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DECENTRALIZED CONTROL OF INTERCONNECTED LINEAR SYSTEMS WITH DELAYED STATES

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This paper addresses the problems of stability analysis and decentralized control of interconnected linear systems with constant time-delays in the state of each subsystems as well as in the interconnections. We develop delay-dependent methods of stability analysis and decentralized stabilization via linear memoryless state-feedback. The proposed methods are given in terms of linear matrix inequalities. Extensions of the decentralized stabilization result to more complex control problems, such as decentralized static output feedback, decentralized $\mathcal{H}_\infty$ control, decentralized robust stabilization, and decentralized robust $\mathcal{H}_\infty$ control are also discussed.

1. INTRODUCTION

Many control problems of modern industrial society are associated with the control of complex large-scale interconnected systems which are in general subject to time-delay in the interconnections. Typical examples can be often encountered in a large spectrum of applications such as electrical power systems, chemical process control systems, etc. During the past 30 years, control problems for interconnected systems have received considerable attention and a very popular way to dealing with these problems is to make use of local or decentralized feedback controllers to stabilize the overall system (see, e. g. [9, 14, 16] and [20]). In recent years, the problems of robust stability analysis and robust stabilization for interconnected uncertain linear systems have been widely studied by many researchers; see, e. g. [2, 5, 8] and [21]. On the other hand, the problems of stability analysis and stabilization of interconnected systems with time-delays have also received a lot of attention and a number of results have been reported in the literature over the past years; see, e. g. [7, 10, 15, 19] and [22]. A common feature of the latter results is that they are independent of the length of the time-delays in the system, i. e. the time-delays are allowed to be arbitrarily large, and as such they cannot be applied in many important applications, more specifically, in situations where the stability or stabilizability of the system depends on the length of the time-delays, which is a fairly common situation. Although increasing attention has recently been devoted to the study of delay-dependent methods of stability analysis and stabilization for "isolated" systems (e.g. [4], [11]–[13], [17] and [18]), the problems of delay-dependent stability and stabilization for interconnected systems have not yet been fully investigated.

This paper addresses the problems of stability analysis and decentralized control of interconnected linear systems with state delays. The time-delays are constant and may appear in state of each subsystems as well as in the interconnections. First, *delay-dependent* methods of stability analysis and decentralized stabilization via linear memoryless state feedback are developed. More precisely, using a Lyapunov functional approach and linear matrix inequalities (LMIs) techniques, we propose a stability criterion and a design method of decentralized stabilizing controllers which incorporate information on the length, or an upper-bound, of the time-delays in the system. The proposed methods have the advantage that they can be implemented numerically very efficiently using recently developed algorithms for solving linear matrix inequalities; see, e.g. [1]. Then, extensions of the decentralized stabilization result to more complex control problems are analysed, including decentralized stabilization via static output, decentralized $\mathcal{H}_\infty$ control, decentralized robust stabilization, and decentralized robust $\mathcal{H}_\infty$ control. This paper extends results of [4] to the context of interconnected linear systems with delayed states.

Notation. $\Re^n$ denotes the n-dimensional Euclidean space, $\Re^{n\times m}$ is the set of $n \times m$ real matrices, diag$\{\cdots\}$ denotes a block-diagonal matrix, and the notation $X > 0$, for a real matrix X, means that X is symmetric and positive definite. $\mathcal{L}_2$ denotes the space of square integrable vector functions on $[0,\infty)$ with norm $\|\cdot\|_2 := \left(\int_0^\infty \|\cdot\|^2 \mathrm{d}t\right)^{\frac{1}{2}}$, where $\|\cdot\|$ stands for the Euclidean vector norm.

2. PROBLEM FORMULATION

Consider the large-scale linear time-delay system composed of the interconnection of N subsystems described by

$$\begin{aligned} (S_i):\ \dot{x}_i(t) &= A_i x_i(t) + \sum_{j=1}^{N} A_{ij} x_j(t-\tau_{ij}) + B_i u_i(t), \quad i = 1,\dots,N, \\ x_i(t) &= \phi_i(t), \ \ \forall\, t \in [-\tau_{\max}, 0]; \quad \tau_{\max} = \max\{\tau_{ij},\ i,j = 1,\dots,N\} \end{aligned} \tag{1}$$

where for the ith subsystem (S_i), $x_i(t) \in \Re^{n_i}$ is the state, $u_i(t) \in \Re^{m_i}$ is the control input, $\tau_{ii} \geq 0$ is the time-delay in the subsystem, $\tau_{ij} \geq 0$, $j = 1,\dots,N$, $j \neq i$ are the time-delays in the interconnections, $\phi_i(\cdot)$ is the initial condition, A_i, A_{ij}, $j = 1,\dots,N$, and B_i are known real constant matrices of appropriate dimensions.

In this paper we shall develop delay-dependent conditions for stability and decentralized stabilization for the interconnected system of (1). The stability problem to be addressed is as follows. *Given scalars $\bar{\tau}_{ij} > 0$, $i, j = 1,\dots,N$, find conditions which ensure that the system of* (1) *with $u_i(t) \equiv 0$, $i = 1,\dots,N$, is globally asymptotically stable for any constant time-delays τ_{ij} satisfying $0 \leq \tau_{ij} \leq \bar{\tau}_{ij}$, $i, j = 1,\dots,N$.*

On the other hand, the stabilization problem under investigation consists in: *Determine a decentralized memoryless control law* $u_i(t) = K_i x_i(t),\ i = 1, \dots, N,$ *for the interconnected system of* (1) *such that the resulting closed-loop system is globally asymptotically stable for any constant time-delays* τ_{ij} *satisfying* $0 \le \tau_{ij} \le \bar{\tau}_{ij},\ i, j = 1, \dots, N$. In this case, the system of (1) is said to be *decentralized stabilizable*. A linear matrix inequality approach will be developed for solving the above problems.

3. MAIN RESULTS

We first deal with the problem of stability analysis for the unforced system of (1) with $u_i(t) \equiv 0,\ i = 1, \dots, N$. A criterion for global asymptotic stability is provided by the following theorem.

Theorem 1. Consider the unforced system of (1) with $u_i(t) \equiv 0,\ i = 1, \dots, N,$ and let $\bar{\tau}_{ij} > 0,\ i, j = 1, \dots, N,$ be given scalars. Then this system is globally asymptotically stable for any constant time-delays τ_{ij} satisfying $0 \le \tau_{ij} \le \bar{\tau}_{ij},\ i, j = 1, \dots, N,$ if there exist $n_i \times n_i$ symmetric positive definite matrices X_i, R_{ij} and S_{ijk}, $i, j, k = 1, \dots, N$, satisfying the following LMI:

$$\begin{bmatrix} \Phi & \Lambda^T & \Omega^T \\ \Lambda & -\mathcal{R} & 0 \\ \Omega & 0 & -J \end{bmatrix} < 0 \tag{2}$$

where

$$\Phi = \begin{bmatrix} \Phi_1 & A_{12}X_2 + X_1A_{21}^T & \cdots & A_{1N}X_N + X_1A_{N1}^T \\ A_{21}X_1 + X_2A_{12}^T & \Phi_2 & \cdots & A_{2N}X_N + X_2A_{N2}^T \\ \vdots & \vdots & & \vdots \\ A_{N1}X_1 + X_NA_{1N}^T & A_{N2}X_2 + X_NA_{2N}^T & \cdots & \Phi_N \end{bmatrix} \tag{3}$$

$$\Phi_i = (A_i + A_{ii})X_i + X_i(A_i + A_{ii})^T + \sum_{j=1}^{N} \bar{\tau}_{ij} A_{ij} \hat{R}_{ij} A_{ij}^T \tag{4}$$

$$\hat{R}_{ij} = R_{ij} + \sum_{j=1}^{N} S_{jki} \tag{5}$$

$$\Lambda = \operatorname{diag}\{\Lambda_1, \dots, \Lambda_N\}, \qquad \Omega = \operatorname{diag}\{\Omega_1, \dots, \Omega_N\} \tag{6}$$

$$\mathcal{R} = \operatorname{diag}\{\mathcal{R}_1, \dots, \mathcal{R}_N\}, \qquad J = \operatorname{diag}\{J_1, \dots, J_N\} \tag{7}$$

$$\Lambda_i^T = \begin{bmatrix} \bar{\tau}_{1i}X_iA_i^T & \bar{\tau}_{2i}X_iA_i^T & \dots & \bar{\tau}_{Ni}X_iA_i^T \end{bmatrix} \tag{8}$$

$$\Omega_i^T = \begin{bmatrix} X_iM_{1i}^T & X_iM_{2i}^T & \dots & X_iM_{Ni}^T \end{bmatrix}, \quad M_{ki}^T = \begin{bmatrix} \bar{\tau}_{k1}A_{1i}^T & \bar{\tau}_{k2}A_{2i}^T & \dots & \bar{\tau}_{kN}A_{Ni}^T \end{bmatrix} \tag{9}$$

$$\mathcal{R}_i = \operatorname{diag}\{\bar{\tau}_{1i}R_{1i},\ \bar{\tau}_{2i}R_{2i},\dots,\bar{\tau}_{Ni}R_{Ni}\} \tag{10}$$

$$J_i = \operatorname{diag}\{J_{1i},\ J_{2i},\dots,J_{Ni}\},\qquad J_{ki} = \operatorname{diag}\{\bar{\tau}_{k1}S_{1ik},\ \bar{\tau}_{k2}S_{2ik},\dots,\bar{\tau}_{kN}S_{Nik}\} \tag{11}$$

Proof. The proof technique is inspired by that used to prove Theorem 1 in [11]. Let $x_i(t)$, $i = 1,\dots,N$, be the trajectory of the unforced system of (1) with $u_i(t) \equiv 0$, $i = 1,\dots,N$. Then we have that for $t \geq \tau_{\max}$:

$$\begin{aligned} x_j(t-\tau_{ij}) &= x_j(t) - \int_{-\tau_{ij}}^{0} \dot{x}_j(t+\theta)\,\mathrm{d}\theta \\ &= x_j(t) - \int_{-\tau_{ij}}^{0} \Big[A_j x_j(t+\theta) + \sum_{k=1}^{N} A_{jk} x_k(t-\tau_{jk}+\theta)\Big]\mathrm{d}\theta. \end{aligned}$$

Substituting $x_j(t-\tau_{ij})$ in (1), we obtain that $x_i(t)$ satisfies:

$$\dot{x}_i(t) = A_i x_i(t) + \sum_{j=1}^{N} A_{ij} x_j(t) - \sum_{j=1}^{N} A_{ij}\int_{-\tau_{ij}}^{0}\left[A_j x_j(t+\theta) + \sum_{k=1}^{N} A_{jk} x_k(t-\tau_{jk}+\theta)\right]\mathrm{d}\theta.$$

In view of the above, it follows that (1) with $u_i(t) \equiv 0$ is a special case of the following system:

$$\begin{aligned} \dot{\xi}_i(t) &= A_i\xi_i(t) + \sum_{j=1}^{N} A_{ij}\xi_j(t) \\ &\quad - \sum_{j=1}^{N} A_{ij}\int_{-\tau_{ij}}^{0}\left[A_j\xi_j(t+\theta) + \sum_{k=1}^{N} A_{jk}\xi_k(t-\tau_{jk}+\theta)\right]\mathrm{d}\theta, \quad i = 1,\dots,N \qquad (12) \\ \xi_i(t) &= \psi_i(t), \quad \forall\, t \in [-2\tau_{\max}, 0], \quad i = 1,\dots,N \qquad (13) \end{aligned}$$

where $\psi_i(\cdot)$ is the initial condition for x_i. Observe that (12) requires initial data on $[-2\tau_{\max}, 0]$.

Notice that any solution of (1) with $u_i(t) \equiv 0$, $i = 1,\dots,N$, is also a solution of (12) – (13); see, e.g. [6]. Therefore, the global asymptotic stability of (12) – (13) will ensure the global asymptotic stability of (1). In the sequel, we will study the stability of the system of (12) – (13) in order to ascertain the global asymptotic stability of the system of (1).

Let the following Lyapunov functional candidate for the system of (12) – (13)

$$V(\xi) = \sum_{i=1}^{N}\{\xi_i^T(t)P_i\xi_i(t) + W_i(\xi)\} \tag{14}$$

where P_i, $i = 1,\dots,N$, are symmetric positive definite matrices, ξ denotes the vector $[\xi_1^T,\dots,\xi_N^T]^T$ and

$$W_i(\xi) = \sum_{j=1}^{N}\int_{-\tau_{ij}}^{0}\int_{t+\theta}^{t} \xi_j^T(s)A_j^T R_{ij}^{-1} A_j\xi_j(s)\,\mathrm{d}s\mathrm{d}\theta$$

$$+\sum_{j=1}^{N}\sum_{k=1}^{N}\int_{-\tau_{ij}}^{0}\int_{t-\tau_{ij}+\theta}^{t}\xi_k^T(s)A_{jk}^T S_{jki}^{-1}A_{jk}\xi_k(s)\,\mathrm{d}s\mathrm{d}\theta \tag{15}$$

where R_{ij} and S_{ijk}, $i, j, k = 1,\ldots,N$, are symmetric positive definite matrices to be chosen.

The time-derivative of $V(\xi)$ along the solution of (12) – (13) is given by

$$\dot{V}(\xi) = \sum_{i=1}^{N}\Big[\xi_i^T(t)\left(P_iA_i + A_i^TP_i\right)\xi_i(t) + 2\xi_i^T(t)P_i\sum_{j=1}^{N}A_{ij}\xi_j(t)\Big]$$
$$+\sum_{i=1}^{N}\Big[\eta_1^{(i)}(\xi,t) + \eta_2^{(i)}(\xi,t) + \dot{W}_i(\xi)\Big] \tag{16}$$

where

$$\eta_1^{(i)}(\xi,t) = -2\xi_i^T(t)P_i\sum_{j=1}^{N}A_{ij}\int_{-\tau_{ij}}^{0}A_j\xi_j(t+\theta)\,\mathrm{d}\theta$$

$$\eta_2^{(i)}(\xi,t) = -2\xi_i^T(t)P_i\sum_{j=1}^{N}A_{ij}\int_{-\tau_{ij}}^{0}\sum_{k=1}^{N}A_{jk}\xi_k(t-\tau_{jk}+\theta)\,\mathrm{d}\theta.$$

Recalling that for any vectors u, v and any matrix $Q>0$ of appropriate dimensions:

$$-2u^Tv \le u^TQu + v^TQ^{-1}v$$

we have that for any matrices $R_{ij}>0$ and $S_{ijk}>0$, $i, j, k = 1,\ldots,N$,

$$\eta_1^{(i)} \le \sum_{j=1}^{N}\tau_{ij}\xi_i^T(t)P_iA_{ij}R_{ij}A_{ij}^TP_i\xi_i(t) + \sum_{j=1}^{N}\int_{-\tau_{ij}}^{0}\xi_j^T(t+\theta)A_j^TR_{ij}^{-1}A_j\xi_j(t+\theta)\,\mathrm{d}\theta \tag{17}$$

$$\eta_2^{(i)} \le \sum_{j=1}^{N}\sum_{k=1}^{N}\tau_{ij}\xi_i^T(t)P_iA_{ij}S_{jki}A_{ij}^TP_i\xi_i(t)$$
$$+\sum_{j=1}^{N}\sum_{k=1}^{N}\int_{-\tau_{ij}}^{0}\xi_k^T(t-\tau_{jk}+\theta)A_{jk}^TS_{kji}^{-1}A_{jk}\xi_k(t-\tau_{jk}+\theta)\,\mathrm{d}\theta. \tag{18}$$

Next, in view of (15), we have that the time-derivative of $W_i(\xi)$ satisfies

$$\dot{W}_i(\xi) = \sum_{j=1}^{N}\left[\tau_{ij}\xi_j^T(t)A_j^TR_{ij}^{-1}A_j\xi_j(t) - \int_{-\tau_{ij}}^{0}\xi_j^T(t+\theta)A_j^TR_{ij}^{-1}A_j\xi_j(t+\theta)\,\mathrm{d}\theta\right]$$
$$+\sum_{j=1}^{N}\sum_{k=1}^{N}\tau_{ij}\xi_k^T(t)A_{jk}^TS_{jki}^{-1}A_{jk}\xi_k(t)$$
$$-\sum_{j=1}^{N}\sum_{k=1}^{N}\int_{-\tau_{ij}}^{0}\xi_k^T(t-\tau_{jk}+\theta)A_{jk}^TS_{jki}^{-1}A_{jk}\xi_k(t-\tau_{jk}+\theta)\,\mathrm{d}\theta. \tag{19}$$

Hence, using (17) – (19) in (16) we obtain:

$$\dot{V}(\xi) \le \sum_{i=1}^{N} \left\{ \xi_i^T(t) \left[P_i A_i + A_i^T P_i + P_i \sum_{j=1}^{N} \tau_{ij} A_{ij} \hat{R}_{ij} A_{ij} P_i \right] \xi_i(t) \right.$$
$$+ 2\xi_i^T(t) P_i \sum_{j=1}^{N} A_{ij}\xi_j(t) + \sum_{j=1}^{N} \tau_{ij} \xi_j^T(t) A_j^T R_{ij}^{-1} A_j \xi_j(t)$$
$$\left. + \sum_{j=1}^{N} \sum_{k=1}^{N} \tau_{ij} \xi_k^T(t) A_{jk}^T S_{jki}^{-1} A_{kj} \xi_k(t) \right\}. \tag{20}$$

where $\hat{R}_i$ is as in (5).

Now, considering that

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \tau_{ij} \xi_j^T(t) A_j^T R_{ij}^{-1} A_j \xi_j(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} \tau_{ji} \xi_i^T(t) A_i^T R_{ji}^{-1} A_i \xi_i(t)$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \tau_{ij} \xi_k^T(t) A_{jk}^T S_{jki}^{-1} A_{jk} \xi_k(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \tau_{kj} \xi_i^T(t) A_{ji}^T S_{jik}^{-1} A_{ji} \xi_i(t)$$

we have that

$$\dot{V}(\xi) \le \sum_{i=1}^{N} \left\{ \xi_i^T(t) \Psi_i(\tau) \xi_i(t) + 2\xi_i^T(t) P_i \sum_{\substack{j=1 \\ j \ne i}}^{N} A_{ij} \xi_j(t) \right\} \tag{21}$$

where τ denotes $\{\tau_{i1}, \ldots, \tau_{iN},\ i = 1, \ldots, N\}$ and

$$\Psi_i(\tau) = P_i(A_i + A_{ii}) + (A_i + A_{ii})^T P_i + P_i \sum_{j=1}^{N} \tau_{ij} A_{ij} \hat{R}_{ij} A_{ij}^T P_i$$
$$+ \sum_{j=1}^{N} \tau_{ji} A_i^T R_{ji}^{-1} A_i + \sum_{j=1}^{N} \sum_{k=1}^{N} \tau_{kj} A_{ji}^T S_{jik}^{-1} A_{ji}. \tag{22}$$

Next, introducing

$$\Theta(\tau) = \begin{bmatrix} \Psi_1(\tau) & P_1 A_{12} + A_{21}^T P_2 & \cdots & P_1 A_{1N} + A_{N1}^T P_N \\ P_2 A_{21} + A_{12}^T P_1 & \Psi_2(\tau) & \cdots & P_2 A_{2N} + A_{N2}^T P_N \\ \vdots & \vdots & & \vdots \\ P_N A_{N1} + A_{1N}^T P_1 & P_N A_{N2} + A_{2N}^T P_2 & \cdots & \Psi_N(\tau) \end{bmatrix} \tag{23}$$

the inequality of (21) can be rewritten as

$$\dot{V}(\xi) \le \xi^T(t) \Theta(\tau) \xi(t). \tag{24}$$

Since $\Theta(\tau)$ is monotonic increasing (in the sense of positive definiteness) with respect to τ_{ij}, $i, j = 1, \ldots, N$, i.e. $\Theta(\bar{\tau}) - \Theta(\tau) \geq 0$, for $\tau = \{\tau_{i1}, \ldots, \tau_{iN},\ i = 1, \ldots, N\}$ and $\bar{\tau} = \{\bar{\tau}_{i1}, \ldots, \bar{\tau}_{iN},\ i = 1, \ldots, N\}$ with $\bar{\tau}_{ij} \geq \tau_{ij}$, we have that if for some scalars $\bar{\tau}_{ij} > 0$, $i, j = 1, \ldots, N$, there exist symmetric positive definite matrices P_i, R_{ij} and S_{ijk}, $i, j, k = 1, \ldots N$, such that

$$\Theta(\bar{\tau}) < 0 \tag{25}$$

where $\bar{\tau}$ denotes $\{\bar{\tau}_{i1}, \ldots, \bar{\tau}_{iN},\ i = 1, \ldots, N\}$, then the system of (12)–(13) is globally asymptotically stable for any constant time-delays τ_{ij} satisfying $0 \leq \tau_{ij} \leq \bar{\tau}_{ij}$, $i, j = 1, \ldots, N$. This in turns implies the global asymptotic stability of the system of (1) for any constant time-delays τ_{ij} satisfying $0 \leq \tau_{ij} \leq \bar{\tau}_{ij}$, $i, j = 1, \ldots, N$.

Now, introduce the new variables, $X_i := P_i^{-1}$, $i = 1, \ldots, N$, and denote $\hat{X} = \text{diag}\{X_1, \cdots, X_N\}$. Pre-multiplying and post-multiplying $\Theta(\bar{\tau})$ by $\hat{X}$ and using Schur complements, it can be readily verified that the condition of (25) is equivalent to the LMI of (2), which concludes the proof. □

Remark 1. Theorem 1 provides a delay-dependent criterion of global asymptotic stability for the class of interconnected time-delay systems of (1) in terms of the solvability of linear matrix inequalities. This stability criterion can be tested numerically very efficiently using interior point algorithms, which have been recently developed for solving linear matrix inequalities; see, e.g. [1].

Remark 2. Note that the matrices R_{ij} and S_{ijk}, $i, j, k = 1, \ldots, N$, in Theorem 1 are scaling matrices to be found, which are used to minimize the upper-bounds for $\eta_1^{(i)}$ and $\eta_2^{(i)}$, $i = 1, \ldots, N$, in (17) and (18), respectively. In the case where N is large, the computational effort required to solve the feasibility problem for the LMI of (2) can become very high and maybe prohibitive. In such situations, a strategy to reduce the computational effort is to reduce the number of scaling matrices; for instance, we could set $R_{ij} = R_i$ and/or $S_{ijk} = S_{ij}$, for $i, j, k = 1, \ldots, N$. However, it is likely that reducing the number of scaling matrices, the result obtained may be more conservative. This tradeoff between the number of different scaling matrices, R_{ij} and S_{ijk}, to be found and the conservatism of the result is an important issue that the author is currently investigating.

In the light of the result of Theorem 1, we are now able to present our main result on decentralized stabilization via linear memoryless state feedback for the interconnected system of (1).

Theorem 2. Consider the interconnected system of (1) and let $\bar{\tau}_{ij} > 0$, $i, j = 1, \ldots, N$, be given scalars. Then this system is decentralized stabilizable for any constant time-delays τ_{ij} satisfying $0 \leq \tau_{ij} \leq \bar{\tau}_{ij}$, $i, j = 1, \ldots, N$, if there exist $n_i \times n_i$ symmetric positive definite matrices X_i, R_{ij} and S_{ijk}, $i, j, k = 1, \ldots, N$, and $m_i \times n_i$ matrices Y_i, $i = 1, \ldots, N$, satisfying the following LMI:

$$\begin{bmatrix} \Phi_c & \Lambda_c^T & \Omega^T \\ \Lambda_c & -\mathcal{R} & 0 \\ \Omega & 0 & -J \end{bmatrix} < 0 \tag{26}$$

where

$$\Phi_c = \begin{bmatrix} \Phi_{c1} & A_{12}X_2 + X_1A_{21}^T & \cdots & A_{1N}X_N + X_1A_{N1}^T \\ A_{21}X_1 + X_2A_{12}^T & \Phi_{c2} & \cdots & A_{2N}X_N + X_2A_{N2}^T \\ \vdots & \vdots & & \vdots \\ A_{N1}X_1 + X_NA_{1N}^T & A_{N2}X_2 + X_NA_{2N}^T & \cdots & \Phi_{cN} \end{bmatrix} \tag{27}$$

$$\Lambda_c = \text{diag}\{\Lambda_{c1}, \ldots, \Lambda_{cN}\} \tag{28}$$

$$\Phi_{ci} = (A_i + A_{ii})X_i + X_i(A_i + A_{ii})^T + B_iY_i + Y_i^TB_i^T + \sum_{j=1}^{N} \bar{\tau}_{ij}A_{ij}\hat{R}_{ij}A_{ij}^T \tag{29}$$

$$\Lambda_{ci}^T = \left[\ \bar{\tau}_{1i}(X_iA_i^T + Y_i^TB_i^T) \quad \bar{\tau}_{2i}(X_iA_i^T + Y_i^TB_i^T) \ \ldots \ \bar{\tau}_{Ni}(X_iA_i^T + Y_i^TB_i^T)\ \right] \tag{30}$$

and $\hat{R}_{ij}$, Ω, $\mathcal{R}$ and J are as in (5) – (7).

Moreover, a suitable decentralized control law is given by $u_i(t) = Y_iX_i^{-1}x_i(t)$.

Proof. With the decentralized control law $u_i(t) = K_ix_i(t)$, $i = 1, \ldots, N$, where the state feedback gain matrices $K_i \in \Re^{m_i \times n_i}$, $i = 1, \ldots, N$, are to be found, the system of (1) becomes

$$S_i: \quad \dot{x}_i(t) = A_{ci}x_i(t) + \sum_{j=1}^{N} A_{ij}x_j(t - \tau_{ij}) \tag{31}$$

where $A_{ci} = A_i + B_iK_i$. Hence, the result follows immediately by applying Theorem 1 to the closed-loop system of (31) and setting $Y_i = K_iX_i$. □

Remark 3. Theorem 2 provides an LMI method for the design of a delay-dependent decentralized state feedback control law that stabilize the class of interconnected time-delay systems of (1). Since the proposed decentralized control design is dependent on the length, or an upper-bound, of the time-delays in the system, it is expected that this design finds a larger spectrum of applications than the delay-independent control designs, especially in situations where the existing time-delays are not allowed to be arbitrarily large, which is often the case in many applications.

4. EXTENSIONS

As the proposed decentralized stabilization method is given in terms of LMIs, it can be easily extended, by using standard LMIs techniques [1], to more complex control problems, such as decentralized stabilization via static output feedback, decentralized $\mathcal{H}_\infty$ control and decentralized robust stabilization. In the sequel we shall consider several extensions of the decentralized stabilization result developed in the previous section.

4.1. Decentralized stabilization via static output feedback

For the problem of decentralized stabilization via static output feedback, we shall consider the interconnected system of (1) together with the local output measurements:

$$y_i(t) = C_i x_i(t), \quad i = 1, \dots, N \tag{32}$$

where $y_i(t) \in \Re^{r_i}$ is the output of the ith subsystem (S_i) and C_i, $i = 1, \dots, N$, are known real constant matrices of appropriate dimensions. Without loss of generality, the following assumption is adopted:

Assumption 1. The matrices C_i, $i = 1, \dots, N$, are of full row-rank.

Note that Assumption 1, which accounts for the linear independence of the components of the local measurement vectors y_i, can always be achieved by discarding redundant measurement components.

In the case of decentralized static output feedback stabilization, the desired control law has the following structure

$$u_i(t) = G_i y_i(t), \quad i = 1, \dots, N$$

or equivalently

$$u_i(t) = K_i x_i(t), \quad i = 1, \dots, N$$

with the constraints

$$K_i = G_i C_i, \quad i = 1, \dots, N.$$

In the light of Theorem 1, given scalars $\bar{\tau}_{ij} > 0$, $i, j = 1, \dots, N$, the system $(\mathcal{S})$ is stabilizable via decentralized output feedback for any constant time-delays τ_{ij} satisfying $0 \le \tau_{ij} \le \bar{\tau}_{ij}$, $i, j = 1, \dots, N$, if there exist symmetric positive definite matrices X_i, G_i, R_{ij} and S_{ijk}, $i, j, k = 1, \dots, N$, satisfying the inequality of (2), where the matrices Φ_i and Λ_i of (4) and (8), respectively, are now given by:

$$\Phi_i = (A_i + A_{ii})X_i + X_i(A_i + A_{ii})^T + B_i G_i C_i X_i + X_i C_i^T G_i^T B_i^T + \sum_{j=1}^{N} \bar{\tau}_{ij} A_{ij} \hat{R}_{ij} A_{ij}^T$$

$$\Lambda_i^T = \left[\ \bar{\tau}_{1i} X_i (A_i + B_i G_i C_i)^T \quad \bar{\tau}_{2i} X_i (A_i + B_i G_i C_i)^T \ \dots \ \bar{\tau}_{Ni} X_i (A_i + B_i G_i C_i)^T \ \right].$$

Now, the problem of numerically solving the inequality of (2) for X_i and G_i, becomes a very difficult one because it is non-convex in general. Motivated by this fact and inspired by the work of [3], in the sequel we present a sufficient condition for decentralized static output feedback stabilization which has the advantage to be convex.

Theorem 3. Consider the system (1) with the measurements of (32) and let $\bar{\tau}_{ij} > 0$, $i, j = 1, \ldots, N$, be given scalars. Then this system is stabilizable via decentralized static output feedback for any constant time-delays τ_{ij} satisfying $0 \leq \tau_{ij} \leq \bar{\tau}_{ij}$, $i, j = 1, \ldots, N$, if there exist symmetric positive definite matrices X_i, R_{ij} and S_{ijk}, $i, j, k = 1, \ldots, N$, and matrices D_i and E_i, $i = 1, \ldots, N$, such that:

$$\begin{bmatrix} \hat{\Phi}_c & \hat{\Lambda}_c^T & \Omega^T \\ \hat{\Lambda}_c & -\mathcal{R} & 0 \\ \Omega & 0 & -J \end{bmatrix} < 0 \tag{33}$$

$$D_i C_i = C_i X_i, \quad i = 1, \ldots, N \tag{34}$$

where

$$\hat{\Phi}_c = \begin{bmatrix} \hat{\Phi}_{c1} & A_{12}X_2 + X_1 A_{21}^T & \cdots & A_{1N}X_N + X_1 A_{N1}^T \\ A_{21}X_1 + X_2 A_{12}^T & \hat{\Phi}_{c2} & \cdots & A_{2N}X_N + X_2 A_{N2}^T \\ \vdots & \vdots & & \vdots \\ A_{N1}X_1 + X_N A_{1N}^T & A_{N2}X_2 + X_N A_{2N}^T & \cdots & \hat{\Phi}_{cN} \end{bmatrix} \tag{35}$$

$$\hat{\Lambda}_c = \mathrm{diag}\left\{\hat{\Lambda}_{c1}, \ldots, \hat{\Lambda}_{cN}\right\} \tag{36}$$

$$\hat{\Phi}_{ci} = (A_i + A_{ii})X_i + X_i(A_i + A_{ii})^T + B_i E_i C_i + C_i^T E_i^T B_i^T + \sum_{j=1}^{N} \bar{\tau}_{ij} A_{ij} \hat{R}_{ij} A_{ij}^T \tag{37}$$

$$\hat{\Lambda}_{ci}^T = \left[\, \bar{\tau}_{1i}(X_i A_i^T + C_i^T E_i^T B_i^T) \;\; \bar{\tau}_{2i}(X_i A_i^T + C_i^T E_i^T B_i^T) \;\ldots\; \bar{\tau}_{Ni}(X_i A_i^T + C_i^T E_i^T B_i^T) \,\right] \tag{38}$$

and $\hat{R}_{ij}$, Ω, $\mathcal{R}$ and J are as in (5) – (7).

Moreover, a suitable stabilizing output gain is given by

$$G_i = E_i D_i^{-1}, \quad i = 1, \ldots, N. \tag{39}$$

Proof. In view of (34) and considering that C_i, $i = 1, \ldots, N$, are of full row-rank, it follows that the matrices D_i, $i = 1, \ldots, N$, are also of full row-rank and thus non-singular. Using this fact together with (34) and the feedback gain expression of (39), we obtain that

$$E_i C_i = G_i C_i X_i, \quad I = 1, \ldots, N.$$

Hence, it follows that the inequality (33) is equivalent to the inequality (2) of Theorem 1 for the closed-loop of system (1) with the output feedback gain G_i of (39), which concludes the proof. □

Remark 4. Theorem 3 provides a method of designing a stabilizing control law for interconnected linear state-delayed systems via decentralized memoryless output feedback. This method has the advantage to be convex and in terms of LMIs, and as such can be tested with efficient and reliable algorithms for solving LMIs.

In the case where state feedback is concerned, i.e. $C_i = I$, $i = 1, \dots, N$, the equality constraints of (34) become redundant and Theorem 3 reduces to the result of Theorem 2 for decentralized stabilization via state feedback.

Observe that the feasibility of (33) and (34) in Theorem 3 is dependent on the state-space realization of the system.

4.2. Decentralized state feedback $\mathcal{H}_\infty$ control

Consider the large scale system composed of the interconnection of the subsystems (S_i), $i = 1, \dots, N$, described by

$$\begin{aligned} (S_i): \quad & \dot{x}_i(t) = A_i x_i(t) + \sum_{j=1}^{N} A_{ij} x_j(t - \tau_{ij}) + B_{w_i} w_i(t) + B_i u_i(t) \\ & z_i(t) = C_{z_i} x_i(t) + D_{z_i} u_i(t) \\ & x_i(t) = \phi_i(t), \quad \forall\, t \in [-\tau_{\max}, 0]; \quad \tau_{\max} = \max\{\tau_{ij},\ i, j = 1, \dots, N\} \end{aligned} \tag{40}$$

where for the ith subsystem (S_i), $x_i(t) \in \Re^{n_i}$ is the state, $u_i(t) \in \Re^{m_i}$ is the control input, $w_i(t) \in \Re^{p_i}$ is the disturbance input, $z_i(t) \in \Re^{q_i}$ is the controlled output, $\tau_{ii} \geq 0$ is the time-delay in the subsystem, $\tau_{ij} \geq 0$, $j = 1, \dots, N$, $j \neq i$ are the time-delays in the interconnections, $\phi_i(\cdot)$ is the initial condition, A_i, A_{ij}, $j = 1, \dots, N$, B_i, B_{w_i}, C_{z_i} and D_{z_i} are known real constant matrices of appropriate dimensions.

The decentralized $\mathcal{H}_\infty$ control problem under consideration consists on: *Given scalars $\bar{\tau}_{ij} > 0$ and $\gamma_i > 0$, $i, j = 1, \dots, N$, determine a decentralized memoryless control law $u_i(t) = K_i x_i(t)$, $i = 1, \dots, N$, for the interconnected system of* (40) *such that the following conditions hold for any constant time-delays τ_{ij} satisfying $0 \leq \tau_{ij} \leq \bar{\tau}_{ij}$, $i, j = 1, \dots, N$:*

- *The closed-loop system is globally asymptotically stable;*
- *The ith subsystem has a level γ_i of disturbance attenuation, namely, under zero initial conditions, $\|z_i\|_2 < \gamma_i \|w_i\|_2$ for any non-zero $w_i(t) \in \mathcal{L}_2$, $i = 1, \dots, N$.*

In view of Theorem 2 and using the LMI approach of delay-dependent $\mathcal{H}_\infty$ control for "isolated" state-delayed systems as proposed in [4], we have the following result.

Theorem 4. Given scalars $\bar{\tau}_{ij} > 0$ and $\gamma_i > 0$, $i, j = 1, \dots, N$, the decentralized $\mathcal{H}_\infty$ control problem for the system (40) is solvable if there exist $n_i \times n_i$ symmetric positive definite matrices X_i, R_{ij}, Q_{ij} and S_{ijk}, $i, j, k = 1, \dots, N$, and $m_i \times n_i$

matrices Y_i, $i = 1, \ldots, N$, satisfying the following LMI:

$$\begin{bmatrix} \Phi_c & \Lambda_c^T & \Omega^T & \Upsilon \\ \Lambda_c & -\mathcal{R} & 0 & 0 \\ \Omega & 0 & -J & 0 \\ \Upsilon^T & 0 & 0 & -Z \end{bmatrix} < 0 \tag{41}$$

where

$$\Upsilon = \operatorname{diag}\{\Upsilon_1, \ldots, \Upsilon_N\}, \qquad Z = \operatorname{diag}\{Z_1, \ldots, Z_N\} \tag{42}$$

$$\Upsilon_i = [\, \bar{\tau}_{i1} A_{i1} \ \ldots \ \bar{\tau}_{iN} A_{iN} \ \ (C_{z_i} X_i + D_{z_i} Y_i)^T X_i \ \ B_{w_i} \,] \tag{43}$$

$$Z_i = \operatorname{diag}\left\{\bar{\tau}_{i1} Q_{i1}, \ \ldots, \ \bar{\tau}_{iN} Q_{iN}, \ I, \ \gamma_i^2 I - B_{w_i}^T \hat{Q}_i B_{w_i}\right\} \tag{44}$$

$$\hat{Q}_i = \sum_{j=1}^{N} \bar{\tau}_{ji} Q_{ji} \tag{45}$$

and Ω, $\mathcal{R}$, J are as in (6) – (11), whereas Φ_c and Λ_c are given in (27) – (30).
Moreover, a suitable decentralized control law is given by $u_i(t) = Y_i X_i^{-1} x_i(t)$.

4.3. Decentralized robust stabilization and $\mathcal{H}_\infty$ Control

In relation to the problems of decentralized robust stabilization and decentralized robust $\mathcal{H}_\infty$ control, using the $\mathcal{S}$-procedure (see, e. g. [1]) the results of Theorems 2 – 4 can be easily extended to the case of systems subject to either norm-bounded or IQC-type parameter uncertainty in the matrices of the system state-space model. On the other hand, using standard LMI techniques [1], Theorems 2 – 4 can also be readily extended to interconnected systems of the form of (1) and (40), where A_i, A_{ij} and B_i, $i, j = 1, \ldots, N$, are uncertain matrices belonging to a given polytope $\mathcal{P}$, which is described by the vertices:

$$[A_1^k \ \ldots \ A_N^k \ \ A_{11}^k \ \ldots \ A_{1N}^k \ \ldots \ A_{N1}^k \ \ldots \ A_{NN}^k \ \ B_1^k \ \ldots \ B_N^k], \quad k = 1, \ldots, n_v. \tag{46}$$

The corresponding decentralized robust stabilization and decentralized robust $\mathcal{H}_\infty$ control methods are similar to those of Theorems 2, 3 and 4, except that now we have n_v LMIs of the form of (26), (33) and (41), respectively, one for each of the vertices in (46).

5. EXAMPLES

Example 1. Consider the decentralized stabilization problem for an interconnected system of the form of (1) with:

$$A_1 = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} -1 & 0 \\ -0.8 & -1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0.1 & 0.3 \\ 0 & 0.3 \end{bmatrix}, \tag{47}$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Note that the above system is not stable as the subsystem 2 is unstable for $\tau_{22} = 0$. Further, delay-independent methods of decentralized stabilizable cannot be applied to this system as (A_2, B_2) is not stabilizable.

Assuming the time-delays τ_{ij}, $i, j = 1, 2$, to be identical, say $\tau_{ij} = \tau$, it was found by Theorem 2 that the above system is decentralized stabilizable for values of τ up to 0.295. For instance, when $\tau = 0.25$, the following stabilizing gains are obtained:

$$K_1 = \begin{bmatrix} -6.2215 & -16.8060 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 446.3563 & 797.3686 \end{bmatrix}.$$

Note that the gain K_2 of a "centralized" stabilizing control law for the above system with zero time-delays is of the order of 100.

In order to illustrate the fact that the scaling matrices R_{ij} and S_{ijk}, $i, j, k = 1, \ldots, N$, are crucial in terms of conservatism of the result of Theorem 1, this Theorem was applied to the system of (47) with the constraints $S_{ijk} = S_{ij}$, for $i, j, k = 1, \ldots, N$. In this case the maximum value of τ for decentralized stabilization is 0.265. On the other hand, when Theorem 1 is applied to the above system with the constraints $R_{ij} = R_i$ and $S_{ijk} = S_{ij}$, for $i, j, k = 1, \ldots, N$, the maximum value of τ for decentralized stabilization is 0.214.

Example 2. Consider the decentralized $\mathcal{H}_\infty$ control problem for the interconnected system of (40) with the same matrices as in (47) and

$$\begin{gathered} B_{w_1} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad B_{w_2} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \\ C_{z_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_{z_2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D_{z_1} = D_{z_2} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}. \end{gathered} \tag{48}$$

Applying Theorem 4 to the above system with $\bar{\tau}_{ij} = 0.2$, $i, j = 1, 2$, and minimizing $\mu = \gamma_1^2 + \gamma_2^2$, it was found that the minimum achievable μ is $\mu = 382.58$ and the corresponding optimal values of γ_1 and γ_2 are $\gamma_1 = 9.154$ and $\gamma_2 = 17.286$.

6. CONCLUSIONS

This paper focused on the problems of decentralized control of interconnected linear state-delayed systems. Systems with constant time-delays in the state of each subsystems as well as in the interconnections have been considered. First, delay-dependent LMI conditions for stability and decentralized stabilization via memoryless state feedback have been developed. Then, several extensions of the decentralized stabilization result to more complex control problems have been analysed, including decentralized stabilization via static output feedback, decentralized $\mathcal{H}_\infty$ control, decentralized robust stabilization, and decentralized robust $\mathcal{H}_\infty$ control.

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