

# CONTROL OF DISTRIBUTED DELAY SYSTEMS WITH UNCERTAINTIES: A GENERALIZED POPOV THEORY APPROACH

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The paper deals with the generalized Popov theory applied to uncertain systems with distributed time delay. Sufficient conditions for stabilizing this class of delayed systems as well as for  $\gamma$ -attenuation achievement are given in terms of algebraic properties of a Popov system via a Liapunov–Krasovskii functional. The considered approach is new in the context of distributed linear time-delay systems and gives some interesting interpretations of  $H^\infty$  memoryless control problems in terms of Popov triplets and associated objects. The approach is illustrated via numerical examples.

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## 1. INTRODUCTION

In the last decade a lot of attention has been paid to the stability and stabilization problems of uncertain systems with delayed state involving unknown constant or time-varying delay (see, for instance, Kolmanovskii and Nosov [11], Hale and Verduyn Lunel [4], Kojima et al [10]) and the references therein).

For the simplest case in which the delayed argument appears only in the state variable, a Liapunov–Krasovskii approach or a Liapunov–Razumikhin like theory was developed (Cheres et al [1], Xie and de Souza [18], Lee et al [12]).

Let us mention that some results have been reported about  $\mathcal{H}_\infty$  analysis and control of delay systems. For example, memoryless  $\mathcal{H}_\infty$  control problem has been investigated by Lee et al where a frequency domain approach is adopted for linear constant delay systems without uncertainty. Niculescu et al [13] deal with robust  $\mathcal{H}_\infty$  memoryless control problem for linear systems with time-varying state delay and norm-bounded time-varying uncertainty which appears in the state matrices and input matrix of the state equation.

Since real systems often have distributed parameters, it is of interest to study such a class of systems. However few works have been done in this field (see for example Dugard and Verriest [3] and the references therein).

In this paper, we consider the control problem of a large class of distributed time-delay systems using the generalized Popov theory (see Ionescu et al [7]). Our objectives are to achieve simultaneously closed-loop stability and disturbance attenuation for systems described in terms of linear retarded functional differential equations. This paper provides a unified approach for the considered problems. Further extensions (other types of control laws) can be achieved using the framework of the generalized Popov theory. The development is essentially based on the 'generalized' Riccati theory pioneered by Prof. Ionescu and co-authors starting with the eighties (see, e.g. Ionescu, Oara and Weiss [7], and the references therein). This theory represents an extension of the famous Popov's positivity theory to the indefinite sign case, usually encountered in game-theory situations.

To be more specific, our results are based on the necessary and sufficient condition for the existence of the stabilizing solution to an adequate Kalman–Yakubovich–Popov system of indefinite sign. This approach, combined with the Krasovskii theory for time-delay systems, leads to explicit representation formulae. The delay system class considered includes discrete and distributed delay terms and can be seen as a special case of infinite-dimensional systems described by distributional convolution equations. Note that the results presented here extend previous results obtained by some of the authors (Dion et al [2], Niculescu and Ionescu [14], Niculescu et al [16]).

The paper is organized as follows: one first presents, in Section 2 and 3 the problem statement and basic definitions and results of the generalized Popov theory for linear non delayed systems. The main results are given in Section 4 and are used in Section 5 for  $H^\infty$  control. Section 6 is devoted to the robust control problem. Some worked examples are given in Sections 5 and 6. Concluding remarks end the paper.

## 2. PROBLEM STATEMENT AND PRELIMINARY RESULTS

Consider the following state-delayed system including discrete and distributed delays:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_1x_t(-\tau_1) \\ &\quad + \int_0^{\tau_2} A_2(\theta)x_t(-\theta) d\theta + B_1u_1(t) + B_2u_2(t)\end{aligned}\quad (1)$$

$$y_1 = C_1x + D_{11}u_1 + D_{12}u_2 \quad (2)$$

where  $x_t$  represents the translation operator  $x_t(\theta) = x(t+\theta)$ , and  $A_2(\theta)$  is a piecewise continuous matrix function,  $x(t) \in \mathbf{R}^n$  is the state,  $u_1(t) \in \mathbf{R}^{m_1}$ ,  $u_2(t) \in \mathbf{R}^{m_2}$  are the disturbance and control inputs,  $y_1(t) \in \mathbf{R}^{p_1}$  is the controlled output,  $A$ ,  $A_1$ ,  $B_i$ ,  $D_{1i}$   $i = 1, 2$  are constant matrices of appropriate dimensions.

We are first interested in finding a memoryless controller

$$u_2(t) = F_2x(t) \quad (3)$$

that simultaneously stabilizes the system (1) and achieves the  $\gamma$ -attenuation property, i. e.,  $\|T_{y_1u_1}\| < \gamma$  where  $T_{y_1u_1}$  is the  $L^2$ -linear bounded input-output operator defined by the closed-loop configuration obtained by coupling (1) and (3).

Consider now the robust control problem for the same system with uncertainties:

$$\begin{aligned}\dot{x} &= (A + \Delta A)x + (A_1 + \Delta A_1)x(t - \tau_1) \\ &\quad + \int_0^{\tau_2} A_2(\theta)x_t(-\theta)d\theta + Bu.\end{aligned}\quad (4)$$

The uncertainty  $[\Delta A \ \Delta A_1]$  satisfies the following norm-bounded conditions  $\forall t \geq t_0$ :

$$\Delta A = DF(t)E \quad F^T(t)F(t) \leq I, \quad (5)$$

$$\Delta A_1 = D_1F_1(t)E_1 \quad F_1^T(t)F_1(t) \leq I, \quad (6)$$

with known matrices  $D \in \mathbf{R}^{n \times i}$ ,  $D_1 \in \mathbf{R}^{n \times i_1}$ . The matrices  $E$  and  $E_1$  are given weighting matrices.

For simplicity only the robust control problem is considered but the joint, robust and  $\gamma$ -attenuation problem.

Unlike the techniques previously mentioned, our interest is directed towards the tools offered by the generalized Popov theory. The interest of such approach is twofold: first to extend the Popov theory developed in the linear case to delay systems and second to provide some alternative to handle control problems for delay case.

First, we define a general quadratic cost function which corresponds to the disturbance attenuation requirement. We show that the stabilization problem is solvable if the corresponding Kalman–Popov–Yakubovich system has a stabilizing solution and a stabilizing controller can be derived from the partition. If further conditions related to the general quadratic cost function are satisfied, then the stabilizing controller also achieves disturbance attenuation requirement.

We will use a Popov theory approach combined with the Liapunov–Krasovskii stability theorem. The direct application of this theorem is not easy because it is difficult to build such functionals. A *general form* for this functional is:

$$v(x_t) = x^T(t)Px(t) + 2x^T(t) \int_{-\tau}^0 Q(\theta)x(t+\theta)d\theta \quad (7)$$

$$+ \int_{-\tau}^0 x^T(t+\theta)R(\theta)x(t+\theta) \quad (8)$$

$$+ \int_{-\tau}^0 \int_{-\tau}^0 x^T(t+\theta_1)S(\theta_1, \theta_2)x(t+\theta_2)d\theta_1d\theta_2, \quad (9)$$

where  $P$ ,  $Q(\theta)$ ,  $R(\theta)$  and  $S(\theta_1, \theta_2)$  are weighting matrices. In the sequel, we will drop the explicit time dependence of  $x(t)$ ,  $u_1(t)$  and  $u_2(t)$  on  $t$  for brevity.

### 3. SOME BASIC RESULTS ON THE GENERALIZED POPOV THEORY

In this section several basic notions and results concerning the general Riccati theory are presented. The present development is essentially based on the theory exposed in Ionescu and Weiss, [8] or in Ionescu et al [5].

**Definition 1.** Call  $\Sigma = (A, B; P)$  where  $A \in \mathbf{R}^{n \times n}$ ,  $B \in \mathbf{R}^{n \times m}$  and

$$P = \begin{bmatrix} Q & L \\ L^T & R \end{bmatrix} = P^T \in \mathbf{R}^{(n+m) \times (n+m)}$$

a Popov triplet with  $Q \in \mathbf{R}^{n \times n}$ .

Frequently, we will use the extensive notation  $\Sigma = (A, B; Q, L, R)$ .

Let  $\Sigma = (A, B; Q, L, R)$  be a Popov triplet and let

$$J = \begin{bmatrix} -I_{m_1} & \\ & I_{m_2} \end{bmatrix}, \quad m_1 + m_2 = m \quad (10)$$

be an arbitrary sign matrix. We associate with  $\Sigma$  the following objects:

- (1) *The Kalman–Popov–Yakubovich system* in  $J$  form ( $\text{KPYS}(\Sigma, J)$ ) is the following nonlinear system with unknown  $X, V, W$ :

$$\begin{aligned} R &= V^T J V \\ L + X B &= W^T J V \\ Q + A^T X + X A &= W^T J W. \end{aligned} \quad (11)$$

This system is usually denoted as  $\text{KPYS}(\Sigma, J)$ .

- (2) The extended Hamiltonian pencil  $\text{EHP}(\Sigma)$   $\lambda M - N$  where

$$\left\{ \begin{aligned} M &= \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ N &= \begin{bmatrix} A & 0 & B \\ -Q & -A^T & -L \\ L^T & B^T & R \end{bmatrix}, \\ M, N &\in \mathbf{R}^{(2n+m) \times (2n+m)}. \end{aligned} \right. \quad (12)$$

**Definition 2.** Any triplet  $(X, V, W)$  for which (11) is fulfilled and in addition  $X = X^T$ ,  $V$  is nonsingular and of lower-left block triangular form

$$V = \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} \quad (13)$$

partitioned in accordance with  $J$  in (10) and  $A + BF$  is exponentially stable for

$$F = -V^{-1}W, \quad (14)$$

called the stabilizing feedback gain, is called a stabilizing solution to the  $\text{KPYS}(\Sigma, J)$ .

Recall that  $\mathcal{V}$  is said to be a *stable proper deflating subspace* (see Oara [17]) of an arbitrary matrix pencil  $\lambda M - N$  if  $NV = MVS$ ,  $MV$  is monic,  $S$  is Hurwitzian and  $\mathcal{V} = \langle V \rangle$  ( $V$  is any basis matrix for  $\mathcal{V}$ ).

**Definition 3.** The EHP( $\Sigma$ ) is said disconjugate if it has a stable proper deflating subspace  $\mathcal{V}$  of dimension  $n$  and, in addition, if

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \begin{matrix} n \\ n \\ m \end{matrix}$$

is any basis matrix for  $\mathcal{V}$  ( $\mathcal{V} = \langle V \rangle$ ), then  $V_1$  is nonsingular.

A relevant result of the generalized Popov theory is:

**Theorem 1.** Let  $\Sigma = (A, B; Q, L, R)$  be a Popov triplet and  $J$  any sign matrix as in (10). Then the following statements are equivalent:

1.  $R$  is nonsingular and the KPYS( $\Sigma, J$ ) has a stabilizing solution  $(X, V, W)$ ;
2. The EHP( $\Sigma$ ) is regular and disconjugate and, in addition, if  $R$  is partitioned in accordance with  $J$  in (10), i.e.,

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} \quad (15)$$

then

$$R_{22} > 0, \quad \text{sgn } R = J. \quad (16)$$

If 2 is true, then (see Definition 3)  $X = V_2 V_1^{-1}$  and  $F = V_3 V_1^{-1}$ .

#### 4. MAIN RESULTS

In a first step, let us consider the following linear distributed time-delay system:

$$\begin{cases} \dot{x} = Ax + A_1 x_t(-\tau_1) \\ \quad + \int_0^{\tau_2} A_2(\theta) x_t(-\theta) d\theta + Bu, \\ x = \phi \text{ on } [-\tau, 0] \end{cases} \quad (17)$$

where  $x \in \mathbf{R}^n$  is the state,  $u \in \mathbf{R}^m$  is the input,  $A, A_1 \in \mathbf{R}^{n \times n}$ ,  $B \in \mathbf{R}^{n \times m}$ ,  $A_2(\theta)$  is a piecewise continuous function,  $\tau = \max\{\tau_1, \tau_2\}$  is the delay and  $\phi$  is any continuous  $n$ -valued function on  $[-\tau, 0]$ .

Let  $\Sigma = (A, B; Q, L, R)$  be a Popov triplet where the entries  $A$  and  $B$  coincide with  $A$  and  $B$  in (17). Consider the extended time-varying Popov triplet associated with (17):

$$\Sigma_v = \left( A, [A_1 \ A_2(\cdot) \ B]; Q, [0 \ 0 \ L], \begin{bmatrix} R_{d1} & 0 & 0 \\ 0 & R_{d2}(\cdot) & 0 \\ 0 & 0 & R \end{bmatrix} \right), \quad (18)$$

where  $R_{d1} \in \mathbb{R}^{n \times n}$  and  $R_{d2}$  is a continuous time-varying function with some sign constraints. Such extended Popov triplet allows us to reduce the control problem of a time-varying delay system to a time-varying system free of delay (here the retarded terms are seen like a perturbation in the system). For the sake of simplicity we shall not address such problem here. Note also that if  $A_2$  is a constant matrix, one recovers the *time-invariant* Popov triplet used in Niculescu et al [16], with  $R_{d2}$  a symmetric and strictly negative-definite matrix.

Consider also the *extended* time-invariant Popov triplet

$$\Sigma_e = \left( A, [A_1 \quad M^{\frac{1}{2}}(\tau_2) \quad B]; Q, [0 \quad 0 \quad L], \begin{bmatrix} R_{d1} & 0 & 0 \\ 0 & -I_n & 0 \\ 0 & 0 & R \end{bmatrix} \right), \quad (19)$$

where

$$M(\tau_2) = \int_0^{\tau_2} A_2(\theta) Y^{-1}(\theta) A_2(\theta)^T d\theta \quad (20)$$

for some  $Y(\cdot)$  continuous time-varying function (seen as a *parameter*). We shall see in the proof given in the appendix how the considered control problem (2) for (17) is solved if some algebraic properties of the extended triplet  $\Sigma_e$  are satisfied. The idea is to interpret such problem as a control problem of an appropriate system free of delay.

Consider also the following (extended) sign matrix

$$J_e = \left[ \begin{array}{c|c} -I_{2n} & \\ \hline & J \end{array} \right] = \left[ \begin{array}{c|c} -I_{2n} & \\ \hline -I_{m_1} & I_{m_2} \end{array} \right], \quad (21)$$

where ( $m_1 + m_2 = m$ ), be considered. Let  $B$ ,  $L$  and  $R$  be partitioned in accordance with  $J$  in (21):

$$B = [B_1 \quad B_2] \quad L = [L_1 \quad L_2] \quad R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix}. \quad (22)$$

The basic result of this section is

**Theorem 2.** Assume that the KPYS( $\Sigma_e, J_e$ ) given by (19) and (21) has a stabilizing solution  $(X, V_e, W_e)$ . Let the stabilizing feedback gain  $F_e$  be partitioned in accordance with  $J_e$  in (21), that is,

$$F_e = -V_e^{-1} W_e = \begin{bmatrix} F_d \\ F_1 \\ F_2 \end{bmatrix}. \quad (23)$$

Let also  $u$  be split in accordance with  $B$  in (22):

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \begin{matrix} m_1 \\ m_2 \end{matrix}.$$

Let  $Y$  be a continuous time varying function. Assume further that

$$X \geq 0 \quad (24)$$

$$R_{11} < 0 \quad (25)$$

$$R_{d2}(\tau_2) < 0 \quad (26)$$

$$Q + L_2 F_2 + F_2^T L_2^T + F_2^T R_{22} F_2 + R_{d1} + R_{d2}(0) > 0 \quad (27)$$

where

$$R_{d2}(\theta) = R_2 + \int_0^\theta Y(\theta) d\theta, \quad \theta \in [0, \tau_2]. \quad (28)$$

Then the state feedback

$$u_2 = F_2 x \quad (29)$$

stabilizes (17) for all  $\tau \leq \tau_2$ , i. e.,

$$\begin{cases} \dot{x} = \tilde{A}x + A_1 x_t(-\tau_1) + \int_0^{\tau_2} A_2(\theta) x_t(-\theta) d\theta \\ x = \phi \text{ on } [-\tau, 0] \end{cases} \quad (30)$$

defines an exponentially stable solution for all  $\phi$ . Here  $\tilde{A} = A + B_2 F_2$ .

The proof is given in the appendix and makes use of the following Liapunov-Krasovskii functional:

$$\begin{aligned} V(x_t) = & x^T(t) X x(t) + \int_{t-\tau_1}^t x^T(\theta) (-R_{d1}) x(\theta) d\theta \\ & + \int_{t-\tau_2}^t x^T(\theta) (-R_{d2}(t-\theta)) x(\theta) d\theta, \end{aligned} \quad (31)$$

where  $X = X^T \geq 0$  and  $R_{d1} = R_{d1}^T < 0$  are given before; the time-varying matrix function  $R_{d2}(\cdot)$  is given in (28). Note that since the inequality (26) is satisfied, it follows that  $-R_{d2}(\xi)$  is a symmetric and positive-definite matrix for each  $\xi \in [0, \tau_2]$ .

The idea is that one may see (31) as a quadratic index for an appropriate *time-invariant* linear system free of delay, and, thus to apply the generalized Popov theory to such system.

**Remark 1.** Since the Liapunov-Krasovskii functional (31) is very general, one may construct various *S-parametrizations* (not only linear!) of the time-varying matrix function  $R_{d2}(\cdot)$ , for which Theorem 2 is still true.

Thus, due to the particular form of the *distributed delay*, if, for example,  $R_{d2}$  is a continuous increasing (decreasing) function, one needs "strong" conditions only in 2 points: 0 and  $\tau_2$ , etc.

**Remark 2.** Using the results developed in Niculescu and Ionescu [14], it follows that one may relax the condition  $R_{d1} < 0$  to  $R_{d1} \leq 0$ , and thus to use more general forms for the corresponding  $J$  matrix.

**Remark 3.** It is easy to see that if  $\tau_1$  is a continuous time-varying function, with bounded derivative:

$$\dot{\tau}_1(t) \leq \beta_1 < 1, \quad \beta \in \mathbf{R}, \quad (32)$$

then the theorem holds if one changes  $R_{d1}$  to  $\frac{1}{1-\beta_1}R_{d1}$ . Note that the corresponding Liapunov–Krasovskii functional changes similarly. A similar technique can be used if one assumes that  $\tau_2$  is a continuous time-varying function. For the sake of simplicity, we shall not develop such extension here.

A natural consequence of this theorem is:

**Corollary 1.** If all the conditions in the statement of Theorem 2 hold, then

$$\begin{cases} \dot{x} = \tilde{A}x + A_1x_t(-\tau_1) \\ \quad + \int_0^{\tau_2} A_2(t-\theta)x_t(-\theta)d\theta + B_1u_1, \\ x = 0 \text{ on } [-\tau, 0] \end{cases} \quad (33)$$

defines a linear bounded input-state operator from  $L_+^{2,m_1}$  into  $L_+^{2,n}$ .

**Proof.** By  $L_+^{2,r}$  we mean the Hilbert space of norm square integrable  $\mathbf{C}^r$ -valued functions defined on  $[0, \infty)$ . The proof is a trivial consequence of the exponentially stable evolution defined by (30) (see also Hale and Verduyn Lunel [4]). Taking into account Theorem 1, an equivalent form of Theorem 2 can be stated as follows:

**Theorem 3.** Assume that the  $\text{EHP}(\Sigma_e)$  is disconjugate. Assume also that

$$R_{22} > 0, \quad \text{sgn } R = J, \quad R_{d1} < 0. \quad (34)$$

If

$$V_2V_1^{-1} \geq 0 \quad (35)$$

and both (25) and (27) hold, then (29) stabilizes (17). Here

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \begin{matrix} n \\ n \\ n+m \end{matrix}$$

is any basis matrix for the maximal stable proper deflating subspace of the  $\text{EHP}(\Sigma_e)$  and

$$F_e = V_3V_1^{-1}$$

partitioned as in (23). □



**Remark 4.** Theorem 3 provides easy checkable sufficient conditions for the stabilizability of the state-delayed system (17) in terms of algebraic properties of the associated matrix pencil.

Let  $\hat{Q}$  be any  $n \times n$  symmetric matrix. Let  $\Sigma = (A, B; \hat{Q}, L, R)$  be the Popov triplet constructed with  $\hat{Q}$  and with entries of  $\Sigma_e$ . Associate with  $\Sigma$  the “usual” Popov index:

$$J_{\Sigma}(\phi, u) = \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} \hat{Q} & L \\ L^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle \quad (36)$$

where  $(x, u) \in L_+^{2,n} \times L_+^{2,m}$  and  $x$  and  $u$  are linked via (17) for some  $\phi$ .

Then we have:

**Proposition 1.** Let us consider a symmetric matrix  $\hat{Q}$  satisfying

$$\begin{bmatrix} \hat{Q} & L_2 \\ L_2^T & R_{22} \end{bmatrix} \geq 0. \quad (37)$$

Assume also that all the conditions in the statement of Theorem 2 hold except (27) which is modified as

$$Q + R_{d1} + R_{d2}(0) > \hat{Q}. \quad (38)$$

If the controller (29) stabilizes the delay system (33), then there exists  $\bar{\zeta} > 0$  such that

$$J_{\Sigma}(0, u_1) \leq -\bar{\zeta} \|u_1 - F_1 x\|_2^2, \quad \forall u_1 \in L_+^{2,m_1} \quad (39)$$

where

$$J_{\Sigma}(0, u_1) := J_{\Sigma}(0, u)|_{u_2=F_2 x}. \quad (40)$$

**Proposition 2.** Assume that all conditions in the statement of Theorem 2 hold. Assume additionally that

$$\bar{Q} + R_{d1} + R_{d2}(0) > 0 \quad (41)$$

where

$$\bar{Q} := Q + LF + F^T L^T + F^T R F.$$

Then

$$\begin{cases} \dot{x} = \tilde{A}x + A_1 x_t(-\tau_1) \\ \quad + \int_0^{\tau_2} A_2(\theta) x_t(-\theta) d\theta + B_1 u_1 \\ v_1 = -F_1 x + u_1 \end{cases} \quad (42)$$

(with  $x = 0$  on  $[-\tau, 0]$ ) defines a linear bounded invertible operator on  $L_+^{2,m_1}$ .

Using all the results presented before, we shall state and prove the main result of this paper.

**Theorem 4.** Let (17) together with the quadratic cost defined by the right-hand side of (36) be given. For arbitrary  $m_1, m_2$  such that  $m_1 + m_2 = m$ , let  $B, L$  and  $R$  be partitioned as in (22).

Assume that there exists two  $n \times n$  symmetric matrices  $Q, R_{d1}$  and a positive scalar  $\varepsilon$  such that the KPYS( $\Sigma_e, J_e$ ), where  $\Sigma_e$  and  $J_e$  are defined by (19) and (21), respectively, has a stabilizing solution  $(X, V_e, W_e)$  and let the stabilizing feedback  $F_e$  be partitioned in accordance with (23).

Let us consider a symmetric matrix  $\bar{Q}$  and assume also that the following conditions all hold:

- |   |                                       |
|---|---------------------------------------|
| 1. $X \geq 0$   | 4. $R_{d2}(\tau_2) < 0$               |
| 2. $\begin{bmatrix} \hat{Q} & L_2 \\ L_2^T & R_{22} \end{bmatrix} \geq 0$ | 5. $Q + R_{d1} + R_{d2}(0) > \hat{Q}$ |
| 3. $R_{11} < 0$   | 6. $\bar{Q} + R_{d1} + R_{d2}(0) > 0$ |

where

$$\begin{aligned}\bar{Q} &= Q + LF + F^T L^T + F^T R F \\ R_{d2}(\theta) &= R_2 + \int_0^\theta Y(\theta) d\theta \quad \theta \in [0, \tau_2].\end{aligned}$$

Then

- a)  $u_2 = F_2 x$  stabilizes (17) for all  $\tau \leq \tau_2$ .
- b) There exists  $c_0 > 0$  such that

$$J_\Sigma(0, u_1) \leq -c_0 \|u_1\|_2^2 \quad \forall u_1 \in L_+^{2, m_1}$$

where  $J_\Sigma(0, u_1)$  has been defined by (40), (36).

**Proof.** a) follows directly from Theorem 2 combined with 2. and 4. in the statement (see the proof of Proposition 1).

b) From Proposition 2 it follows that there exists  $\zeta_1 > 0$  such that

$$\|v_1\|_2^2 = \|u_1 - F_1 x\|_2^2 \geq \zeta_1 \|u_1\|_2^2. \quad (43)$$

Using Proposition 1 the conclusion follows by substituting (43) in (39) and putting  $c_0 = \zeta_1 \bar{\zeta}$ .  $\square$

## 5. $\mathcal{H}^\infty$ -CONTROL

In this section the theory developed in Section 3 will be applied for solving the following  $\mathcal{H}^\infty$ -control problem formulated for state-delayed systems. Such a problem is stated as follows.

Let the system

$$\begin{cases} \dot{x} = Ax + A_1 x_t(-\tau_1) + \int_0^{\tau_2} A_2(\theta) x_t(-\theta) d\theta \\ + B_1 u_1 + B_2 u_2 \\ y_1 = C_1 x + D_{11} u_1 + D_{12} u_2 \end{cases} \quad (44)$$

(where  $x = 0$  on  $[-\tau, 0]$ ) be given. Here  $x$  is the state (in the usual sense),  $u_1$  and  $u_2$  are the disturbance and control inputs, respectively, and  $y_1$  is the output to be controlled. The state  $x$  is assumed to be accessible for measurement. We are looking for a state feedback law

$$u_2 = F_2 x \quad (45)$$

which stabilizes (44) and achieves  $\gamma$ -attenuation property for the closed-loop system, i.e., there exists  $c_0 > 0$  such that

$$\begin{aligned} -\gamma^2 \|u_1\|_2^2 + \|y_1\|_2^2 &\leq -c_0 \|u_1\|_2^2 \\ \forall u_1 \in L_+^{2, m_1} \end{aligned} \quad (46)$$

or equivalently the system

$$\begin{cases} \dot{x} = (A + B_2 F_2) x + A_1 x_t(-\tau_1) \\ + \int_0^{\tau_2} A_2(\theta) x_t(-\theta) d\theta + B_1 u_1 \\ y_1 = (C_1 + D_{12} F_2) x + D_{11} u_1 \end{cases} \quad (47)$$

(where  $x = 0$  on  $[-\tau, 0]$ ) defines a  $\gamma$ -strictly contractive input-output map. Here  $\gamma$  is a prescribed tolerance for the attenuation level. Then we have:

**Corollary 2.** Assume that there exist two  $n \times n$  symmetric matrices  $Q$  and  $R_d$  such that all the conditions of Theorem 4 hold with respect to the following particular data:

$$\begin{aligned} B &= \begin{bmatrix} m_1 & m_2 \\ B_1 & B_2 \end{bmatrix} \\ \hat{Q} &= C_1^T C_1, \quad L = [L_1 \quad L_2] = C_1^T [D_{11} \quad D_{12}] \\ R &= \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} = \begin{bmatrix} -\gamma^2 I + D_{11}^T D_{11} & D_{11}^T D_{12} \\ D_{12}^T D_{11} & D_{12}^T D_{12} \end{bmatrix}. \end{aligned} \quad (48)$$

Then for  $F_2$  given in Theorem 4,  $u_2 = F_2 x$  is a solution to the  $H^\infty$ -control problem stated above.

**Proof.** Let  $\Sigma = (A, B; \hat{Q}, L, R)$ . Then

$$J_\Sigma = -\gamma^2 \|u_1\|_2^2 + \|y_1\|_2^2 \quad (49)$$

as directly follows by simple computation from (48). Apply Theorem 4 to (49) and the conclusion follows easily.  $\square$

## 6. AN EXAMPLE

In this section, two numerical examples are presented for comparing these stability criteria using a general form for  $R_{d2}(\theta)$  with the existing ones given in Niculescu et al [16] with a particular function. First let us calculate a maximal size of the distributed delay  $\tau_2$  for a fixed  $\gamma$ . It should be pointed out that there are few results in the literature addressed to this problem (Dugard and Verriest [3]). Let the following unstable distributed state-delayed system

$$\begin{aligned}\dot{x} &= x + \sigma^{-\tau} x + \int_0^{\tau_2} \theta x_t(-\theta) d\theta + u_1 + u_2 \\ y_1 &= x + u_2\end{aligned}\quad (50)$$

be given. Here  $x$ ,  $u_1$ ,  $u_2$  and  $y_1$  are all scalars. The problem is to find a memoryless controller

$$u_2 = F_2 x \quad (51)$$

that achieves simultaneously closed-loop stability and  $\gamma$ -attenuation. The prescribed tolerance is

$$\gamma = 2. \quad (52)$$

The input data are

$$\begin{aligned}A &= 1, & A_1 &= 1, & B_1 &= 1, & B_2 &= 1 \\ C_1 &= 1, & D_{11} &= 0, & D_{12} &= 1\end{aligned}\quad (53)$$

as directly follows from (50). Choose  $R_d = -10$ .

Then the extended Popov triplet (see (19)) is

$$\Sigma_e = \left( (1, [1 \ M(\tau_2)^{\frac{1}{2}} \ 1 \ 1]); Q, [0 \ 0 \ 0 \ 1], \begin{bmatrix} -10 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right). \quad (54)$$

We consider two values for  $Y(\theta)$ . In the first case, we take non-linear form  $Y(\theta) = \theta^2 + 1$  and with (20) we find that  $M(\tau_2) = \tau_2 - \text{atan}(\tau_2)$ . But (54) is equivalent to the algebraic Riccati equation (ARE) associated with  $\Sigma_e$ , that is,

$$0 = A^T X + X A + Q - ([0 \ 0 \ L_1 \ L_2] + X[A_1 \ M(\tau_2)^{\frac{1}{2}} \ B_1 \ B_2]) \left( \begin{bmatrix} R_d & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & R_{11} & R_{12} \\ 0 & 0 & R_{12}^T & R_{22} \end{bmatrix}^{-1} \left( \begin{bmatrix} 0 \\ 0 \\ L_1^T \\ L_2^T \end{bmatrix} + \begin{bmatrix} A_1^T \\ M(\tau_2)^{\frac{1}{2}} \\ B_1^T \\ B_2^T \end{bmatrix} X \right) \right).$$

After computations we find the stabilizing solution:

$$X = \sqrt{\frac{Q - 1}{0.65 - M(\tau_2)}} \quad (55)$$

which is greater than 0 when  $Q > 1$ , and we can still find a solution for  $\tau_{2\max} = 1.69$ . Let us verify the conditions stated in Theorem 4. The first condition is fulfilled when  $Q > 1$  as (55) shows. For the next ones, we obtain

$$R_{d2}(\tau_2) < 0, \quad Q - 1 > -R_{d1} - R_{d2}(0).$$

There are verified for all  $R_2 < 0$ . Let  $R_2 = -0.1$  and we obtain  $Q > 7.87$ . Therefore all of the conditions stated in Theorem 4 hold for  $Q > 7.87$  and consequently

$$u_2 = - \left( 1 + \sqrt{\frac{Q-1}{0.65-M}} \right) x$$

is the desired feedback law. For the second case, we consider  $A_2(\theta) = \theta$  and  $Y(\theta) = \theta$ . With these values we obtain  $M(\tau_2) = \frac{\tau_2^2}{2}$ . This time we find  $\tau_{2\max} = 1.1$ , which is a more conservative one. And all of the conditions stated in Theorem 4 hold for  $Q > 10.45$

## 7. CONTROL OF THE UNCERTAIN SYSTEM

Let us consider the following uncertain state-delayed system:

$$\dot{x} = (A + \Delta A)x + (A_1 + \Delta A_1)x(t - \tau_1) + \int_0^{\tau_2} A_2(\theta)x_t(-\theta)d\theta + Bu. \quad (56)$$

Let  $\Sigma = (A, B; Q, L, R)$  be a Popov triplet where the entries  $A$  and  $B$  coincide with  $A$  and  $B$  in (56). The uncertainty  $[\Delta A \ \Delta A_1]$  satisfies the following norm-bounded conditions  $\forall t \geq t_0$ :

$$\Delta A = DF(t)E \quad F^T(t)F(t) \leq I, \quad (57)$$

$$\Delta A_1 = D_1F_1(t)E_1 \quad F_1^T(t)F_1(t) \leq I, \quad (58)$$

with known matrices  $D \in \mathbb{R}^{n \times i}$ ,  $D_1 \in \mathbb{R}^{n \times i_1}$ . The matrices  $E$  and  $E_1$  are given weighting matrices.

We can obtain an equivalent of Theorem 2 (uncertainty case), using an appropriate extended Popov triplet:

Let  $R_{d1} \in \mathbb{R}^{n \times n}$  and consider the extended Popov triplet for the system (56)–(58):

$$\Sigma_{ep} = (A, [A_1 \ D \ D_1 \ M(\tau_2)^{1/2} \ B]; Q, [0 \ 0 \ 0 \ 0 \ L], \begin{bmatrix} R_{dp} & 0 \\ 0 & R \end{bmatrix}), \quad (59)$$

with

$$R_{dp} = \begin{bmatrix} R_{d1} & 0 & 0 & 0 \\ 0 & -I_1 & 0 & 0 \\ 0 & 0 & -I_d & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}.$$

**Remark 5.** In this section of the paper, because we only deal with the control problem, we will use a simplified KYP-system with  $J \equiv I$ , but we can easily extend these results for the  $H^\infty$  problem (according to Sections 3 and 4).

**Theorem 5.** Assume that the KPYS( $\Sigma_{ep}$ ) is given by

$$\Sigma_{ep} = \left( A, [A_1 \ D \ D_1 \ M(\tau_2)^{1/2} \ B]; Q, [0 \ 0 \ 0 \ 0 \ L], \begin{bmatrix} R_{dp} & 0 \\ 0 & R \end{bmatrix} \right), \quad (60)$$

has a stabilizing solution  $(X, V_{ep}, W_{ep})$ . Let the stabilizing feedback gain  $F_{ep}$  be partitioned in accordance with  $\Sigma_{ep}$ , that is,

$$F_e = -V_{ep}^{-1}W_{ep} = \begin{bmatrix} F_{dp} \\ F_1 \\ F_2 \end{bmatrix}. \quad (61)$$

Let also  $u$  be split in accordance with  $B$  in (22):

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{matrix} m_1 \\ m_2 \end{matrix}.$$

Let  $Y$  be a continuous time varying function. Assume further that

$$X \geq 0 \quad (62)$$

$$R_{d2}(\tau_2) < 0 \quad (63)$$

$$Q - E^T E - E_1^T E_1 + R_{d1} + R_{d2}(0) > 0 \quad (64)$$

where

$$R_{d2}(\theta) = R_2 + \int_0^\theta Y(\theta) d\theta, \quad \theta \in [0, \tau_2]. \quad (65)$$

Then the state feedback

$$u_2 = F_2 x \quad (66)$$

stabilizes (56) for all  $\tau \leq \tau_2$ , i. e.,

$$\begin{cases} \dot{x} = \tilde{A}x + A_1 x_t(-\tau_1) + \int_0^{\tau_2} A_2(\theta) x_t(-\theta) d\theta \\ x = \phi \text{ on } [-\tau, 0] \end{cases} \quad (67)$$

defines an exponentially stable solution for all  $\phi$ . Here  $\tilde{A} = A + B_2 F_2$ .

**An example.**

$$\begin{aligned} \dot{x} &= (1 + 0.2 \sin(t))x + (1 + 0.2 \cos(t))x(t - \tau_1) + \int_0^{\tau_2} \theta x_t(-\theta) d\theta + u_2 \\ y &= x + 2u_2. \end{aligned} \quad (68)$$

Here  $D = D_1 = 0.2$  and  $E = E_1 = 1$ . The problem is to find a memoryless controller

$$u_2 = F_2 x \quad (69)$$

that achieves the closed-loop stability.

We choose  $R_{d1} = -1, R = 1$ . The extended Popov triplet (see (59)) is

$$\Sigma_{ep} = \left( 1, [1 \ 0.2 \ 0.2 \ M(\tau_2)^{\frac{1}{2}} \ 2]; Q, 0, \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right). \quad (70)$$

We consider for  $Y(\theta)$  a non-linear form  $Y(\theta) = \theta^2 + 1$ . After computations the KPY system is reduced to:

$$x^2(2.92 - M(\tau_2)) + 2x + Q = 0. \quad (71)$$

We find a stabilizing solution whenever  $0 < Q$ . The maximal  $\tau$  for which the solution is still positive is  $\tau^* = 4.21s$ . For these values, we find  $x = 35.84$  and the feedback law  $F_2 = -94.041$ . The conditions of the theorem become

$$R_{d2}(\theta) = R_{d2}(0) + \frac{\tau^3}{3} + \tau < 0$$

$$Q + R_{d2}(0) - 3 > 0$$

which are fulfilled for  $R_{d2}(0) = -29$  and  $\forall Q \geq 32$ .

## 8. CONCLUSIONS

An extension of the generalized Popov theory to the case of delay system with discrete and distributed delays has been done. Our interest has been focused on the memoryless controller design for  $\mathcal{H}^\infty$ -control problem. As future research direction we suggest the investigation of observer-based compensation technique.

## APPENDIX: PROOF OF THEOREM 2

Since  $(X, V_e, W_e)$  is a stabilizing solution to the KPYS( $\Sigma_e, J_e$ ), it has the following form:

$$\begin{aligned} \begin{bmatrix} R_{d1} & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & R \end{bmatrix} &= V_e^T J_e V_e \\ [0 \ 0 \ L] + X[A_1 \ M(\tau_2)^{\frac{1}{2}} \ B] &= W_e^T J_e V_e \\ Q + A^T X + X A &= W_e^T J_e W_e. \end{aligned} \quad (72)$$

Taking into account Definition 2 in conjugation with (21), the first equation in (72) leads to the following structure for  $V_e$ :

$$V_e = \left[ \begin{array}{c|c} V_{dp} & V \\ \hline & \end{array} \right] = \left[ \begin{array}{c|cc} V_{dp} & V_{11} & 0 \\ \hline & V_{21} & V_{22} \end{array} \right], \quad (73)$$

where

$$V_{dp} = \begin{bmatrix} V_{d1} & \\ & V_{d2} \end{bmatrix}.$$

Let  $W_e$  be partitioned accordingly, i. e.,

$$W_e = \begin{bmatrix} \frac{W_{dp}}{W} \end{bmatrix} = \begin{bmatrix} \frac{W_{dp}}{W_1} \\ W_2 \end{bmatrix}, \quad (74)$$

where  $W_{dp}^T = [W_{d1}^T \ W_{d2}^T]$ . Substituting (73), (74) in (72), leads to the following form:

$$\begin{aligned} \begin{bmatrix} R_{d1} & -I & R \end{bmatrix} &= \begin{bmatrix} -V_{d1}^T V_{d1} & -V_{d2}^T V_{d2} & V^T J V \end{bmatrix} \\ \begin{bmatrix} X A_1 & X M(\tau_2)^{\frac{1}{2}} \end{bmatrix} &= \begin{bmatrix} -W_{d1}^T V_{d1} & -W_{d2}^T V_{d2} \end{bmatrix} \\ L + X B &= W^T J V \\ Q + A^T X + X A &= -W_{d1}^T W_{d1} - W_{d2}^T W_{d2} + W^T J W. \end{aligned} \quad (75)$$

Using (23) one gets

$$F_e = \begin{bmatrix} \frac{F_{dp}}{F} \end{bmatrix} = \begin{bmatrix} \frac{-V_{dp}^{-1} W_{dp}}{-V^{-1} W} \end{bmatrix} = \begin{bmatrix} \frac{F_{dp}}{F_1} \\ F_2 \end{bmatrix} = \begin{bmatrix} \frac{-V_{dp}^{-1} W_{dp}}{-V_{11}^{-1} W_1} \\ V_{22}^{-1} V_{21} V_{11}^{-1} W_1 - V_{22}^{-1} W_2 \end{bmatrix} \quad (76)$$

where the structure (13) has been also taken into account and where

$$A + \begin{bmatrix} A_1 & M(\tau_2)^{\frac{1}{2}} & B \end{bmatrix} \begin{bmatrix} F_{d1} \\ F_{d2} \\ F \end{bmatrix}$$

is uniformly asymptotically stable.

Let now

$$\tilde{F}_{e,2} := \begin{bmatrix} 0 \\ \tilde{F}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{F}_2 \end{bmatrix} \quad (77)$$

and let  $\tilde{\Sigma}_{e,2}$  be the  $\tilde{F}_{e,2}$ -equivalent of  $\Sigma_e$  in (19). Following Theorem 1 and taking into account the zero structure of  $\tilde{F}_{e,2}$  in (77), the updated form of the last equation in (75) corresponding to  $\tilde{\Sigma}_{e,2}$  is

$$\tilde{A}^T X + X \tilde{A} = -\tilde{Q} - W_{d1}^T W_{d1} - W_{d2}^T W_{d2} + (W + V \tilde{F}_2)^T J (W + V \tilde{F}_2). \quad (78)$$

Furthermore

$$\begin{aligned} W + V \tilde{F}_2 &= \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} + \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} 0 \\ V_{22}^{-1} V_{21} V_{11}^{-1} W_1 - V_{22}^{-1} W_2 \end{bmatrix} \\ &= \begin{bmatrix} W_1 \\ V_{21} V_{11}^{-1} W_1 \end{bmatrix} \end{aligned}$$



from which

$$\begin{aligned}(W + V\tilde{F}_2)^T J(W + V\tilde{F}_2) &= -W_1^T W_1 + W_1^T V_{11}^{-T} V_{21}^T V_{11}^{-1} W_1 \\ &= W_1^T V_{11}^{-T} (-V_{11}^T V_{11} + V_{21}^T V_{21}) V_{11}^{-1} W_1.\end{aligned}\quad (79)$$

Taking into account (22), the second equation in (75) yields, by equating the (1, 1) entries

$$R_{11} = -V_{11}^T V_{11} + V_{21}^T V_{21}.\quad (80)$$

With (80) in (79) and then with (79) in (78), one gets eventually

$$\tilde{A}^T X + X \tilde{A} = -\tilde{Q} - W_{d1}^T W_{d1} - W_{d2}^T W_{d2} - W_1^T V_{11}^{-T} \tilde{R}_{11} V_{11}^{-1} W_1\quad (81)$$

where

$$\tilde{R}_{11} := -R_{11} > 0\quad (82)$$

as follows from (25).

We introduce the Lyapunov functional

$$\begin{aligned}V(x_t) &= x^T(t) X x(t) + \int_{t-\tau_1}^t x^T(\theta) (-R_{d1}) x(\theta) d\theta \\ &\quad + \int_{t-\tau_2}^t x^T(\theta) (-R_{d2}(t-\theta)) x(\theta) d\theta,\end{aligned}\quad (83)$$

where  $X = X^T \geq 0$  and  $R_{d1} = R_{d1}^T < 0$  are given before; the time-varying matrix function  $R_{d2}(\cdot)$  is given in (28). Note that since the inequality (26) is satisfied, it follows that  $-R_{d2}(\xi)$  is a symmetric and positive-definite matrix for each  $\xi \in [0, \tau_2]$ . Simple computations prove that there exist two positive numbers  $d_1, d_2$  such that

$$d_1 \|x(t)\|^2 \leq V(t, x_t) \leq d_2 \sup_{\theta \in [t-\tau, t]} \|x(\theta)\|^2.\quad (84)$$

Taking the Lyapunov derivative of (83) with respect to (30), one obtains:

$$\begin{aligned}\dot{V}(x_t) &= x^T(\tilde{A}^T X + X \tilde{A}) x + x(t-\tau_1)^T A_1^T X x + x^T X A_1 x(t-\tau_1) \\ &\quad + \left( \int_0^{\tau_2} A_2(\theta) x(t-\theta) d\theta \right)^T X x + x^T X \left( \int_0^{\tau_2} A_2(\theta) x(t-\theta) d\theta \right) \\ &\quad + x^T(-R_{d1}) x - x^T(t-\tau_1)(-R_{d1}) x(t-\tau_1) + x^T(-R_{d2}(0)) x \\ &\quad - x^T(t-\tau_2)(-R_{d2}(\tau_2)) x(t-\tau_2) + \int_{-\tau_2}^0 x^T(\theta+t) \left( \frac{dR_{d2}(\theta)}{d\theta} \right) x(\theta+t) d\theta.\end{aligned}\quad (85)$$

Since

$$\begin{aligned}&2x^T X \int_0^{\tau_2} A_2(\theta) x(t-\theta) d\theta \\ &\leq x^T X \left( \int_0^{\tau_2} A_2(\theta) Y(\theta) - 1 A_2^T(\theta) d\theta \right) X x + \int_0^{\tau_2} x^T(t+\theta) Y(\theta) x(t+\theta) d\theta \\ &= x^T X M^{1/2} M^{1/2} X x - \int_{-\tau_2}^0 x^T(\theta+t) \left( \frac{dR_{d2}(\theta)}{d\theta} \right) x(\theta+t) d\theta,\end{aligned}$$

an since we have (75), we can write the corresponding equations of the KPYS( $\tilde{\Sigma}_{e,2}, J_e$ ) (see the structure of  $\tilde{F}_{e,2}$  in (77)), and (85) becomes:

$$\begin{aligned} \dot{V}(x_t) &\leq -x^T \tilde{Q} x - x^T W_{d1}^T W_{d1} x - x^T W_{d2}^T W_{d2} x - x^T W_1^T V_{11}^{-T} \tilde{R}_{11} V_{11}^{-1} W_1 x \\ &\quad - x^T W_{d1}^T V_{d1} x(t - \tau_1) - x(t - \tau_1)^T V_{d1}^T W_{d1} x - x(t - \tau_1)^T V_{d1}^T V_{d1} x(t - \tau_2) \\ &\quad - x^T R_{d1} x - x^T R_{d2}(0) x - x^T(t - \tau_2)(-R_{d2}(\tau_2)) x(t - \tau_2) + x^T X M^{1/2} M^{1/2} X x \\ &\leq -x^T (\tilde{Q} + R_{d1} + R_{d2}(0)) x(t) \end{aligned} \quad (86)$$

where both (27) and (81) have been used. With (84) and after some algebraic manipulations the proof is completed via the Krasovskii stability theorem (Hale and Verduyn Lunel [4]).

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