, $0 \leq t_0 < t_1 < \cdots$, $\delta(t - t_i)$ is a delayed Dirac function, $h_i \in \mathbb{C}$, $h_a \in L^1$ and $\sum_{i=0}^\infty |h_i| < \infty$.
The norm on $\mathcal{A}$ is defined by $\|h\|_{\mathcal{A}} = \|h_a\|_{L^1} + \sum_{i=0}^\infty |h_i|$.

$\hat{\mathcal{A}}$ denotes the space of Laplace transforms of functions in $\mathcal{A}$.

We recall that BIBO-stability of a system P with convolution kernel h (with vanishing singular part) is defined as $\sup_{x \in L^\infty, x \neq 0} \frac{\|h*x\|_{L^\infty}}{\|x\|_{L^\infty}} < \infty$ which is equivalent to $\|h\|_{\mathcal{A}} = \|P\|_{\hat{\mathcal{A}}} < \infty$. It is well known that this implies that P lies in H_∞, the space of bounded analytic functions on the right half plane $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$.

P is said to have a *coprime factorization* (N, D) over $\hat{\mathcal{A}}$ if $P = ND^{-1}$, $D \neq 0$, N, $D \in \hat{\mathcal{A}}$ and there exists X, $Y \in \hat{\mathcal{A}}$ such that $-NX + DY = 1$.

P analytic in $\{\operatorname{Re} s > 0\}$ and continuous on $i\mathbb{R}$ is said to be *strictly proper* on $\{\operatorname{Re} s \geq 0\}$ if $\lim_{\rho \longrightarrow \infty} \left(\sup_{\{\operatorname{Re} s \geq 0, |s| \geq \rho\}} |P(s)| \right) = 0$.

P analytic in $\{\operatorname{Re} s > 0\}$ and continuous on $i\mathbb{R}$ is said to have a limit at infinity in $\{\operatorname{Re} s \geq 0\}$ if there exists a complex constant P_∞ such that
$\lim_{\rho \longrightarrow \infty} \left(\sup_{\{\operatorname{Re} s \geq 0, |s| \geq \rho\}} |P(s) - P_\infty| \right) = 0$.

Let P be a function that is meromorphic in $\mathbb{C} \setminus \mathbb{R}_-$ and has a branch point at $s = 0$. The point $s = 0$ is defined to be a pole of fractional order $\alpha > 0$ of P if there is a non-zero constant c such that $f(s) = s^{-\alpha}(c + o(1))$ as $s \to 0$ in $\mathbb{C} \setminus \mathbb{R}_-$. It is easy to see that this definition is independent of our choice of a branch of $s^{-\alpha}$ in $\mathbb{C} \setminus \mathbb{R}_-$.

Remark 2.1. The function e^{-s^r} where r is a real number, $0 < r < 1$, has a branch point at $s = 0$. To study this function we make a cut in the complex plane at $\mathbb{R}^-$ and consider the domain $\mathbb{C} \setminus \mathbb{R}^-$. In this domain, we can extend the function by continuity at $s = 0$, so that the function is analytic in $\{\mathrm{Re}\, s > 0\}$ and continuous on $\imath\mathbb{R}$. In fact the domain of analyticity is even bigger than $\{\mathrm{Re}\, s > 0\}$ and it will be useful to exploit the fact that the function can be defined to be analytic and proper in $\mathcal{D} = \{s \in \mathbb{C},\ s \neq 0,$ such that $|\arg s| < a\pi\}$ with $\frac{1}{2} < a < 1$ and $ar < \frac{1}{2}$.

A system is said to be nuclear if its sequence (σ_n) of Hankel singular values satisfies $\sum \sigma_n < \infty$ (see, for example [5]).

We recall three theorems due respectively to Wiener, Hardy and Littlewood, and Peller as they will be of use later:

Theorem 2.1. ([7], theorem 4.18.6) Let f be in $\hat{\mathcal{A}}$, f has an inverse in $\hat{\mathcal{A}} \Longleftrightarrow \inf_{\{\mathrm{Re}\, s \geq 0\}} |f(s)| > 0$.

Theorem 2.2. [6] Let $r \in L^1_{loc}$ have a Laplace transform $\hat{r}(s)$ that is defined (as an absolutely convergent integral) in the open half plane $\{\mathrm{Re}\, s > 0\}$. Moreover, suppose that $\hat{r}$ is bounded and has a bounded continuous extension to the closed right half plane $\{\mathrm{Re}\, s \geq 0\}$, and that the boundary function $\tilde{r}(\omega) = \lim_{\sigma \longrightarrow 0} \hat{r}(\sigma + i\omega)$ is locally absolutely continuous and satisfies $\tilde{r}' \in L^1(\mathbb{R})$. Then, $r \in L^1(\mathbb{R}^+)$, and $\|r\|_{L^1(\mathbb{R}^+)} \leq \frac{1}{2}\|\tilde{r}'\|_{L^1(\mathbb{R})}$.

Theorem 2.3. [11] Let G be an H_∞ transfer function. Then G is nuclear if and only if

$$\int\int_{\mathbb{C}_+} |G''(s)|\, \mathrm{d}A(s) < \infty,$$

where the integral is with respect to standard plane measure.

3. STABILITY ANALYSIS

We consider the class of fractional systems with scalar transfer function given by

$$\begin{aligned} P(s) &= \frac{q_0(s) + \sum_{i=1}^{n_2} q_i(s)e^{-\beta_i s} + \sum_{i=1}^{\tilde{n}_2} \tilde{q}_i(s)e^{-v_i(s)}}{p_0(s) + \sum_{i=1}^{n_1} p_i(s)e^{-\gamma_i s} + \sum_{i=1}^{\tilde{n}_1} \tilde{p}_i(s)e^{-u_i(s)}} \\ &= \frac{h_2(s)}{h_1(s)} \end{aligned} \qquad (1)$$

where $0 < \gamma_1 \cdots < \gamma_{n_1}$, $0 < \beta_1 \cdots < \beta_{n_2}$, the p_i, q_i, $\tilde{p}_i$, $\tilde{q}_i$ being polynomials of the form $\sum_{k=0}^{l_i} a_k s^{\alpha_k}$ with $\alpha_k \in \mathbb{R}^+$ and u_i, v_i being polynomials of the form $\sum_{k=1}^{m_i} b_k s^{\delta_k}$

with $0 < \delta_k \leq 1$ and $b_k \geq 0$. We suppose of course that u_i and v_i are not of the form αs that is, are not standard polynomials of degree one.

We will assume throughout that h_2 and h_1 have no common zeroes in $\{\mathrm{Re}\, s \geq 0\} \setminus \{0\}$.

Note that, for $s \neq 0$ and $\delta \in \mathbb{R}$, we define s^δ to be $\exp(\delta(\log|s| + i \arg s))$, and a continuous choice of $\arg s$ in a domain leads to an analytic branch of s^δ. In this work we shall normally make the choice $-\pi < \arg s < \pi$, for $s \in \mathbb{C} \setminus \mathbb{R}_-$.

We shall consider two different classes of systems. The first one, satisfying Condition 1 below will be referred as the class of fractional exponential delay systems of retarded type, and the second one, satisfying Condition 2 below, as the class of fractional exponential delay systems of neutral type.

Condition 1. $\deg\ p_0 > \deg\ p_i$ for $i = 1, \cdots, n_1$ and $\deg\ p_0 > \deg\ q_i$ for $i = 0, \cdots, n_2$.

Condition 2. $\deg\ p_0 \geq \deg\ p_i$ for $i = 1, \cdots, n_1$ (with equality for at least one polynomial p_i) and $\deg\ p_0 > \deg\ q_i$ for $i = 0, \cdots, n_2$.

Those conditions imply that we deal here with strictly proper systems.

We will need later to characterize the behaviour of h_1 and h_2 at zero and infinity, so let us remark that for $s \in \{\mathrm{Re}\, s \geq 0\}$

$$h_1(s) = s^\alpha(c_1 + o(1)) \text{ at zero with } \alpha \geq 0, \tag{2}$$
$$h_2(s) = s^\beta(c_2 + o(1)) \text{ at zero with } \beta \geq 0. \tag{3}$$

Let us write also

$$\gamma = \deg p_0 > 0, \qquad \text{and} \quad \delta = \max_{i=0,\dots n_2} \deg q_i \geq 0.$$

By Conditions 1 and 2 we have that $\gamma > \delta$.

The stability of systems described by (1) has been studied in some particular cases. We recall here the results previously obtained as they will be of use proving the general case.

We begin with results concerning systems such as $G(s) = \frac{\exp(-a\sqrt{s})}{s}$ or $G(s) = \frac{2e^{-a\sqrt{s}}}{b(1-e^{-2a\sqrt{s}})}$.

Theorem 3.1. [3] Let P be defined as in (1). In the particular case where $P(s) = \frac{q_0(s)+\sum_{i=1}^{i=\tilde{n}_2} \tilde{q}_i(s)e^{-\beta_i s^r}}{p_0(s)+\sum_{i=1}^{i=\tilde{n}_1} \tilde{p}_i(s)e^{-\gamma_i s^r}}$, we have:

P is BIBO stable if and only if P has no poles in $\{\mathrm{Re}\, s \geq 0\}$ (in particular, no poles of fractional order at $s = 0$).

The next theorems concern respectively fractional delay systems of retarded and neutral type.

Theorem 3.2. [2] Let P be defined as in (1) and satisfying Condition 1. In the particular case where $P(s) = \frac{\sum_{i=0}^{n_2} q_i(s)e^{-\beta_i s}}{\sum_{i=0}^{n_1} p_i(s)e^{-\gamma_i s}}$, we have:

P is BIBO stable if and only if P has no poles in $\{\mathrm{Re}\, s \geq 0\}$ (in particular, no poles of fractional order at $s = 0$).

Theorem 3.3. [2] Let P be defined as in (1) and satisfying Condition 2. In the particular case where $P(s) = \frac{\sum_{i=0}^{n_2} q_i(s)e^{-\beta_i s}}{\sum_{i=0}^{n_1} p_i(s)e^{-\gamma_i s}}$, we have:

If there exists $a < 0$ such that P has no poles in $(\mathbb{C} \setminus \mathbb{R}_-) \cap \{\mathrm{Re}\, s > a\} \cup \{0\}$ (in particular, no poles of fractional order at $s = 0$) then P is BIBO-stable.

We can see that in the retarded case, we obtain the same necessary and sufficient condition as in Theorem 3.1 (which is the 'usual' condition that P has no poles in $\{\mathrm{Re}\, s \geq 0\}$) whereas in the neutral case the condition on the location of poles ensuring stability is stronger and moreover is only a sufficient condition. From what we can see about standard delay systems, we cannot hope for a stronger general result like in Theorem 3.3. In fact, let us consider $P_1(s) = \frac{1}{s+1+se^{-s}}$. We have that P_1 has no poles in $\{\mathrm{Re}\, s \geq 0\}$ but is not BIBO-stable so that we cannot hope to obtain a sufficient condition of the type 'no poles in $\{\mathrm{Re}\, s \geq 0\}$' in the neutral case. Considering now $P_2(s) = \frac{1}{(s+1)^5(s+1+se^{-s})}$ which is BIBO-stable but has poles z_n satisfying $\mathrm{Re}\, z_n < 0$ and $\mathrm{Re}\, z_n \xrightarrow[n\to\infty]{} 0$, we see that we cannot hope that the condition '$\exists\, a < 0$ such that P has no poles in $(\mathbb{C} \setminus \mathbb{R}_-) \cap \{\mathrm{Re}\, s > a\} \cup \{0\}$' is necessary (see [2] for further details).

We now state the main results of this section which naturally extend the previous theorems.

Theorem 3.4. The system P defined as in (1) and satisfying Condition 1 is BIBO stable if and only if P has no poles in $\{\mathrm{Re}\, s \geq 0\}$ (in particular, no poles of fractional order at $s = 0$).

Proof. The 'only if' part is obvious.

For the 'if' part, let us write

$$P(s) = \left(\frac{(s+1)^{[\alpha]}(s^{\{\alpha\}}+1)h_2(s)}{s^\alpha(s+1)^{[\gamma]}(s^{\{\gamma\}}+1)}\right) \Big/ \left(\frac{(s+1)^{[\alpha]}(s^{\{\alpha\}}+1)h_1(s)}{s^\alpha(s+1)^{[\gamma]}(s^{\{\gamma\}}+1)}\right) = \tilde{h}_2(s)/\tilde{h}_1(s).$$

The proof of Theorem 3.2 can be extended to the present case.

The fact that $\tilde{h}_1$ and $\tilde{h}_2$ are in $\hat{\mathcal{A}}$ is not immediate due to the presence of the term s^α in both denominators. Moreover we cannot directly use the derivative test in Theorem 2.2 on $h = \tilde{h}_1$ or $\tilde{h}_2$ because we need more than h being proper at infinity to prove that h is in $\hat{\mathcal{A}}$ so we decompose h into the sum $h(s) = \frac{h(s)}{(s+1)^2} + \frac{s(s+2)h(s)}{(s+1)^2} = h_I + h_{II}$ and now consider the derivative test in Theorem 2.2 for h_I.

We have that $h_I'(s) = \frac{2h(s)}{(s+1)^3} + \frac{h'(s)}{(s+1)^2}$. As $\tilde{h}_1 = c_1 + o(1)$ and $\tilde{h}_2 = s^{\beta-\alpha}(c_2 + o(1))$, we have that each h is bounded near zero and this ensures that the integrals of both terms in the sum converge at zero. Now, as h is proper (in fact $\tilde{h}_1$ is proper and $\tilde{h}_2$ is strictly proper) it is easily verified that the integrals of $\frac{2h(s)}{(s+1)^3}$ and $\frac{h'(s)}{(s+1)^2}$ converge at infinity. We can conclude that h_I is in $\hat{\mathcal{A}}$. Now, the second term $h_{II}(s)$ has one more zero at zero than h so may now be in $\hat{\mathcal{A}}$, otherwise we decompose h_{II} (which is still proper or strictly proper as $\tilde{h}_1$ or $\tilde{h}_2$) according to the same algorithm until we eventually find an h_{II} in $\hat{\mathcal{A}}$. □

Theorem 3.5. Let P be defined as in (1) and satisfying Condition 2. If there exists $a < 0$ such that P has no poles in $(\mathbb{C} \setminus \mathbb{R}_-) \cap \{\mathrm{Re}\, s > a\} \cup \{0\}$ (in particular, no poles of fractional order at $s = 0$) then P is BIBO-stable.

Proof. This theorem can be proved in the same way as the previous one. The fact that $\inf_{\{\mathrm{Re}\, s \geq 0\}} \left| \frac{(s+1)^{[\alpha]}(s^{\{\alpha\}}+1)h_1(s)}{s^{\alpha}(s+1)^{[\delta]}(s^{\{\delta\}}+1)} \right| > 0$ relies on a straightforward modification of the proof of Theorem 3.3. □

4. ROBUST STABILIZATION AND NUCLEARITY

In the view of constructing robust stabilizing controllers for our systems using the well-known Youla parametrization, we consider now the determination of a coprime factorization (N, D) of P and associated Bézout factors X and Y which satisfy $-NX + DY = 1$. We address here the case of fractional exponential delay systems of retarded type. The next results extend the corresponding result of [1, 2, 3]. They are given without proof as this goes as in [3]. We will just give some hints to the reader.

Proposition 4.1. A coprime factorization (N, D) in $\hat{\mathcal{A}}$ of P defined as (1) and satisfying Condition 1 is given by

$$N(s) = \frac{(s+1)^{[\mu]}(s^{\{\mu\}}+1)h_2(s)}{s^{\mu}(s+1)^{[\gamma]}(s^{\{\gamma\}}+1)}, \qquad D(s) = \frac{(s+1)^{[\mu]}(s^{\{\mu\}}+1)h_1(s)}{s^{\mu}(s+1)^{[\gamma]}(s^{\{\gamma\}}+1)}.$$

Sketch of proof. We construct N and D satisfying the necessary and sufficient condition of coprimeness in $\hat{\mathcal{A}}$ which is $\inf_{\{\mathrm{Re}\, s > 0\}}(|N(s)| + |D(s)|) > 0$ (see [4]). From the BIBO stability characterization of Theorem 3.4, we look for functions N and D which have no poles in $\{\mathrm{Re}\, s > 0\}$ and no problem of boundedness at zero and infinity. Taking $N(s) = \frac{h_2(s)}{s^{\mu}(s+1)^{[\gamma]}(s^{\{\gamma\}}+1)}$ and $D(s) = \frac{h_1(s)}{s^{\mu}(s+1)^{[\gamma]}(s^{\{\gamma\}}+1)}$ we avoid problems of boundedness at zero and infinity and then taking $N(s) = \frac{(s+1)^{[\mu]}(s^{\{\mu\}}+1)h_2(s)}{s^{\mu}(s+1)^{[\gamma]}(s^{\{\gamma\}}+1)}$, and $D(s) = \frac{(s+1)^{[\mu]}(s^{\{\mu\}}+1)h_1(s)}{s^{\mu}(s+1)^{[\gamma]}(s^{\{\gamma\}}+1)}$, we ensure moreover that one term is proper while the other is strictly proper and this together with the fact that N and D are not zero at the same time around zero and have no commun zeroes in $\{\mathrm{Re}\, s > 0\}$ gives the result.

The case of neutral systems is much more difficult to handle. To verify that functions N, D are in $\hat{\mathcal{A}}$ is easy when the condition is 'no poles in $\{\mathrm{Re}\, s \geq 0\}$' as it requires, as we have seen above, to verify that there is no poles in $\{\mathrm{Re}\, s > 0\}$ and no problem of boundedness at zero and infinity. It is much more difficult to verify that $\exists\, a < 0$ such that N (or D) has no poles in $(\mathbb{C} \setminus \mathbb{R}_-) \cap \{\mathrm{Re}\, s > a\} \cup \{0\}$ as it requires an explicit calculation of zeroes of transcendental functions. □

Theorem 4.1. Let $\sigma_1, \cdots, \sigma_m$ be the m nonzero unstable zeroes of h_1 and let

$$\begin{aligned} T_1(s) &= s^{\mu}(s+1)^{[\gamma]}(s^{\{\gamma\}}+1), \\ T_2(s) &= (s+1)^{[\mu]}(s^{\{\mu\}}+1)h_2(s), \\ T_3(s) &= (s+1)^{[\mu]}(s^{\{\mu\}}+1)h_1(s). \end{aligned}$$

Now, let us define

$$\begin{aligned} Y(s) &= \frac{T_1(s)+T_2(s)X(s)}{T_3(s)} \quad \text{and} \\ X(s) &= \frac{f_0 + f_{\lambda_1}s^{\lambda_1} + \cdots f_{\lambda_n}s^{\lambda_n}}{(s+1)^M} + \frac{f_{M-m+1}s^{M-m+1} + \cdots f_M s^M}{(s+1)^M} \end{aligned}$$

where $\lambda_n \in \mathbb{R}$ and $M \in \mathbb{N}$ is chosen such that $M > \lambda_n + m$, the coefficients $f_0, f_{\lambda_1}, \ldots, f_{\lambda_n}$ are chosen in order to satisfy that $T_1(s)+T_2(s)X(s)$ is of fractional order α at zero, and the coefficients $f_{M-m+1}, \cdots, f_M$ are chosen so that $T_1(\sigma_i) + T_2(\sigma_i)X(\sigma_i) = 0$ for $1 \le i \le m$.

Then (X,Y) are Bézout factors associated to the coprime factors N and D of P.

Sketch of proof. We choose X in $\hat{A}$ such that the unstable zeroes of $1+NX$ are also those of D so that $Y = \frac{1+NX}{D}$ is analytic in $\{\mathrm{Re}\, s > 0\}$. We handle separately unstable zeroes at the origin and those in the right half plane, so that X contains fractional polynomials (to deal with zeroes of fractional order at the origin) and classical polynomials (to interpolate the remaining unstable zeroes). □

Example. We shall consider here a retarded version of the heat equation considered in [3], that is, the transfer function $P(s) = \frac{2e^{-s}(1-e^{-\sqrt{s}})}{\sqrt{s}s(1+e^{-\sqrt{s}})}$.

Of course, the coprime factors are easily deduced from those of the standard heat equation, however, applying the formulae of Proposition 4.1 we find that $N(s) = \frac{2e^{-s}(1-e^{-\sqrt{s}})}{\sqrt{s}(s+1)}$ and $D(s) = \frac{\sqrt{s}s(1+e^{-\sqrt{s}})}{(s+1)}$ is a coprime factorization of P.

Using now the algorithm described in Theorem 4.1, we take $X(s) = \frac{f_0+f_{1/2}\sqrt{s}+f_1 s}{s+1}$ and $Y(s) = \frac{\sqrt{s}(s+1)^2+e^{-s}(1-e^{-\sqrt{s}})(f_0+f_{1/2}\sqrt{s}+f_1 s)}{\sqrt{s}s(s+1)(1+e^{-\sqrt{s}})}$ where f_0, $f_{1/2}$ and f_1 have to be chosen such that Y is bounded near zero, that is, $f_0 = -1/2$, $f_{1/2} = -1/4$ and $f_1 = -37/24$.

We now continue the analysis of our systems by characterizing their nuclearity properties. Here again, a similar study was done in the case of fractional delay systems. Recall that nuclearity is an important notion when considering model reduction: nuclear systems always possess a balanced realization and truncations of these realizations have been proved to produce good approximations of the initial system in the H_∞-norm [5].

We begin with a technical lemma, which together with Theorem 2.3 allows to characterize the nuclearity of systems with transfer function involving polynomial and exponential in fractional powers of s. The case of systems which also involve delay terms appears more complicated to describe.

Lemma 4.1. Suppose that $f \in H^\infty(\mathbb{C}_+)$ and $|f(s)| = O(s^{-\alpha})$ as $|s| \to \infty$ in $\mathbb{C}_+$. Then

$$\text{(i)} \qquad \int\int_{\mathbb{C}_+} |f(s)|\, \mathrm{d}A(s) < \infty \qquad \text{if } \alpha > 2;$$

$$\text{(ii)} \qquad \int\int_{\mathbb{C}_+} |f(s)| e^{-s^r}\, \mathrm{d}A(s) < \infty \qquad \text{for all } 0 < r < 1, \quad \alpha \in \mathbb{R}.$$

Proof. (i) This is a simple calculation in polar coordinates. See [2].
(ii) Switch to polar coordinates. We observe that, writing $s = Re^{i\theta}$ for s in the right-hand half plane, we have

$$\mathrm{Re}(-s^r) = -R^r \cos r\theta \le -R^r \cos r\pi/2.$$

Thus $|\exp(-s^r)| \le \exp(-R^r)\exp(\cos r\pi/2)$ and now it is easy to see that if $0 < r < 1$ then the double integral

$$\int_{R=1}^{\infty} \int_{\theta=-\pi/2}^{\pi/2} R^\alpha \exp(-R^r \cos r\pi/2)\, \mathrm{d}\theta\, \mathrm{d}R$$

converges for any $\alpha \in \mathbb{R}$, and this implies the result. □

Corollary 4.1. Let P be defined as in (1). In the particular case where $P(s) = \frac{q_0(s)+\sum_{i=1}^{i=\tilde{n}_2} \tilde{q}_i(s)e^{-\beta_i s^r}}{p_0(s)+\sum_{i=1}^{i=\tilde{n}_1} \tilde{p}_i(s)e^{-\gamma_i s^r}}$, we have that P is nuclear if and only if P has no poles in $\{\mathrm{Re} \ge 0\}$.

Proof. The 'if' part is obvious.

The 'only if' part. We have that $f'' = h_2''/h_1 - 2h_2'h_1'/{h_1}^2 + 2h_2h_1'h_1'/h_1^3 - h_2h_1''/h_1^2$. As the terms in h_2 and h_1 with no e^{-s^r} factors are of degrees $\deg q_0$ and $\deg p_0$ with $\deg q_0 < \deg p_0$, the orders of the non exponential terms in P'' at infinity are each $\deg q_0 - \deg q_0 - 2 < -2$. Now, the result follows from Theorem 2.3, Theorem 3.1 and Lemma 4.1. □

5. CONCLUSION

We have handled the robust stabilization of fractional exponential delay systems of retarded type generalizing the study of [2] on the robust stabilization of fractional delay systems of retarded type. The determination of coprime and Bézout factors in the case of neutral systems is under study in both cases.

(Received November 22, 2000.)

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