

# SLIDING MODE CONTROL IN THE PRESENCE OF DELAY

JEAN-PIERRE RICHARD, FRÉDÉRIC GOUAISBAUT AND WILFRID PERRUQUETTI

This paper provides an overview of recent results for relay-delay systems. In a first section, simple examples illustrate the problems induced by delays in the synthesis of sliding mode controllers. Then, a brief overview of the existing results shows the present advances and limits in this domain. The last parts of the paper are devoted to new results: first, for systems with state delay, then for systems with input delay.

## 1. INTRODUCTION TO SLIDING MODE CONTROL

### 1.1. A short and basic recall on SMC for systems without delay

*Variable structure systems* (VSS) theory and practice have a deep historical background: the major part of the studies were concerned with Ordinary Differential Equations (ODE's) which means, with systems without time-delay. In variable structure controllers, the control law commutates between  $d$  different values in order to force the system flow to behave as “a nonsmooth contracting map”, which means the motions converge to the origin with some discontinuity in the time-derivatives of the state variables. One of the historic reasons that made VSS popular is that many physical systems naturally present discontinuity in their dynamics, as for mechanical systems with Coulomb friction, electrical systems with ideal relays ... This has led control theorists to begin (mostly in eastern countries) with the study of these relay-based control systems.

*Sliding mode control* (SMC, [37]) is a particular case of variable structure system control ( $d = 2$ ). Roughly speaking, it is based on the design of an adequate “sliding surface”  $\{x, s(x) = 0\}$  (or “sliding manifold”) which includes the origin (state to be reached,  $s(0) = 0$ ) and divides the state space (vector  $x$ ) in two parts: each of them corresponds to one of the two controls, which commute from one to the other when the state crosses the surface. For systems with single input<sup>1</sup>, this simply corresponds

---

<sup>1</sup>For multi-input systems,  $u \in \mathbb{R}^m$ ,  $m$  manifolds  $s_i(x) = 0$  are to be defined, and all variables have indices  $i$  in the formula. Two strategies are then possible: use a discontinuous control  $u_1$  so to reach remain on surface  $s_1$ , then use  $u_2$  so to reach and remain on  $s_1 \cap s_2$ , and so on... until  $s_m$  and then, 0. In the second strategy, control discontinuities appears only on  $\bigcap_{1,m} s_i$ . First solution is simple since it is solved as  $m$  single-input problems, but in practice it can lead to undesirable constraints on the system.

to

$$u = \begin{cases} u^+ & \text{if } s(x) > 0, \\ u^- & \text{if } s(x) < 0. \end{cases}$$

Sliding mode control techniques are therefore based on a two-stage behavior:

1. Hitting phase (or reaching phase): the state is driven in finite time onto the surface. Roughly speaking, condition

$$s(x(t))\dot{s}(x(t)) < 0$$

is expected in this phase, which condition depends on the control design  $\{u^+, u^-\}$ . During this phase, boundeness of  $x(t)$  must be guaranteed.

2. Sliding phase: the state remains on the surface and, if surface is adequately designed, the state tends to the origin. In sliding phase, control reduces to an *equivalent control*  $u_{eq}$  [37]: since the state remains on  $s(x) = 0$ ,  $u_{eq}$  is defined by mixing equation  $\dot{s}(x) = 0$  with the state equation (see Example 1). It was shown that for single input systems,

$$u^- < u_{eq} < u^+$$

is a NSC for existence of sliding phase.

This second phase may be imperfect: if the actuator does not exhibit ideal commutations (for instance, if there is some inertia or, worst, some delay), or if the control is not adequately designed (for instance, if does not respect the “natural tendencies” of the process), then a vibrating phenomenon appears: it is known as “chattering” (in French, *réticence*) and may have some bad consequences (among them, premature wearing of the actuators). In the major part of the situations (except input delays, as we shall see) this chattering phenomenon can be reduced or avoided.

Both phases are concerned with stability and attractivity concepts since:

- In the first step, the condition ensuring the sliding motions is a contraction property (at least locally around the sliding manifold).
- In the second phase, the choice of the surface (“*shaping procedure*”) is mainly related to stabilization: one has to compute (or “tune”) the parameters involved in the shape of the sliding surface such that the sliding motions achieve some convergence and/or stabilization problem.

**Example 1.** For instance, let us consider the very basic, second-order system

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_1(t)x_2(t) + d(t) + u(t). \end{cases} \quad (1.1)$$

Suppose the system is expected to converge toward  $x = 0$  with exponential rate 2 ( $e^{-2t}$ ). Define the sliding surface as  $s(x) = ax_1 + x_2$ . Now, examine its design: in

the sliding phase, the state remains on  $s(x) = 0$ , then the first state variable satisfies  $\dot{x}_1(t) = x_2(t) = -ax_1(t)$ . One deduces that a necessary condition for asymptotic stability of the origin is  $a > 0$ , which provides admissible surfaces. Tuning  $a = 2$  achieves the exponential convergence as soon as system remains on the surface  $s(x) = x_2 + 2x_1$ .

Concerning the hitting phase, let us define the control  $u^+ = -x_1x_2 - 2x_2 - k$ ,  $u^- = -x_1x_2 - 2x_2 + k$ , this means,

$$u(t) = -x_1(t)x_2(t) - 2x_2(t) - k \operatorname{sign} s(t). \quad (1.2)$$

It is easy to check that:

$$s(t)\dot{s}(t) = -k|s(t)| + s(t)d(t).$$

In other words, if gain  $k$  is high enough (here,  $k > \sup_t |d(t)|$ ), then

$$s(t)\dot{s}(t) < 0$$

and the motions converge toward the surface  $s = 0$  in finite time. The equivalent control

$$u_{eq} = -2x_2(t) - x_1(t)x_2(t) - d(t)$$

is obtained from equation  $\dot{s}(x) = 0$ , thus  $\dot{s}(x) = +2x_2 + x_1(t)x_2(t) + d(t) + u(t) = 0$ . Note that condition  $u^- < u_{eq} < u^+$  holds.

Moreover, the state converges to the surface in finite time (see Figure 1, with gain  $k = 10$  and  $d \equiv 0$ ).

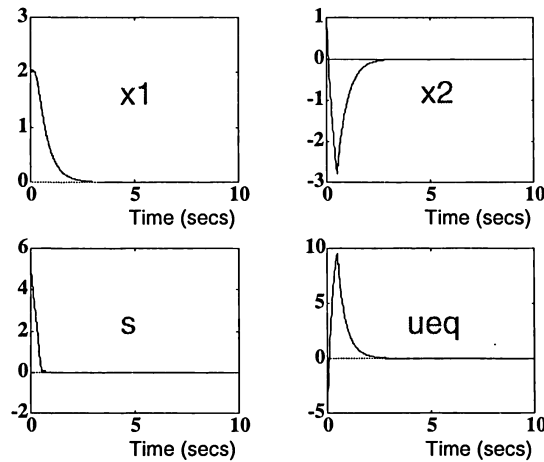


Fig. 1. SMC without delay.

### 1.2. Some advantages of SMC

Designing such sliding mode controllers is a necessity for systems with naturally discontinuous actuators. But, even for systems with continuous actuators, introducing a nonsmooth control algorithm may benefit the behaviors: it enlarges the possibilities of other continuous controllers, and examples were provided of systems stabilizable by means of discontinuous control, which were not verifying the Brockett's necessary conditions of  $\mathcal{C}$ -stabilization<sup>2</sup>.

Another advantage of SMC is its robustness with regard to input and parameter disturbances: SMC is known to provide an efficient way to tackle challenging robust stabilization problems for finite-dimensional dynamic systems. For instance, as soon as a complex system can be stated with a *normal form* (see [11]) as equation

$$\begin{cases} \dot{x}_i = x_{i+1}, & \forall i = 1, \dots, (n-1), \\ \dot{x}_n = f(t, x) + g(t, x)u(t) + d(t), \\ y = x_1, \end{cases} \quad (1.3)$$

it is known that an appropriate sliding mode strategy can achieve stabilization for a wide class of disturbances: the nonlinear terms and the disturbances  $d(t)$  (generally modelling the unknown dynamics) can be “dominated”<sup>3</sup> [7, 35]. Such “domination” was illustrated in Example 1, corresponding to the choice of a sufficiently high gain  $k$ . The controllability-like conditions allowing such rejection of  $d(t)$  for more general systems are known as “matching conditions” [24]; they are satisfied in the particular case of (1.3).

Lastly, let us mention that commutation strategies also provide a way of obtaining finite-time convergence properties, since equations reaching  $s = 0$  within a time  $T(s(0)) < \infty$ , as

$$\dot{s}(t) = -k \operatorname{sign} s(t), \quad T = k^{-1}s(0),$$

can also be worked out.

### 1.3. Two introducing examples to the problem “relay-delay”

However, the modelling of many physical systems has to take into account an irreducible influence of the past: time-delays are natural phenomena in numerous engineering devices [20, 32] and the modelling phasis cannot neglect them anymore when increasing the dynamic performances is aimed at. Consequently, specific models, analysis and controllers (see survey by the authors in [31]) have to take into account the *infinite dimensional* nature of such systems. Even for linear models, the

<sup>2</sup>For instance, the completely controllable system

$$\begin{cases} \dot{x}_1 = u_1, \\ \dot{x}_2 = u_2, \\ \dot{x}_3 = x_2 u_1 - x_1 u_2; \end{cases}$$

has no stabilizing  $C^1$  state feedback.

<sup>3</sup>Note that, in continuous time, a discontinuity in the control law can be interpreted as a high-gain effect.

design of controllers is not obvious, mainly because applying the existing necessary and sufficient stability conditions is very tricky.

The combination of delay phenomenon with relay actuators makes the situation much more complex. For instance, we recall here first an investigation of the notion of steady modes resulting from the relay-delay combination and concluded to the possible existence of a *countable set* of oscillation periods. Then, we reconsider Example 1 with an additional delay.

**Example 2.** [12, 13] Let the prototypical equation

$$\dot{x}(t) = -\text{sign } x(t-1).$$

For adequate initial conditions, it has the 4-periodic solution

$$g_0(t) = g_0(t+4k) = \begin{cases} t & \text{for } -1 \leq t \leq 1, \\ 2-t & \text{for } 1 \leq t \leq 3, \end{cases}$$

but also exhibits any of the  $\frac{4}{4n+1}$ -periodic solutions

$$g_n(t) = \frac{4}{4n+1} g_0((4n+1)t).$$

**Example 3.** [6] Consider again Example 1, but now with an additional input delay  $\tau = 0.1$ . The model becomes

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_1(t) x_2(t) + u(t-\tau), \end{cases} \quad (1.4)$$

while the control law  $u(t)$  is still defined by (1.2). Simulation Figure 2 shows the resulting oscillations. Note that for  $k = 1000$ ,  $\tau = 0.08$ , the oscillation exhibits a triple limit cycle (Figures 3,5) instead of a single one (Figure 4). This simple example points out behavioral changes (bifurcations) arising in relay-delay systems, and motivates the study of specific SMC design for systems with state and/or input aftereffect<sup>4</sup>.

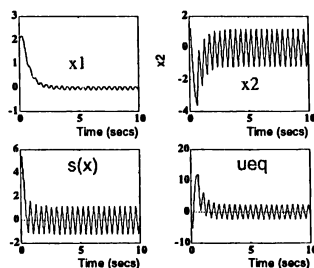


Fig. 2. SMC with input delay  $\tau = 0.1$ .

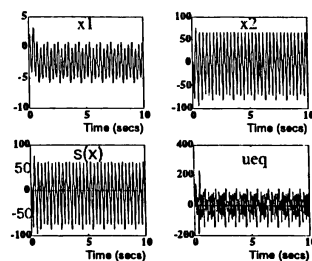
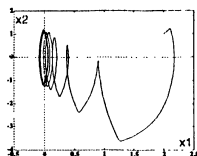
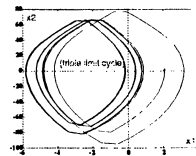


Fig. 3.  $k = 1000$ ,  $\tau = 0.08$ .

<sup>4</sup>The question is also related to the "real sliding behaviour", taking in account both the sampling period and the inertia of the switching devices, by opposition to the "ideal sliding behaviour" that was presented in the introduction. On this question, see for instance [30].

Fig. 4. Phase portrait,  $k = 10$ ,  $\tau = 0.1$ .Fig. 5.  $k = 1000$ ,  $\tau = 0.08$ .

## 2. AN OVERVIEW OF SMC FOR DELAY SYSTEMS

Despite some extension of SMC to infinite dimensional systems [28] [29] and of differential inclusions to aftereffect systems [20], the concrete control results are not so numerous. We shall divide them into two classes: systems with or without input delay.

Most of the papers we found in the literature are considering *systems without input delay*<sup>5</sup>:

– [1, 3, 4, 9, 16, 25, 26, 34, 36, 38] are directly concerned with SMC. The models involve delayed state variables (input and sensors are not submitted to delay). [3] was not directly related to SMC, but turned out to join this class for high gain values. Other ways of designing variable structure controllers still yield computational difficulties: [38] relies on the Fiagbedzi–Pearson approach, with the connected difficulties<sup>6</sup>. In [15], SMC design with unknown time-varying delay is considered, and [14] generalizes the approach to a class of nonlinear systems.

– Lastly, [36] obtained relay-delay identification with application to the control of chemical processes. In this interesting result, relay is involved in the only identification procedure, then replaced by a finite spectrum assignment control [27].

*In what concerns systems with input or sensor delay*, the question is still more challenging. For instance, we speak here about systems as

$$\dot{x}(t) = A_0 x(t) + A_d x(t-h) + Bu(t) + B_1 u(t-h), \quad (2.1)$$

for which the pairs  $(A_0, B)$  or  $(A_0 + A_d, B)$  are not controllable (for instance,  $B = 0$ ), which means one must use the  $B_1 u(t-h)$  term so to obtain an efficient control. To our best knowledge, few results leading to concrete SMC design of have been published in this case:

– In [2], the considered systems have output delay and relay actuators, but the study is limited to the first order processes.

<sup>5</sup>Such “inner delay” phenomenon appears in several cases as in chemical transformations (reaction lags), epidemiology (germ incubation time), population dynamics (average life duration) ...

<sup>6</sup>This theory aims at transforming retarded systems into ordinary ones, as in [10]. The problem is that one has to know the unstable eigenvalues and eigenvectors of a characteristic equation such as  $A = \int_{-h}^0 e^{A\theta} dK(\theta)$  and, then, to implement distributed controls.

– In [5], the aim was to reduce the chattering induced by delayed sensors: a combination with observer-based control was achieved on a concrete process, but without providing the theoretical proof of convergence.

– In [33, 38], the considered systems have an input delay, but no state delay. They use an observer-like control, which will be recalled in the last section of this paper. Note that [33] may need some complementary proofs.<sup>7</sup>

– In a case study [6], the authors considered the above Example 3. We used a Lyapunov–Razumikhin approach leading to overestimation of the chattering amplitude (i.e., the determination of an attracting neighborhood around the sliding manifold). This preliminary result, concerned with the sensitivity of SMC with respect to time-delay effect, was completed by the estimation of its asymptotic stability domain.

– In [17], we recently proposed a control design ensuring a robust convergence of SMC under state-and-input delay. This was achieved by combining the Lyapunov–Krasovskii method with a normal form as (1.3) for delay systems. Such results allow taking into consideration the presence of a delay affecting sensors (observation of  $x(t - \tau)$  instead of  $x(t)$ ) or actuators (control  $u(t - \tau)$  instead of  $u(t)$ ). Note the chattering phenomenon was avoided by using nonlinear gains. But, because of the delay, the additive disturbance  $[d(t) \text{ in } (1.3)]$  could not completely rejected, which implies (in the best case) ultimate boundedness instead of asymptotic convergence.

To give a first conclusion on the possible SMC strategies for LTDS, let us summarize the situation as follows:

1) for systems with *state delay*, situation is the same as for ODEs, even if design and computations are more complicated;

2) the presence of *input delay* under perturbations still leads to open problems.

Concerning the stability study (see generally [8, 19]), the methods than can be used in SMC are mainly based on the time-domain Krasovskii's approach (for linear systems the results are then expressed in terms of Ricatti equations [22] or, equivalently, of LMIs [21]), Razumikhin's approach, and comparison approach (results in terms of matrix norms and measures [18]). They allow handling nonlinear systems, whereas the frequency-domain and complex-plane methods (generally leading to diophantine polynomial equations) need the delays to be constant. But, in sliding control, their use has to be chosen in relation to the phase under consideration:

1. In hitting phase, it is not necessary to study the convergence of a Lyapunov functional  $v(x_t)$  concerning the whole state  $x_t$ , since only the distance from state to the surface (say,  $v(x_t) = s^2(x_t)$ ) has to be involved<sup>8</sup>. However, the method has to be able to guarantee finite-time reaching of the surface. It is important to note that, up to now, *only Krasovskii's approach* can ensure

---

<sup>7</sup>In [33], the input is delayed and a state predictor [27] is defined. But, some ambiguity arises in the proof of this result since the *finite time* convergence to the sliding manifold is not ensured: the proof relies on a Razumikhin's approach.

<sup>8</sup>For nonlinear systems, one must additionally check that the state remains bounded within finite time.

the finite time convergence. Razumikhin's method<sup>9</sup> has never been shown to be admissible and its combination with Filipov's theory has not been deeply studied. The question is illustrated in Figure 6: if finite time convergence is proven under Razumikhin's relaxing conditions (i.e., for trajectories coming out from the set  $v(s) = cte$ ), finite time is ensured for other trajectories (regularly converging, for instance).

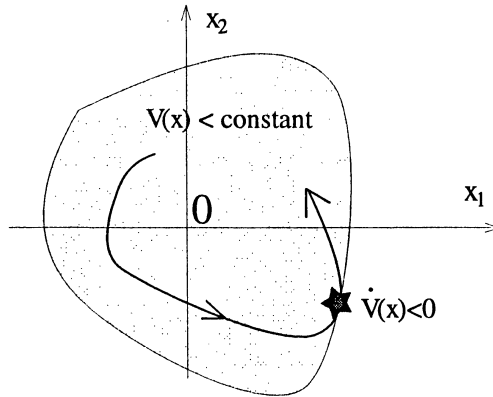


Fig. 6. Can Razumikhin's principle be applied to hitting phase?

2. For the sliding phase: once on the surface, all stability methods can be used. Comparison techniques as well as Razumikhin's approach may be more suitable for invariant domains estimation, constrained control properties, varying delay. Krasovskii's functional are interesting for linear systems with constant delays, since optimization LMI algorithms are well fitted for this case.
3. If sliding regime cannot be reached (because of input delays and perturbations, for instance), then Razumikhin-like techniques can provide interesting information about amplitude of the chattering (see below).

<sup>9</sup>Razumikhin's theorem statement: Let  $u(\rho)$ ,  $v(\rho)$ ,  $w(\rho)$ ,  $p(\rho)$  be scalar, continuous, positive, nondecreasing functions, with  $u(0) = v(0) = 0$  and  $p(\rho) > \rho$  for  $\rho > 0$ . If there is a continuous function  $V(t, x)$  such that

$$u(\|x\|) \leq V(t, x) \leq v(\|x\|),$$

$$\dot{V}(t, x(t)) \leq -w(\|x(t)\|)$$

for states  $x_t$  verifying  $\forall \theta \in [-h, 0], V(t + \theta, x(t + \theta)) < p[V(t, x(t))]$ , then 0 is uniformly asymptotically stable for  $\dot{x}(t) = f(x_t, t)$ .

What we call "relaxing condition" is the sentence "for states  $x_t$  verifying...".



### 3. NEW RESULTS FOR DELAYED STATE, MEMORYLESS INPUT

#### 3.1. Preliminary results

We consider linear time delay systems of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h) + Bu(t) + f_1(t, x_t), & t > 0, \\ x(t) = \phi(t), & t \in [-\tau, 0], \end{cases} \quad (3.1)$$

where  $x(t) \in \mathbb{R}^n$ ;  $x_t$  is the function defined for  $\theta \in [-h, 0]$  by  $x_t(\theta) = x(t + \theta)$ ,  $A$  and  $A_d$  are constant  $n \times n$  matrices;  $B$  is an  $n \times m$  matrix of rank  $m$ ; the control vector  $u(t)$  belongs to  $\mathbb{R}^m$ ; and  $f_1$ , which represents neglected dynamics and/or external disturbances, is a signal satisfying the classical *matching conditions*, i. e.

$$f_1(x(t), t) = Bf(x(t), t). \quad (3.2)$$

We define the following assumptions:

A1) The pair of matrices  $(A + A_d, B)$  is controllable.

A2) The perturbation vector  $f_1$  satisfies an inequality of the form

$$\|f\| < \Psi(x_t). \quad (3.3)$$

where  $\Psi$  is an a priori known functional of  $x_t$  (possibly non vanishing).

Throughout the paper the following notations are used.

- $\|\cdot\|$  denotes both the octahedral norm of an  $n$ -vector  $e$  defined by:  $\|e\| = \sum_{i=1}^n |e_i|$  and its induced matrix norm, i. e.  $\|A\| = \sup_{\|e\|=1} \frac{\|Ae\|}{\|e\|}$  for  $A \in \mathbb{R}^{n \times n}$ .
- Sign is the map defined by  $\text{Sign}(z) = [\text{sign}(z_1), \dots, \text{sign}(z_m)]^T$  for all  $z = [z_1, \dots, z_m] \in \mathbb{R}^m$ .
- $A > 0$  means that the matrix  $A$  is positive definite.

#### 3.2. Regular form

The following results aim at transforming the original system into an appropriate form for the design of a sliding mode controller. This form constitutes an extension of the regular form introduced in [24] to the class of time-delay systems.

**Definition 4.** System (3.1) is said to be in a regular form if  $B = \begin{bmatrix} 0 \\ D \end{bmatrix}$ , where  $D$  is a nonsingular,  $m \times m$  matrix.

Existence of a regular form for linear time-delay systems may be obtained using strictly the same way than for delay-free system (see for instance [37]):

**Lemma 5.** There exists a regular coordinate transformation  $T \in \mathbb{R}^{n \times n}$  such that system (3.1), written with the new variables  $z(t) = (z_1, z_2)^T = Tx(t)$ ,  $z_1 \in \mathbb{R}^{(n-m)}$ ,  $z_2 \in \mathbb{R}^m$ , takes the following *regular form*:

$$\begin{cases} \dot{z}_1(t) = A_{11}z_1(t) + A_{12}z_2(t) + A_{d11}z_1(t-h) + A_{d12}z_2(t-h), \\ \dot{z}_2(t) = A_{21}z_1(t) + A_{22}z_2(t) + A_{d21}z_1(t-h) + A_{d22}z_2(t-h) + Du(t) + Df. \end{cases} \quad (3.4)$$

**Lemma 6.** Under Assumption A1), the pair of matrices  $(A_{11} + A_{12}, A_{d11} + A_{d12})$  is controllable.

Until the end of the paper, we assume that the initial system (3.1) has already been set in regular form and that matrices  $A, A_d, B$  are partitioned into:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_d = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ D \end{bmatrix}$$

where  $A_{11}, A_{d11}$  are  $(n-m) \times (n-m)$  matrices,  $D$  is a regular  $m \times m$  matrix.

### 3.3. Sliding mode synthesis

#### 3.3.1. Case of a finite-dimensional sliding surface

We consider here the choice of a sliding surface  $s(x) = 0$  of the form

$$s(x) = x_2(t) + Kx_1(t) \quad (3.5)$$

where  $K \in \mathbb{R}^{m \times (n-m)}$ . The aim of this section is to design a sliding mode controller steering vector  $x(t)$  toward the hyperplane  $s(x) = 0$ .

An equivalent representation can be obtained using variable  $s$ :

$$\begin{cases} \dot{x}_1(t) = (A_{11} - A_{12}K)x_1(t) + (A_{d11} - A_{d12}K)x_1(t-h) \\ \quad + A_{12}s(x) + A_{d12}s(x(t-h)), \\ \dot{s}(x(t)) = \Phi(x_t) + D_2u(t) + D_2f, \end{cases} \quad (3.6)$$

where functional  $\Phi$  is defined by:

$$\begin{aligned} \Phi(x_t) &= (A_{21} + KA_1)x_1(t) + (A_{22} + KA_{12})x_2(t) \\ &\quad + (A_{d21} + KA_{d11})x_1(t-h) + (A_{d22} + KA_{d12})x_2(t-h). \end{aligned}$$

Let  $X \in \mathbb{R}^{m \times m}$  be a Hurwitz matrix, and denote  $P_2$  the (positive definite) solution of the Lyapunov equation

$$X^T P_2 + P_2 X = -I_m. \quad (3.7)$$

We consider the control law

$$u(t) = -D^{-1}(\Phi + M(x_t) \frac{P_2 s(x(t))}{\|P_2 s(x(t))\|} - Xs(x(t))), \quad (3.8)$$

where  $M(x_t) = m_1 + \|D\| \Psi(x_t)$  for  $m_1 > 0$ .

**Theorem 7.** Under above assumptions A1) and A2), the control law (3.8) makes the surface  $s(x) = 0$  attractive and reached in finite time. The equilibrium  $x = 0$  is then globally asymptotically stable for all delays  $h \in [0, h_{\max})$  where  $h_{\max}$  is the solution of the following optimization problem:

$$h_{\max}^{-1} = \min_{S, W} h^{-1}$$

for matrices  $S \in \mathbb{R}^{(n-m) \times (n-m)}$  symmetric positive definite matrix and  $W \in \mathbb{R}^{m \times (n-m)}$  such that

$$\begin{pmatrix} h^{-1}\Gamma & \varepsilon\Delta^T & (SA_{d11} - W^T A_{d12}^T) \\ \varepsilon\Delta & -\varepsilon S & 0 \\ (A_{d11}S - A_{d12}W) & 0 & -\varepsilon S \end{pmatrix} < 0 \quad (3.9)$$

where  $\varepsilon > 0$ ,  $\Delta = (A_{11} + A_{d11})S - (A_{12} + A_{d12})W$ ,  $\Gamma = \Delta + \Delta^T$ .

The sliding surface (3.5) is then defined by  $K = WS^{-1}$ .

*Proof.* In a first step, we prove the convergence of the solution  $x(t)$  of (3.4) onto the surface  $s(x) = 0$  in finite time. Let us choose the following function:

$$V(t) = s(x(t))^T P_2 s(x(t)). \quad (3.10)$$

Its derivative along the solution of (3.4) is:

$$\dot{V}(t) = 2s(x(t))^T P_2 (\Phi(x_t) + Du(t) + Df). \quad (3.11)$$

If  $u(t)$  is given by (3.8), then

$$\dot{V} < -s(x)^T s(x) - 2m_1 \sqrt{\lambda_{\min}(P_2)} \sqrt{V} < -2m_1 \sqrt{\lambda_{\min}(P_2)} \sqrt{V} \quad (3.12)$$

which proves that  $s(x) = 0$  is a sliding surface, reached in finite time.

On this surface, we get  $s(x) = \dot{s}(x) = 0$  and the reduced system is governed by the following differential equation

$$\dot{x}_1(t) = (A_{11} - A_{12}K)x_1(t) + (A_{d11} - A_{d12}K)x_1(t-h). \quad (3.13)$$

The second step is then to prove the stability of the subsystem (3.13). As the pair  $(A_{11} + A_{d11}, A_{d12} + A_{12})$  is controllable, there exists some  $K$  such that the reduced system (3.13) with  $h = 0$  is asymptotically stable. We now intend to maximize the delay  $h_{\max}$  for which system (3.13) is asymptotically stable (with  $h < h_{\max}$ ). With this aim in mind, we consider the Lyapunov-Krasovskii functional:

$$\begin{aligned} V(x_t) &= z^T(t)Pz(t) + \varepsilon^{-1} \int_{t-h}^t \int_s^t x_1^T(\theta)(A_{d11} - A_{d12}K)^T \\ &\quad PQP(A_{d11} - A_{d12}K)x_1(\theta) d\theta ds \end{aligned}$$

where  $E = A_{11} + A_{d11} - (A_{12} + A_{d12})K$ ,  $z(t) = x(t) + \int_{t-h}^t (A_{d11} - A_{d12}K)x_1(\theta) d\theta$ ,  $P, Q \in \mathbb{R}^{(n-r) \times (n-r)}$  are positive definite matrices, and  $\varepsilon > 0$ .  $\dot{V} < 0$  is ensured by LMI (3.9).  $\square$

### 3.3.2. Case of a functional surface

Another solution is to design a controller steering the solutions of the system (3.1) on the surface  $\Omega(x_t) = 0$ , where the functional  $\Omega$  is given by:

$$\Omega(x_t) = x_2(t) + K_1 x_1(t) + K_2 x_1(t - h), \quad (3.14)$$

$K_1$  and  $K_2$  being matrices of appropriate dimensions.

Using  $(x_1, \Omega(x_t))$  as new state coordinates, an equivalent representation of system (3.4) is obtained, given as follows

$$\begin{cases} \dot{x}_1(t) = (A_{11} - A_{12}K_1)x_1(t) + (A_{d11} - A_{d12}K_1 - A_{12}K_2)x_1(t - h) \\ \quad - A_{d12}K_2x_1(t - 2h) + A_{12}\Omega(x) + A_{d12}\Omega(x(t - h)), \\ \dot{\Omega}(x(t)) = \Sigma(x_t) + Du(t) + Df(t), \end{cases} \quad (3.15)$$

where functional  $\Sigma$  is given by:

$$\begin{aligned} \Sigma(x_t) = & (A_{21} + KA_1)x_1(t) + (A_{22} + KA_{12})x_2(t) + (A_{d21} + KA_{d11})x_1(t - h) \\ & + (A_{d22} + KA_{d12})x_2(t - h) + K_2\dot{x}_1(t - h). \end{aligned}$$

As in the previous section, we consider a control law of the form

$$u(t) = -D^{-1}[\Sigma(x_t) + M(x_t)\frac{P_2\Omega(x_t)}{\|P_2\Omega(x_t)\|} - X\Omega(x_t)], \quad (3.16)$$

where  $M(x_t) = m_1 + \|D\|\Psi(x_t)$  for  $m_1 > 0$ , and, again,  $X \in \mathbb{R}^{m \times m}$  and  $P_2$  denote, respectively, a Hurwitz matrix and the solution of the Lyapunov equation (3.7).

The design of the gain matrices  $K_1, K_2$  can be done using the following theorem.

**Theorem 8.** Under assumptions A1) and A2), the control law (3.16) makes the surface  $\Omega(x_t) = 0$  attractive and reached in finite time. The equilibrium  $x = 0$  is then globally asymptotically stable for all delays  $h \in [0, h_{\max}]$  where  $h_{\max}$  is the solution of the optimization problem

$$h_{\max}^{-1} = \min_{S, W_1, W_2} h^{-1}$$

for matrices  $S \in \mathbb{R}^{(n-m) \times (n-m)}$  a symmetric, positive-definite matrix and  $W_1, W_2 \in \mathbb{R}^{m \times (n-m)}$  such that

$$\begin{pmatrix} h^{-1}\Gamma & \gamma\Delta^T & \gamma W_2^T A_{d12}^T & \Theta \\ \gamma\Delta & -\gamma S & 0 & 0 \\ \gamma A_{d12}W_2 & 0 & \frac{-\gamma S}{2} & 0 \\ \Theta^T & 0 & 0 & \frac{-\gamma S}{3} \end{pmatrix} < 0, \quad (3.17)$$

where  $\Delta = A_{d11}S - A_{d12}W_1 - A_{12}W_2$ ,  $\Theta = (A_{11} + A_{d11})S - (A_{12} + A_{d12})(W_1 + W_2)$ ,  $\Gamma = \Theta + \Theta^T$ .

The sliding surface is then given by:  $K_1 = W_1 S^{-1}$ ,  $K_2 = W_2 S^{-1}$ .

**Proof.** Let us first prove that the solutions of (3.1) reach and remain on the surface  $\Omega(x_t) = 0$  within a finite time. For this, let us consider the functional  $V$  defined by

$$V(x_t) = \Omega(x_t)^T P_2 \Omega(x_t). \quad (3.18)$$

The expression of the derivative of  $V$  along the solutions of (3.15) is:

$$\dot{V}(t) = 2\Omega(x_t)^T P_2 (\Sigma(x_t) + Du(t) + Df(t)). \quad (3.19)$$

If the input  $u(t)$  is given by (3.16), then one obtains the inequality

$$\dot{V}(t) < -\Omega(x_t)^T \Omega(x_t) - 2m_1 \sqrt{\lambda_{\min}(P_2)} \sqrt{V(x_t)} < -2m_1 \sqrt{\lambda_{\min}(P_2)} \sqrt{V(x_t)}. \quad (3.20)$$

This proves that there is an ideal sliding motion on the surface  $\Omega(x_t) = 0$  after a finite time.

It remains to be demonstrated that the solution  $x(t) = 0$  of (3.1), (3.16) is globally asymptotically stable. The ideal sliding dynamics is obtained by writing that  $\Omega(x) = \dot{\Omega}(x) = 0$  which means:

$$\begin{aligned} \dot{x}_1(t) &= (A_{11} - A_{12}K_1)x_1(t) + (A_{d11} - A_{d12}K_1 - A_{12}K_2)x_1(t-h) \\ &\quad - A_{d12}K_2x_1(t-2h). \end{aligned} \quad (3.21)$$

Introducing the intermediate variable

$$z(t) = x_1(t) + E_d \int_{t-h}^t x_1(s) ds + E_p \int_{t-2h}^t x_1(s) ds, \quad (3.22)$$

where  $E_d = A_{d11} - A_{d12}K_1 - A_{12}K_2$  and  $E_p = -A_{d12}K_2$ , we get

$$\dot{z}(t) = (A_{11} + A_{d11} - (A_{12} + A_{d12})(K_1 + K_2))x_1(t) \triangleq Ex_1(t) \quad (3.23)$$

We choose the following Lyapunov–Krasovskii functional

$$\begin{aligned} V(x_t) &= z^T(t)Pz(t) + \gamma \int_{t-h}^t \int_v^t x_1(s)^T E_d^T P R_1 P E_d x_1(s) ds dv \\ &\quad + \gamma \int_{t-2h}^t \int_v^t x_1(s)^T E_p^T P R_2 P E_p x_1(s) ds dv \end{aligned}$$

where  $\gamma > 0$ , and  $P, R_1, R_2$  are positive definite matrices of appropriate dimensions. Writing  $\dot{V} < 0$  leads to LMI (3.17).  $\square$

### 3.4. Example

Consider the system

$$\dot{x}(t) = Ax(t) + A_d x(t-h) + Bu(t) + Bf, \quad (3.24)$$

with

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 1.75 & 0.25 & 1 \\ 0 & 2 & 3 \end{pmatrix}, \quad A_d = \begin{pmatrix} -0.3 & 0 & 0 \\ -0.1 & -0.25 & 0.1 \\ 1 & -2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (3.25)$$

To simulation purpose, we considered the disturbance function  $f = 0.8 \sin(2t)$  and the initial function  $x(t) = [0.5, 1, -2]^T$  for  $t \in [-h, 0]$ .

Since the system is in regular form, we can apply the method directly. Implementing the control (3.8) and using Theorem 4 makes certain that the system (3.25) is asymptotically stable for  $h \leq h_{\max} = 1.31$ . By convex optimization, we find the coefficient of the surface

$$K = \begin{pmatrix} 2.86 & -0.4486 \end{pmatrix}.$$

For control (3.16), we obtain by LMI optimization the following gains

$$K_1 = \begin{pmatrix} 2.21 & -0.116 \end{pmatrix}, \\ K_2 = \begin{pmatrix} -0.30 & 0.084 \end{pmatrix}.$$

Asymptotic stability of the zero solution is now guaranteed for  $h \leq h_{\max} = 3$  which demonstrates the additional possibilities offered by the functional surfaces as (3.14) compared to (3.5).

The simulation results are shown in Figures 7–8. They were obtained using a first order integration scheme with a step size of 0.01 and taking  $X = -2$ .

In the first and second simulation, we use control (3.8) with  $m_1 = 2$ , and  $X = -2$ , with a delay  $h = 1.31$ .

In the third simulation, we use the control (3.16), with a gain  $m_1 = 2$ , and  $X = -2$ , with a delay of  $h = 1.31$ .

## 4. RESULTS FOR DELAYED INPUT

We now consider the following model

$$\dot{x}(t) = Ax(t) + Bu(t-h) + f, \quad (4.1)$$

where  $f$  is a perturbation which has to be rejected. Define a linear transformation  $\Gamma$  by:

$$z(t) = x(t) + \int_{t-h}^t e^{A(t-h-s)} Bu(s) ds. \quad (4.2)$$

The motion of  $z(t)$  is governed by the ordinary differential equation:

$$\dot{z}(t) = Az(t) + e^{-Ah} Bu(t) + f. \quad (4.3)$$

**Theorem 9.** Asymptotic stability of  $z(t)$  implies asymptotic stability of  $x(t)$ .

Using transformation  $\Gamma$  (which is in fact a predictor, since  $x(t+h) = z(t)$ ), and Theorem 9, several authors [23, 33, 38] have developed sliding mode control, which show a robustness equivalent to a classical variable structure control. Note that this kind of control is infinite dimensional, since the the manifold used is of the form:

$$\Omega = Sz = S \left( x(t) + \int_{t-h}^t e^{A(t-h-s)} Bu(s) ds \right). \quad (4.4)$$

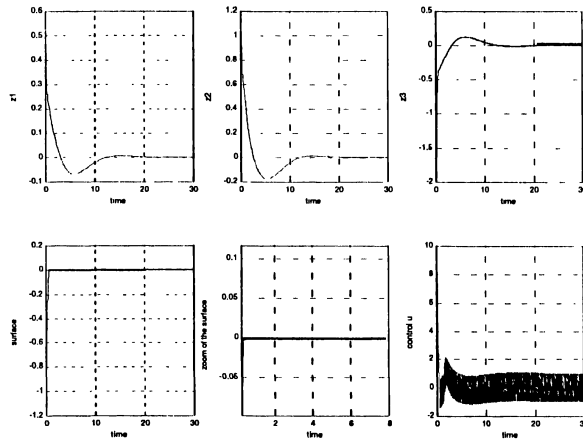


Fig. 7. Simulation of (4.2), (3.8) with  $m_1 = 2$ ,  $X = -2$ ,  $h = 1.31$ .

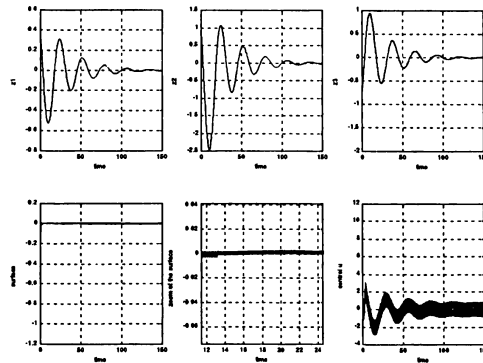


Fig. 8. Simulation of (4.2), (3.16) with  $m_1 = 2$ ,  $X = -2$ ,  $h = 1.31$ .

## 5. CONCLUSIONS

The presence of delay within a sliding mode control can induce oscillations around the design surface: the opening case study has pointed out possible behavioral changes (bifurcations) arising in such relay/delay systems. This motivates the study of specific SMC design for systems with state and/or input aftereffect.

Here, the main contribution lays in the analysis of delay/relay motions (amplitude of the possible oscillations around the sliding manifold and admissible initial conditions<sup>10</sup>) and the design of SMC for systems with state delay. Calculable control laws are provided, together with upper bounds of the delay values preserving the asymptotic stability.

The control implementation is rather simple, even if the proofs may appear as complex: the control law assures the existence of a Lyapunov–Krasovskii functional.

Some extensions are possible:

1. relax constraint of a *linear model*,
2. introduce *parameter uncertainties*,
3. consider *multiple delays* with some computation.

(Received November 22, 2000.)

## REFERENCES

- 
- [1] W. Aggoune: Contribution à la Stabilisation de Systèmes Non Linéaires: Application Aux Systèmes Non Réguliers et Aux Systèmes à Retards. (In French.) INRIA Lorraine/CNRS, University of Metz 1999.
  - [2] C. Bonnet, J. Partington, and M. Sorine: Robust control and tracking in  $l_\infty$  of delay systems equipped with a relay sensor. In: LSS'98, 8th IFAC Conference Large Scale Systems, Patras 1998.
  - [3] E. Cheres, S. Gutman, and Z. Palmor: Stabilization of uncertain dynamic systems including state delay. IEEE Trans. Automat. Control *34* (1989), 11, 1199–1203.
  - [4] H. Choi: An LMI approach to sliding mode control design for a class of uncertain time delay systems. In: European Control Conference ECC'99, Karlsruhe 1999.
  - [5] S. Choi and J. Hedrick: An observer-based controller design method for improving air/fuel characterization of spark ignition engines. IEEE Trans. Control Syst. Technology *6* (1998), 3, 325–334.
  - [6] M. Dambrine, F. Gouaisbaut, W. Perruquetti, and J.-P. Richard: Robustness of sliding mode control under delays effects: a case study. In: Proc. 2nd IEEE-IMACS CESA'98, Comput. Eng. in Systems Applications, Hamamet 1998, pp. 817–821.
  - [7] R. De Carlo, S. Zak, and G. Matthews: Variable structure control of nonlinear variable systems: A tutorial. Proc. IEEE *76* (1988), 212–232.
  - [8] L. Dugard and E.I. Verriest: Stability and Control of Time-Delay Systems. (Lecture Notes in Control and Information Sciences 228.) Springer Verlag, Berlin 1997.
  - [9] R. El-Khazaly: Variable structure robust control of uncertain time-delay systems. Automatica *34* (1998), 3, 327–332.

---

<sup>10</sup>More precisely, determination of an attracting neighbourhood of the sliding manifold and estimation of its asymptotic stability domain.



- [10] Y. Fiagbedzi and A. Pearson: Feedback stabilization of linear autonomous time lag systems. *IEEE Trans. Automat. Control* *31* (1986), 847–855.
- [11] M. Fliess: Generalized controller canonical forms for linear and nonlinear dynamics. *IEEE Trans. Automat. Control* *35* (1990), 9, 994–1000.
- [12] L. Fridman, E. Fridman, and E. Shustin: Steady modes in an autonomous system with break and delay. *J. Differential Equations* *29* (1993), 8.
- [13] L. Fridman, E. Fridman, and E. Shustin: Steady modes and sliding modes in the relay control systems with time delay. In: 35th IEEE CDC'96 (Conference on Decision and Control), Kobe 1996, pp. 4601–4606.
- [14] F. Gouaisbaut, Y. Blanco, and J. Richard: Robust control of nonlinear time delay system: A sliding mode control design. In: IFAC Symposium NOLCOS'01, St Petersburg 2001. Accepted.
- [15] F. Gouaisbaut, M. Dambrine, and J.-P. Richard: Sliding mode control of perturbed systems with time varying delays. In: SCS'01, IFAC Symposium on Structure and Control Systems, Prague 2001.
- [16] F. Gouaisbaut, W. Perruquetti, Y. Orlov, and J.-P. Richard: A sliding mode controller for linear time delay systems. In: Proc. ECC'99 European Control Conference, Karlsruhe 1999.
- [17] F. Gouaisbaut, W. Perruquetti and J.-P. Richard: A sliding mode control for linear systems with input and state delays. In: Proc. CDC'99, 38th Conference on Decision and Control, Phoenix 1999.
- [18] A. Goubet-Bartholoméüs, M. Dambrine, and J.-P. Richard: Stability of perturbed systems with time-varying delay. *Systems Control Lett.* *31* (1997), 155–163.
- [19] V. Kolmanovskii and A. Myshkis: *Applied Theory of Functional Differential Equations* (Math. Appl. 85.) Kluwer, Dordrecht 1992.
- [20] V. Kolmanovskii and A. Myshkis: *Introduction to the Theory and Applications of Functional Differential Equations*. Kluwer, Dordrecht 1999.
- [21] V. Kolmanovskii, S. Niculescu, and J.-P. Richard: On the Liapunov–Krasovskii functionals for stability analysis of linear delay systems. *Internat. J. Control* *72* (1999), 4, 374–384.
- [22] V. Kolmanovskii and J.-P. Richard: Stability of some linear systems with delay. *IEEE Trans. Automat. Control* *44* (1999), 5, 984–989.
- [23] X. Li and S. Yurkovich: Sliding mode control of systems with delayed states and control. *Variable Structure Systems, Sliding Mode and Nonlinear Control* *247* (1999), 1.
- [24] A. Lukyanov and V. Utkin: Methods of reducing equations of dynamics systems to regular form. *Automat. Remote Control* *42* (1981), 413–420.
- [25] N. Luo and M. De la Sen: State feedback sliding mode controls of a class of time-delay systems. In: ACC'92 (American Control Conference), Chicago 1992, pp. 894–895.
- [26] N. Luo and M. De la Sen: State feedback sliding mode control of a class of uncertain time-delay systems. *IEE Proceedings-D* *140* (1993), 4, 261–274.
- [27] A. Manitius and A. Olbrot: Finite spectrum assignment problem for systems with delays. *IEEE Trans. Automat. Control* *24* (1979), 4, 541–553.
- [28] Y. Orlov: Discontinuous unit feedback control of uncertain infinite-dimensional systems. *IEEE Trans. Automat. Control* *45* (2000), 5, 834–843.
- [29] Y. Orlov and V. Utkin: Sliding mode control in infinite-dimensional systems. *Automatica* *6* (1987), 753–757.
- [30] W. Perruquetti and J. Barbot: *Sliding Mode Control in Engineering*. Marcel Dekker, New York 2001. To appear.
- [31] J. Richard: Some trends and tools for the study of time delay systems. In: 2nd Conference IMACS-IEEE CESA'98, Computational Engineering in Systems Applications, Tunisia 1998, pp. 27–43.

- [32] J. Richard and V. E. Kolmanovskii: Delay systems. *Mathematics and Computers in Simulation* (Special issue), *45* (1998), 3–4.
- [33] Y. Roh and J. Oh: Robust stabilization of uncertain input-delay systems by sliding mode control with delay compensation. *Automatica* *35* (1999), 1681–1685.
- [34] K. Shyu and J. Yan: Robust stability of uncertain time-delay systems and its stabilization by variable structure control. *Internat. J. Control* *57* (1993), 237–246.
- [35] H. Sira-Ramirez: Differential geometric methods in variable-structure control. *Internat. J. Control* *48* (1998), 4, 1359–1390.
- [36] K. Tan, Q. Wang, and T. Lee: Finite spectrum assignment control of unstable time delay processes with relay tuning. *Ind. Eng. Chem. Res.* *37* (1998), 4, 1351–1357.
- [37] V. Utkin: *Sliding Modes in Control Optimization*. CCES Springer-Verlag, Berlin 1991.
- [38] F. Zheng, M. Cheng, and W.-B. Gao: Variable structure control of time-delay systems with a simulation study on stabilizing combustion in liquid propellant rocket motors. *Automatica* *31* (1995), 7, 1031–1037.

*Prof. Dr. Jean-Pierre Richard, Dr. Frédéric Gouaisbaut, and Dr. Wilfrid Perruquetti, LAIL (CNRS UPRESA 8021), Ecole Centrale de Lille, BP 48, 59651 Villeneuve d'Ascq CEDEX, France.*

*e-mail: jprichard,gouaisba,perruquetti@ec-lille.fr*