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COMPUTING THE DISTRIBUTION OF A LINEAR COMBINATION OF INVERTED GAMMA VARIABLES[1]

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A formula for evaluation of the distribution of a linear combination of independent inverted gamma random variables by one-dimensional numerical integration is presented. The formula is direct application of the inversion formula given by Gil–Pelaez [4]. This method is applied to computation of the generalized p-values used for exact significance testing and interval estimation of the parameter of interest in the Behrens–Fisher problem and for variance components in balanced mixed linear model.

1. INTRODUCTION

Gil-Pelaez in [4] derived a version of the inversion formula which is particularly useful for numerical evaluation of a general distribution function by one-dimensional numerical integration:

Theorem 1. Let $\phi(t) = \int_{-\infty}^{\infty} e^{itx}\,\mathrm{d}F(x)$ be a characteristic function of the one-dimensional distribution function $F(x)$. Then, for x being the continuity point of the distribution, the following inversion formula holds true:

$$F(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty}\left(\frac{e^{-itx}\phi(t) - e^{itx}\phi(-t)}{2it}\right)\mathrm{d}t$$
$$= \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty}\mathrm{Im}\left(\frac{e^{-itx}\phi(t)}{t}\right)\mathrm{d}t. \qquad (1)$$

Proof. See [4]. □

Furthermore, it is easy to observe that if the distribution belongs to the continuous type (if $\int|\phi(t)|\,\mathrm{d}t < \infty$) then the density function is given by

$$f(x) = \frac{1}{2\pi}\int_0^{\infty}\left(e^{itx}\phi(-t) - e^{-itx}\phi(t)\right)\mathrm{d}t$$

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$$= \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(e^{-itx}\phi(t)\right)\,\mathrm{d}t. \tag{2}$$

The limit properties of the integrand in (1) are given by the following Lemma 1:

Lemma 1. Let $F(x)$ be a distribution function of a random variable X with expectation $E(X)$ and its characteristic function $\phi(t)$. Then

$$\lim_{t\to 0} \mathrm{Im}\left(\frac{e^{-itx}\phi(t)}{t}\right) = E(X) - x, \quad \text{and} \quad \lim_{t\to\infty} \mathrm{Im}\left(\frac{e^{-itx}\phi(t)}{t}\right) = 0. \tag{3}$$

Proof. We will show the first equality:

$$\begin{aligned} \lim_{t\to 0} \mathrm{Im}\left(\frac{e^{-itx}\phi(t)}{t}\right) &= \lim_{t\to 0} \frac{1}{i}\left(\frac{e^{-itx}\phi(t) - e^{itx}\phi(-t)}{2t}\right) \\ &= \frac{1}{i}\left(e^{-itx}\phi(t)\right)'\Big|_{t=0} \\ &= \frac{1}{i}\left((-ix)\,e^{-itx}\phi(t) + e^{-itx}\phi'(t)\right)\Big|_{t=0} \\ &= \frac{1}{i}\left(\phi'(t)|_{t=0} - ix\right) = E(X) - x. \end{aligned} \tag{4}$$

The second equality is direct consequence of the fact that the function $e^{-itx}\phi(t)$ is bounded in modulus. □

Consider now $X = \sum_{k=1}^n \lambda_k X_k$, a linear combination of independent random variables, and let $\phi_{X_k}(t)$ denotes the characteristic function of X_k, $k = 1,\ldots,n$. The characteristic function of X is

$$\phi_X(t) = \phi_{X_1}(\lambda_1 t)\cdots\phi_{X_n}(\lambda_n t), \tag{5}$$

and, the distribution function $F_X(x) = \Pr\{X \le x\}$ is given by (1) with $\phi(t) = \phi_X(t)$. Notice that

$$\lim_{t\to 0} \mathrm{Im}\left(\frac{e^{-itx}\phi_X(t)}{t}\right) = \sum_{k=1}^n \lambda_k E(X_k) - x, \tag{6}$$

$$\lim_{t\to\infty} \mathrm{Im}\left(\frac{e^{-itx}\phi_X(t)}{t}\right) = 0. \tag{7}$$

Formula (1) is readily applicable to numerical approximation of the distribution function $F_X(x)$ using a finite range of integration $0 \le t \le T$, $T < \infty$. In general a complex-valued function should be numerically evaluated. The degree of approximation depends on the error of truncation and the error of integration method.

An interesting application of the above inversion formula was given by Imhof in [5] who derived the formula to calculate the distribution of a linear combination of independent non-central chi-squared random variables $X = \sum_{k=1}^n \lambda_k X_k$, where $X_k \sim \chi^2_{\nu_k}(\delta_k^2)$, with ν_k degrees of freedom and the non-centrality parameter δ_k^2.

Imhof's algorithm does not require evaluation of the complex-valued function. Observing that the characteristic function of X is

$$\phi_X(t) = \prod_{k=1}^{n} \phi_{X_k}(\lambda_k t) = \prod_{k=1}^{n} (1 - 2i\lambda_k t)^{-\frac{1}{2}\nu_k} \exp\left\{ \frac{i\delta_k^2 \lambda_k t}{1 - 2i\lambda_k t} \right\}, \tag{8}$$

Imhof applied (1) and derived the distribution function of X as

$$F_X(x) = \Pr\{X \leq x\} = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\sin\theta(u)}{u\varrho(u)}\, du, \tag{9}$$

where

$$\begin{aligned} \theta(u) &= \frac{1}{2}\sum_{k=1}^{n} \left(\nu_k \arctan(\lambda_k u) + \frac{\delta_k^2 \lambda_k u}{1 + \lambda_k^2 u^2} \right) - \frac{1}{2} x u, \\ \varrho(u) &= \prod_{k=1}^{n} (1 + \lambda^2 u^2)^{\frac{1}{4}\nu_k} \exp\left\{ \frac{(\delta_k \lambda_k u)^2}{2(1 + \lambda_k^2 u^2)} \right\}, \end{aligned} \tag{10}$$

are real-valued functions.

In [12] the inversion formula (1) was used for exact computation of the density and of the quantiles of linear combinations of t and F random variables.

2. INVERTED GAMMA DISTRIBUTION

Let $Z \sim G(\alpha, \beta)$ be a gamma random variable with the shape parameter $\alpha > 0$ and the scale parameter $\beta > 0$. Random variable $Y = Z^{-1}$, known as an inverted gamma variable, $Y \sim IG(\alpha, \beta)$, has its probability density function $f_Y(y)$ defined for $y \geq 0$ by

$$f_Y(y) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(\frac{1}{y} \right)^{\alpha+1} \exp\left\{ -\frac{1}{\beta y} \right\}. \tag{11}$$

Theorem 2. Let $Y \sim IG(\alpha, \beta)$ be an inverted gamma random variable with its probability density function $f_Y(y)$ given by (11). Then the characteristic function of Y is

$$\phi_Y(t) = E\left(e^{itY}\right) = \frac{2(-it\beta)^{\frac{1}{2}\alpha} K_\alpha \left\{ \frac{2}{\beta}(-it\beta)^{\frac{1}{2}} \right\}}{\beta^\alpha \Gamma(\alpha)}, \tag{12}$$

where $K_\alpha(z)$ denotes the modified Bessel function of second kind.

Proof. Using the result of Prudnikov et al, see the formula 2.3.16.1 in [7]:

$$\int_0^\infty y^{\nu-1} e^{-py-\frac{q}{y}}\, dy = 2\left(\frac{q}{p} \right)^{\frac{\nu}{2}} K_\nu \left\{ 2(pq)^{\frac{1}{2}} \right\}, \tag{13}$$

where ν, p, q are complex numbers with $\mathrm{Re}(p) > 0$, and $\mathrm{Re}(q) > 0$, and $K_\nu(z)$ denotes the modified Bessel function of second kind (see [1], p. 374), we directly get the Laplace transform of Y:

$$E\left(e^{-tY}\right) = \frac{2(t\beta)^{\frac{1}{2}\alpha} K_\alpha \left\{\frac{2}{\beta}(t\beta)^{\frac{1}{2}}\right\}}{\beta^\alpha \Gamma(\alpha)}. \tag{14}$$

Substitute t by $\varepsilon - it$, ε being a small positive real number. Then, for ε approaching 0, we get that the characteristic function $\phi_Y(t)$ of Y is given by (12). □

Lemma 2. Let $Y \sim IG(\alpha, \beta)$ be an inverted gamma random variable with characteristic function $\phi_Y(t)$ given by (12). Consider $Z = \lambda Y$, where λ be a real number. Let $\kappa_Z(t)$ denote the cumulant generating function of Z, $\kappa_Z(t) = \log \phi_Z(t) = \log \phi_Y(\lambda t)$. Then the first and second derivative of $\kappa_Z(t)$ are

$$\kappa_Z'(t) = \frac{\alpha}{t} + \frac{i\lambda}{(-it\lambda\beta)^{\frac{1}{2}}} R(t), \tag{15}$$

$$\kappa_Z''(t) = -\frac{\alpha}{t^2} + \frac{i\lambda}{t\beta}\left(R^2(t) - \frac{(1+\alpha)\beta}{(-it\lambda\beta)^{\frac{1}{2}}} R(t) - 1\right), \tag{16}$$

where

$$R(t) = \frac{K_{\alpha+1}\left\{\frac{2}{\beta}(-it\lambda\beta)^{\frac{1}{2}}\right\}}{K_\alpha\left\{\frac{2}{\beta}(-it\lambda\beta)^{\frac{1}{2}}\right\}}. \tag{17}$$

Proof. The result is easy to obtain by using the following property:

$$[K_\alpha(z)]' = -K_{\alpha+1}(z) + \frac{\alpha}{z} K_\alpha(z). \tag{18}$$

See [1], p. 376, equation 9.6.26. □

Consequently, the expectation and variance of Z are given by

$$E(Z) = \lim_{t\to 0} \frac{\kappa_Z'(t)}{i} = \frac{\lambda}{(\alpha-1)\beta}, \quad \text{for } \alpha > 1, \tag{19}$$

$$Var(Z) = \lim_{t\to 0} \frac{\kappa_Z''(t)}{i^2} = \frac{\lambda^2}{(\alpha-1)^2\beta^2(\alpha-2)}, \quad \text{for } \alpha > 2. \tag{20}$$

The following Lemma 3 gives simple recursive relation for evaluation of the characteristic function of the inverted gamma random variable $IG(\alpha, \beta)$ with $\alpha = n + \frac{1}{2}$, where $n = 0, 1, 2, \ldots$. This could avoid calling of the modified Bessel function $K_\alpha\{z\}$ during the numerical calculation.

Lemma 3. Let $Y_n \sim IG(\alpha_n, \beta)$ be an inverted gamma random variable with $\alpha_n = n + \frac{1}{2}$ and $\beta > 0$ for $n = 0, 1, 2, \ldots$. Let $w = \frac{2}{\beta}(-2it)^{\frac{1}{2}}$. Then the characteristic function $\phi_n(t)$ of Y_n is given as

$$\begin{aligned} \phi_0(t) &= \exp\{-w\} \\ \phi_1(t) &= \exp\{-w\}(1+w) \\ \phi_2(t) &= \exp\{-w\}\left(1 + w + \frac{1}{3}w^2\right). \end{aligned} \tag{21}$$

For $n \geq 2$, $\phi_{n+1}(t)$ is given by the recursive relation:

$$\phi_{n+1}(t) = \frac{w^2}{(2n+1)(2n-1)}\phi_{n-1}(t) + \phi_n(t). \tag{22}$$

Proof. Equation 10.2.17, [1] p. 444, states that

$$\begin{aligned} K_{\frac{1}{2}}(z) &= \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \exp -z, \\ K_{\frac{3}{2}}(z) &= \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \exp -z\left(1 + z^{-1}\right), \\ K_{\frac{5}{2}}(z) &= \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \exp -z\left(1 + 3z^{-1} + 3z^{-2}\right). \end{aligned} \tag{23}$$

Define

$$f_n(z) = (-1)^{n+1}\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} K_{n+\frac{1}{2}}(z), \tag{24}$$

then, according to the equation 10.2.18 in [1]

$$f_{n-1}(z) - f_{n+1}(z) = (2n+1)z^{-1}f_n(z). \tag{25}$$

From (12) we observe that for $n \geq 1$

$$K_{n+\frac{1}{2}}(w) = [2(n-1)+1]!!\left(\frac{\pi}{2w}\right)^{\frac{1}{2}} w^{-n}\phi_n(t), \tag{26}$$

where $w = \frac{2}{\beta}(-2it)^{\frac{1}{2}}$, and we get the required result. □

Consider now a sample of independent variables $Y_{(\alpha_1,\beta_1)}, \ldots, Y_{(\alpha_n,\beta_n)}$, where $Y_{(\alpha_k,\beta_k)} \sim IG(\alpha_k, \beta_k)$, with $\alpha_k > 0$ and $\beta_k > 0$, $k = 1, \ldots, n$, and define $X = \sum_{k=1}^{n} \lambda_k Y_{(\alpha_k,\beta_k)}$ a linear combination of n inverted gamma variables, with real coefficients λ_k. Let $\phi_k(t) = E(\exp\{itY_{(\alpha_k,\beta_k)}\})$ denote a characteristic function of the distribution of $Y_{(\alpha_k,\beta_k)}$.

The characteristic function $\phi_X(t)$ of X is given by (5) and the formula for evaluation of $F_X(x)$ is given by (1), using $\phi(t) = \phi_X(t)$. From (6) and (19) we get also

$$\lim_{t\to 0} \operatorname{Im}\left(\frac{e^{-itx}\phi_X(t)}{t}\right) = \sum_{k=1}^{n} \frac{\lambda_k}{(\alpha_k - 1)\beta_k} - x, \tag{27}$$

and from (7) we get

$$\lim_{t\to\infty} \operatorname{Im}\left(\frac{e^{-itx}\phi_X(t)}{t}\right) = 0. \tag{28}$$

If $\alpha_k \in (0,1\rangle$ for some subset of indices k, $k = 1,\ldots,n$, the limit (27) does not exist and the result becomes more complicated as the limit of the integrand could be $+\infty$, $-\infty$, or a finite number (depending on the coefficients α_k, β_k, and λ_k). This suggest that the numerical integration in the range close to zero should be carried out very carefully if $\alpha_k \in (0,1\rangle$ for some k.

3. SOME NUMERICAL RESULTS

Davies in [3] gave a general method for selecting the sampling interval which ensures the maximum allowable error ε. He suggested approximation of the integral (1) using the trapezoidal rule

$$\Pr\{X \le x\} \approx \frac{1}{2} - \frac{1}{\pi}\sum_{k=0}^{K} \operatorname{Im}\left(\frac{\exp\{-i(k+\frac{1}{2})\Delta x\}\phi_X\{(k+\frac{1}{2})\Delta\}}{(k+\frac{1}{2})}\right), \tag{29}$$

where $\Delta > 0$ is chosen so that

$$\max\left[\Pr\left\{X \le x - \frac{2\pi}{\Delta}\right\}, \Pr\left\{X \le x + \frac{2\pi}{\Delta}\right\}\right] < \frac{\varepsilon}{2}, \tag{30}$$

and K is chosen so that the truncation error is also less then $\frac{\varepsilon}{2}$, i. e.

$$\frac{1}{\pi}\sum_{k=K+1}^{\infty} \operatorname{Im}\left(\frac{\exp\{-i(k+\frac{1}{2})\Delta x\}\phi_X\{(k+\frac{1}{2})\Delta\}}{(k+\frac{1}{2})}\right) < \frac{\varepsilon}{2}. \tag{31}$$

For more details on finding the bounds Δ and K see [3]. For other details on obtaining distribution functions by numerical inversion of characteristic functions see [9].

Table 1 presents some results of numerical evaluation of the distribution function of different linear combinations of independent inverted gamma random variables. In fact, the table presents the probabilities, rounded to the fifth decimal place, that the random variable X exceeds given number x. The algorithm is a adirect application of (1), (27), and (28).

The integral was computed on the finite interval $\langle 0, T_{ub}\rangle$ if the integrand has a finite limit as t approaches 0, or on the interval $\langle 10^{-12}, T_{ub}\rangle$ if such limit does not exist. The upper bound T_{ub} was chosen such that the integrand function is in absolute value less then 10^{-7} for $t > T_{ub}$.

The algorithm was realized in MATLAB environment where the package for numerical evaluation of Bessel functions of a complex argument and nonnegative order is implemented, see [2].

Table 1. Probability that X, the linear combination of independent inverted gamma random variables, exceeds x. Notice that $\Pr\{X > x\} = 1 - \Pr\{X \leq x\}$. $\lim_{t\to 0}$ stands for a limit of the integrand as t approaches zero. T_{ub} stands for the upper bound of integration.

$X = \sum \lambda_k Y_{(\alpha_k,\beta_k)}$	x	$\lim_{t\to 0}$	T_{ub}	$\Pr\{X > x\}$
$X_1 = Y_{(0.5,2)}$	1	$+\infty$	104.74	0.68269
$X_2 = Y_{(0.5,2)} + Y_{(0.5,2)}$	1	$+\infty$	36.97	0.95450
$X_3 = Y_{(1.5,2)} + Y_{(2.5,2)}$	1	0.3334	73.91	0.34260
$X_4 = 3Y_{(1.5,2)} - 5Y_{(2.5,2)}$	0	1.3334	32.75	0.53515
$X_5 = 5Y_{(2.5,2)} + Y_{(1,2)} - Y_{(1,2)}$	1	0.6667	26.13	0.57869
$X_6 = 2Y_{(1,1.5)} + Y_{(1,2.5)}$	2	$+\infty$	43.98	0.69683
$X_7 = 332.313Y_{(4.5,2)} + 733.949Y_{(3,2)}$	100	130.9605	0.48	0.93429
$X_8 = 1265.96Y_{(1,2)} + 668.634Y_{(9,2)}$	500	$+\infty$	0.29	0.74890
$X_9 = X_7 - X_8$	0	$-\infty$	0.17	0.05341
$X_{10} = X_1 + \cdots + X_9$	0	$+\infty$	0.05	0.67722

4. APPLICATIONS

In this section we briefly mention two applications on testing hypotheses and interval estimation based on the generalized p-values which lead to the problem of evaluation of the distribution function of a linear combination of independent inverted chi-squared random variables. As χ^2_ν is a special case of gamma random variable with $\alpha = \frac{\nu}{2}$ and $\beta = 2$ the above mentioned method of evaluation could be used.

4.1. Definition of generalized p-values

The concept of generalized p-values has been introduced in [8, 10]. Several applications for testing variance components in mixed linear models were given in [13]. For more details see also [6] and [11].

Consider an observable random vector X such that its distribution depends on the vector parameter $\xi = (\theta, \vartheta)$, where θ is the scalar parameter of interest and ϑ is a vector of the other nuisance parameters. Further, consider the problem of testing one-sided hypothesis

$$H_0 : \theta \leq \theta_0, \quad \text{vs.} \quad H_1 : \theta > \theta_0, \tag{32}$$

where θ_0 is a prespecified value of θ. Let x be an observed value of the random variable X. Then the observed significance level for hypothesis testing is defined on the basis of a data-based generalized extreme region, a subset of the sample space, with x on its boundary. In order to define such an extreme region a stochastic ordering of the sample space according to the possible values of θ is required. This could be accomplished by means of generalized test variable, say $T(X, x, \xi)$. $T(X, x, \xi)$ denotes a random variable which functionally depends on the random variable X

and also on the (nonstochastic) observed value x of X and the vector of parameters $\xi = (\theta, \vartheta)$.

A random variable $T(X, x, \xi)$ is said to be a generalized test variable if it has the following properties:

1. $t_{\text{obs}} = T(x, x, \xi)$ does not depend on unknown parameters.
2. The probability distribution of $T(X, x, \xi)$ is free of nuisance vector parameter ϑ.
3. For fixed x and ϑ, and for any given t, $\Pr\{T(X, x, \xi) \leq t\}$ is a monotonic function of θ.

   If $\Pr\{T(X, x, \xi) > t\} = 1 - \Pr\{T(X, x, \xi) \leq t\}$ is a nondecreasing function of θ, then $T(X, x, \xi)$ is said to be stochastically increasing in θ. If $\Pr\{T(X, x, \xi) > t\}$ is a nonincreasing function of θ, then $T(X, x, \xi)$ is said to be stochastically decreasing in θ.

If $T(X, x, \xi)$ is a stochastically increasing test variable then the subset of the sample space $C_x(\xi) = \{y : T(y, x, \xi) > T(x, x, \xi)\}$ is said to be a generalized extreme region for testing H_0 against H_1 and $p = \sup_{\theta \leq \theta_0} \Pr\{X \in C_x(\xi) | \theta\} = \sup_{\theta \leq \theta_0} \Pr\{T(X, x, \xi) > T(x, x, \xi) | \theta\}$ is said to be its generalized p-value for testing H_0. Notice that if $T(X, x, \xi)$ is stochastically increasing then $p = \Pr\{T(X, x, \xi) > T(x, x, \xi) | \theta = \theta_0\}$ and this p-value is computable, since it is free of the nuisance parameter ϑ. If $T(X, x, \xi)$ is stochastically decreasing then the p-value is $p = \Pr\{T(X, x, \xi) \leq T(x, x, \xi) | \theta = \theta_0\}$.

If the null hypothesis is right-sided, then the generalized p-value for testing H_0 is $p = \Pr\{T(X, x, \xi) \leq T(x, x, \xi) | \theta = \theta_0\}$, if $T(X, x, \xi)$ is stochastically increasing, or $p = \Pr\{T(X, x, \xi) > T(x, x, \xi) | \theta = \theta_0\}$, if $T(X, x, \xi)$ is stochastically decreasing.

4.2. The Behrens–Fisher problem

Let $X = (X_1, \ldots, X_m) \sim N(\mu_1, \sigma_1^2)$ and $Y = (Y_1, \ldots, Y_n) \sim N(\mu_2, \sigma_2^2)$ be two independent random samples from two normal populations characterized by parameters μ_1, μ_2, σ_1^2, and σ_2^2. Let $\bar{X} = \frac{1}{m} \sum X_k$, $\bar{Y} = \frac{1}{n} \sum Y_k$ denote the sample means and $S_1^2 = \frac{1}{m} \sum (X_k - \bar{X})^2$, $S_2^2 = \frac{1}{n} \sum (Y_k - \bar{Y})^2$ denote the sample variances. $(\bar{X}, \bar{Y}, S_1^2, S_2^2)$ consist a sufficient statistic for the parameters of the distribution. Notice that

$$\bar{X} \sim N\left(\mu_1, \frac{\sigma_1^2}{m}\right) \quad \text{and} \quad \bar{Y} \sim N\left(\mu_2, \frac{\sigma_2^2}{n}\right), \tag{33}$$

$$\frac{m}{\sigma_1^2} S_1^2 \sim \chi^2_{m-1} \quad \text{and} \quad \frac{n}{\sigma_2^2} S_2^2 \sim \chi^2_{n-1}, \tag{34}$$

are mutually independent random variables.

Let $\theta = \mu_1 - \mu_2$ and $\vartheta = (\sigma_1^2, \sigma_2^2)$. The hypothesis of interest is

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta \neq \theta_0. \tag{35}$$

In this testing problem the parameter of interest is θ and ϑ is the vector of nuisance parameters.

Let $x = (x_1, \ldots, x_m)$ be observed X and $y = (y_1, \ldots, y_n)$ be observed Y. For testing H_0 and interval estimation of θ we shall define a generalized test variable

$$T(X, Y, x, y, \theta, \vartheta) = \frac{(\bar{X} - \bar{Y} - \theta)^2}{\left(\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}\right)} \left(\frac{\sigma_1^2}{m} \frac{s_1^2}{S_1^2} + \frac{\sigma_2^2}{n} \frac{s_2^2}{S_2^2}\right). \tag{36}$$

Notice that for any given $\theta = \theta_0$, $t_{\text{obs}} = (\bar{x} - \bar{y} - \theta_0)^2$ does not depend on unknown parameters and under H_0 denote $T_0 = T(X, Y, x, y, \theta_0, \vartheta)$. Then the distribution of T_0 is

$$T_0 \sim \chi_1^2 \left(\frac{s_1^2}{\chi_{m-1}^2} + \frac{s_2^2}{\chi_{n-1}^2}\right), \tag{37}$$

where χ_1^2, χ_{m-1}^2 and χ_{n-1}^2 symbolically denote independent random variables with chi-squared distribution with 1, $m - 1$ and $n - 1$ degrees of freedom. For fixed x, y, and $\vartheta = (\sigma_1^2, \sigma_2^2)$, T is stochastically decreasing for $\theta > \bar{x} - \bar{y}$ and stochastically increasing for $\theta < \bar{x} - \bar{y}$.

For any θ the generalized p-value is defined as $p(\theta) = \Pr\{T > t_{\text{obs}} | \theta\}$. The significance test of the hypothesis H_0 is based on $p(\theta_0)$:

$$p(\theta_0) = \Pr\left\{\frac{s_1^2}{\chi_{m-1}^2} + \frac{s_2^2}{\chi_{n-1}^2} - \frac{(\bar{x} - \bar{y} - \theta_0)^2}{\chi_1^2} > 0\right\}. \tag{38}$$

We reject H_0 if the p-value is small (smaller than chosen critical p-value, say $p_{\text{crit}} = 0.05$).

The $100(1 - p_{\text{crit}})\,\%$ generalized p-value interval estimator of θ is

$$(\bar{x} - \bar{y}) \pm \delta_{\text{crit}} \tag{39}$$

where the δ_{crit} is given by the following identity:

$$p_{\text{crit}} = \Pr\left\{\frac{s_1^2}{\chi_{m-1}^2} + \frac{s_2^2}{\chi_{n-1}^2} - \frac{\delta_{\text{crit}}^2}{\chi_1^2} > 0\right\}. \tag{40}$$

Example. We have generated two random samples $X_i \sim N(3, 4)$, $i = 1, \ldots, 7$, and $Y_j \sim N(5, 9)$, $j = 1, \ldots, 10$, and observed $\bar{x} = 2.871$, $\bar{y} = 5.8685$, $s_1^2 = 4.1014$, and $s_2^2 = 7.5135$.

Then according to (38) the p-value for significance testing of the hypothesis $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$ is

$$p = \Pr\left\{\frac{4.1014}{\chi_6^2} + \frac{7.5135}{\chi_9^2} - \frac{(-2.9975)^2}{\chi_1^2} > 0\right\} = 0.0424, \tag{41}$$

so, for $p_{\text{crit}} = 0.05$, we reject the null hypothesis that $\theta = 0$. According to (39) and (40) the generalized p-value $95\,\%$ interval estimate of $\theta = \mu_1 - \mu_2$ is $(-5.8732; -0.1218)$.

4.3. Variance components in balanced mixed linear model

Zhou and Mathew, [13], considered a problem that deals with hypothesis testing for variance components in balanced mixed linear model where exact F-tests do not exist. Satterthwaite's approximation of the distribution of the test statistic is a standard solution to the problem. The other possibility is the test using generalized p-values.

Let σ_l^2, $l = 1, \ldots, r$ denote the variance components in balanced mixed model that has r random effects. Denote $\theta = \sigma_1^2$. The generalized testing problem is

$$H_0 : \theta \leq \theta_0 \quad \text{vs.} \quad H_1 : \theta > \theta_0, \tag{42}$$

where θ_0 is a given constant. Let SS_k, $k = 1, \ldots, m$, denote the required analysis of variance sum of squares such that

$$SS_1 \sim (EMS_1)\chi^2_{\nu_1}, \quad SS_k \sim (EMS_k)\chi^2_{\nu_k}, \quad k = 2, \ldots, m, \tag{43}$$

where $EMS_1 = \left(a_1\theta + \sum_{l=2}^r a_l\sigma_l^2\right)$ and $EMS_k = \left(\sum_{l=2}^r b_{kl}\sigma_l^2\right)$, a_l and b_{kl} are known nonnegative scalars, and $\chi^2_{\nu_k}$, $k = 1, \ldots, m$ are independent central χ^2 random variables with ν_k, $k = 1, \ldots, m$, degrees of freedom. We shall suppose that the variables SS_k, $k = 2, \ldots, m$, are sorted and denoted such that the unbiased analysis of variance estimator of θ could be expressed as

$$\hat{\theta} = \frac{1}{a_1}\left(\frac{SS_1}{\nu_1} + \sum_{k=2}^{q}\frac{SS_k}{\nu_k} - \sum_{k=q+1}^{m}\frac{SS_k}{\nu_k}\right). \tag{44}$$

Let ss_k be the observed values of SS_k. Denote $SS = (SS_1, \ldots SS_m)$, $ss = (ss_1, \ldots ss_m)$, and $\vartheta = (\sigma_2^2, \ldots, \sigma_r^2)$. Then, the random variable

$$T(SS, ss, \theta, \vartheta) = \frac{a_1\theta + \sum_{k=q+1}^m (EMS_k)\frac{ss_k}{SS_k}}{\sum_{k=1}^q (EMS_k)\frac{ss_k}{SS_k}} \tag{45}$$

is the generalized test variable for testing H_0 against H_1.

Notice, that $t_{\text{obs}} = 1$, so it does not depend on the unknown parameters, and that the distribution of T does not depend on the nuisance parameters $\sigma_2^2, \ldots, \sigma_r^2$, as

$$T(SS, ss, \theta, \vartheta) \sim \frac{a_1\theta + \sum_{k=q+1}^m \frac{ss_k}{\chi^2_{\nu_k}}}{\sum_{k=1}^q \frac{ss_k}{\chi^2_{\nu_k}}}. \tag{46}$$

Finally, since θ appears with a positive coefficient in the numerator of T, it is clear that T satisfies the condition 3, and T is stochastically increasing in θ.

For any θ the test variable T is used to derive the generalized p-value

$$p(\theta) = \Pr\{T > 1|\theta\} = \Pr\left\{\frac{1}{a_1}\left(\sum_{k=1}^{q}\frac{ss_k}{\chi^2_{\nu_k}} - \sum_{k=q+1}^{m}\frac{ss_k}{\chi^2_{\nu_k}}\right) < \theta\right\} \tag{47}$$

Table 2. A study of the efficiency of workers in assembly lines in several plants. The sum of squares, degrees of freedom, and the expected values of the mean sum of squares obtained by applying Khuri's transformation.

Sum of squares	DF	Expected mean squares
$SS_\alpha = 1265.96$	2	$12\sigma_\alpha^2 + 3\sigma_\beta^2 + 4\sigma_\gamma^2 + \sigma_{\beta\gamma}^2 + \sigma^2$
$SS_\beta = 332.313$	9	$3\sigma_\beta^2 + \sigma_{\beta\gamma}^2 + \sigma^2$
$SS_\gamma = 733.949$	6	$4\sigma_\gamma^2 + \sigma_{\beta\gamma}^2 + \sigma^2$
$SS_{\beta\gamma} = 668.634$	18	$\sigma_{\beta\gamma}^2 + \sigma^2$
$SS_\varepsilon = 246.245$	47	σ^2

For significance testing of H_0 we use $p(\theta_0)$. We reject H_0 if the p-value is small (smaller than chosen critical p-value, say $p_{\text{crit}} = 0.05$).

The $100(1 - p_{\text{crit}})\,\%$ generalized p-value interval estimator of θ is

$$(\theta_L; \theta_U) \cap (0; \infty), \tag{48}$$

where the lower and upper bound are given by the following identities:

$$\begin{aligned} p_1 &= \Pr\left\{ \frac{1}{a_1}\left(\sum_{k=1}^{q} \frac{ss_k}{\chi^2_{\nu_k}} - \sum_{k=q+1}^{m} \frac{ss_k}{\chi^2_{\nu_k}} \right) < \theta_U \right\} \\ 1 - p_2 &= \Pr\left\{ \frac{1}{a_1}\left(\sum_{k=1}^{q} \frac{ss_k}{\chi^2_{\nu_k}} - \sum_{k=q+1}^{m} \frac{ss_k}{\chi^2_{\nu_k}} \right) < \theta_L \right\}, \end{aligned} \tag{49}$$

for given p_1 and p_2, such that $p_1 + p_2 = p_{\text{crit}}$, $p_{\text{crit}} \in (0; 0.5)$.

Example. A problem that deals with a study of the efficiency of workers in assembly lines in several plants was considered in [13]. The original data were unbalanced, with unequal cell frequencies in the last stage, however, by using the transformation given by Khuri, see [6], the exact F-test can be constructed for testing the significance of all the variance components except σ_α^2. Table 2 gives the sum of squares and the expected values of the mean sum of squares obtained by applying Khuri's transformation.

The generalized p-value for testing $H_0 : \sigma_\alpha^2 = 0$ against the alternative $H_1 : \sigma_\alpha^2 > 0$ is according to (47) equal to $p = \Pr\{-X_9 < 0\} = 0.0534$, X_9 is given in Table 1. Thus, comparing with $p_{\text{crit}} = 0.05$, the data do not provide strong evidence against H_0. Choosing $p_1 = p_2 = 0.025$, and according to (48) and (49), the generalized p-value 95 % interval estimate of σ_α^2 is $(0; 2067.8)$.

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