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ON THE STRUCTURE OF THE CORE OF BALANCED GAMES

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The uniform competitive solutions (u.c.s.) are basically stable sets of proposals involving several coalitions which are not necessarily disjoint. In the general framework of NTU games, the uniform competitive solutions have been defined in two earlier papers of the author (Stefanescu [5]) and Stefanescu [6]). The general existence results cover most situations formalized in the framework of the cooperative game theory, including those when the coalitional function is allowed to have empty values.

The present approach concerns the situation when the coalition configurations are balanced. One shows, that if the coalitional function has nonempty values, the game admits balanced u.c.s. To each u.c.s. one associated an "ideal payoff vector" representing the utilities that the coalitions promis to the players. One proves that if the game is balanced, then the core and the strong core consist of the ideal payoff vectors associated to all balanced u.c.s.

1. INTRODUCTION

As usual in the cooperative game theory a non-transferable utility (NTU) game is represented as a pair (N, V), where $N = \{1, 2, \dots, n\}$ is the set of the players and V is the coalitional function defined on the family of all coalitions, 2^N. For V we adopt one of the commonly used representations; if $C \in 2^N$, then $V(C)$ is a subset of the $|C|$-dimensional Euclidean space, denoted by $\boldsymbol{R}^C$. Any element of the set $V(C)$ is referred to as a C-*effective payoff*. An N-effective payoff will be also referred to as a *feasible payoff*.

Some special notations will be used in the following. For any vector $x = (x_1, \dots, x_n) \in \boldsymbol{R}^N$, $x_C = (x_i)_{i \in C}$ stands for its projection onto $\boldsymbol{R}^C$, and for any subset $A \subseteq \boldsymbol{R}^N$, one denotes $\mathrm{pr}_C A = \{x_C \mid x \in A\}$ (when $C = \{i\}$, then we will write, simply, $\mathrm{pr}_i A$.) If x and y are two vectors of the same dimension, then $x \geq y$ means $x_i \geq y_i$ for all i; $x > y$ means $x \geq y$ but $x \neq y$, and $x \gg y$ means $x_i > y_i$ for all i.

Turning back to the formal definition of a game, note that almost everywhere in the literature, $V(C)$ is described as the set of all vectors representing utilities that the coalition C can guarantee for its members. Particularly, this means that any such vector can be achieved by a suitable play. Since it is hard to believe that a particular coalition can determine a game output and the grand coalition cannot do it, then we are forced to consider that if $x \in V(C)$ then x must have an extension in $V(N)$, i.e., $x \in \mathrm{pr}_C V(N)$. In view of this fact, a quite natural property for cooperative game is expressed by the following definition:

Definition 1. The coalitional function V (the game (N, V)) is pseudo-monotonic if $V(C) \subseteq \mathrm{pr}_C V(N)$ for every $C \in 2^N$.

2. BALANCED UNIFORM COMPETITIVE SOLUTIONS

A uniform competitive solution is basically a stable set of proposals involving several coalitions which are not necessarily disjoint. For proposals we adopt the definition in [4].

Definition 2. A proposal of the game (N, V) is any pair (x, C), where $C \in 2^N$ and $x \in V(C) \cap \mathrm{pr}_C V(N)$.

Denote by $\mathcal{P}(N, V)$ the set of all proposals of the game (N, V).

Intuitively, a proposal is an offer that a coalition can make for its members. The associated payoff is effective for the coalition and, at the same time it must be extended up to a feasible payoff, that is, it actually can be achieved by the players if the grand coalition will be formed.

Remark 1. If the game is pseudo-monotonic then the meaning of this notion basically reduces to that of C-effectiveness.

In the general framework of NTU games, the uniform competitive solutions have been defined in [5]. A weaker version of this notion first appears in [6].

Let us consider a finite collection of proposals, $\mathcal{K} = \{(x^C, C) | C \in \mathcal{C}\}$, where $\mathcal{C} \subseteq 2^N \backslash \{\emptyset\}$ covers N, i.e. $\cup_{C \in \mathcal{C}} C = N$.

Definition 3. $\mathcal{K}$ is a uniform competitive solution (u.c.s.) if it satisfies the following two conditions:

$$x_i^C = x_i^D, \text{ whenever } i \in C \cap D, \text{ for any two coalitions } C,\ D \in \mathcal{C} \tag{1}$$

and

$$(u, S) \in \mathcal{P}(N, V),\ u_{S \cap C} > x^C_{S \cap C}, \text{ for some } C \in \mathcal{C} \Rightarrow x_j^D > u_i \text{ for some } D \in \mathcal{C} \text{ and } j \in S \cap D. \tag{2}$$

Weakening the second condition one obtains a new variant:

Definition 4. $\mathcal{K}$ is a weakly uniform competitive solution (w.u.c.s.) if it satisfies (1) and

$$(u,S) \in \mathcal{P}(N,V),\ u_{S\cap C} \gg x^C_{S\cap C} \text{ for some } C \in \mathcal{C} \Rightarrow x^D_j \geq u_i \text{ for some } D \in \mathcal{C} \text{ and } j \in S \cap D. \tag{3}$$

The basic ideas in the above definitions are to ensure that any (weakly) u.c.s. is a stable proposal-configuration.

This stability has two components: the internal stability and the external stability. The internal stability requires by (1) that each player has identical preferences for the offers coming from different coalitions when he is a potential member of them. Therefore, there are no objections from inside.

The external stability expressed in two variants by (2) and, (corresponding to the weaker version of the Pareto principle) by (3), says that no coalition outside the solution can threaten the existing configuration, because there does not exist a coalition which can make its members better off.

The main motivation of these solution-concepts is the emptiness of the core. If the grand coalition is not able to propose a non-objectable output, a u.c.s. proposes a selection of outputs for the negotiation process. All players support, in some sense, these proposals, because each player is member of at least one coalition in the configuration. Likewise in the case of von Neumann–Morgenstern solution, the players should choose finally only one of these proposals, but a precise prediction is impossible.

The u.c.s. have also some interesting properties of rationality type. Any solution is coalitionally-rational as it immediately follows from the definitions.

Proposition 2.1. If $\mathcal{K} = \{(x^C, C) | C \in \mathcal{C}\}$ is a u.c.s. (w.u.c.s.), then x^C is a Pareto optimum (weakly Pareto optimum) of $V(C) \cap \mathrm{pr}_C V(N)$, for every $C \in \mathcal{C}$.

Remark 2. For pseudo-monotonic games the coalitional rationality says that every coalition involved in a solution offers to its members an optimum effective payoff.

The usual meaning of the individual rationality is that each player should receive at least the best utility he can obtain himself. In the present framework, we define the individually-rational payoff level of the player $i \in N$ by:

$$v_i = \begin{cases} \sup V(\{i\}) \cap \mathrm{pr}_i V(N), & \text{if } V(\{i\}) \neq \emptyset \\ -\infty, & \text{if } V(\{i\}) = \emptyset. \end{cases}$$

Now, let us associate to each u.c.s. (w.u.c.s.) a utility vector w of components:

$$w_i = x^C_i, \quad \text{whenever } i \in C \in \mathcal{C}.$$

By (2) w is well-defined. Call it *the ideal payoff vector* associated to $\mathcal{K}$. Obviously, the ith component of w is exactly the utility promised to the player i by all coalitions of $\mathcal{C}$ where he is a potential member.

Proposition 2.2. If w is the ideal payoff vector associated to the u.c.s. (w.u.c.s.) $\mathcal{K}$, then

$$w_i \geq v_i, \quad \text{for all } i \in N.$$

In summary, each u.c.s. (w.u.c.s.) is individually-rational because it predicts to each player a utility not less than the one he can guarantee for himself, and coalitionally-rational because each coalition directly involved can offer to its members an optimal effective payoff.

As it was proved in [6], the general existence results cover most situations formalized in the framework of the cooperative game theory, including those when the coalitional function is allowed to have empty values. For the classical formalism, when the values of V are nonempty, one can point out the existence of u.c.s. (w.u.c.s.) with a special structure. An interesting situation occurs when the coalition configurations are balanced.

We recall that a collection of coalitions $\mathcal{C} \subseteq 2^N$ is said to be *balanced* if there exists the non-negative weights r_C, $C \in \mathcal{C}$, such that $\sum_{C\in\mathcal{C}(i)} r_C = 1$, for every $i \in N$, where $\mathcal{C}(i) = \{C \in \mathcal{C} | i \in C\}$.

Definition 5. The u.c.s. (w.u.c.s.) $\mathcal{K} = \{(x^C, C) | C \in \mathcal{C}\}$ is said to be balanced if $\mathcal{C}$ is balanced.

Obviously, if $\mathcal{K}$ is balanced and $(r_C)_{C\in\mathcal{C}}$ is any system of weights associated to $\mathcal{C}$, then the ideal payoff vector associated to $\mathcal{K}$ may be also defined by:

$$w_i = \sum_{C\in\mathcal{C}(i)} r_C u_i^C, \quad i \in N.$$

We will show that within the general framework of this section, every game satisfying the following five axioms:

(A_1) *nonemptiness:* $V(C) \neq \emptyset \Leftrightarrow C \neq \emptyset$.

(A_2) *closedness:* $V(C)$ is closed in $\boldsymbol{R}^C$, for every C.

(A_3) *boundedness*: For every C the set $\{x \in \boldsymbol{R}^C | x \in V(C), x_i \geq b_i, \ \forall i \in C\}$ is bounded, where $b_i = \sup V(\{i\})$, $i \in N$.

(A_4) *comprehesiveness*: For every C, if $x \in V(C)$ and $y \leq x$ for some $y \in \boldsymbol{R}^C$, then $y \in V(C)$.

(A_5) *pseudo-monotonicity*: V is pseudo-monotonic,

admits balanced w.u.c.s.

If, the following additional condition

(A_6) For any $\varepsilon > 0, S \subseteq N, x \in V(S)$ and $i \in S$, there exists $y \in V(S)$ such that $y_i = x_i - \varepsilon$, and $y_j > x_j$ for all $j \in S \setminus \{i\}$

is also fulfilled, then the game has balanced u.c.s.

Basically, the above condition states that, within the set of effective payoffs of a given coalition, it is always possible to improve the utility of other players if one player accepts to diminish his own utility.

To prove the existence results one can use the arguments in the compact proof of Scarf's theorem concerning the non-emptiness of the core for balanced games. In fact, removing the balancedness condition, several proofs of this theorem known in the literature, represent direct proofs of the existence of balanced w.u.c.s. (see for instance [1], Theorem 1.5.9., or [3], Theorem 5.4.1.) For the sake of the completeness let us sketch the proof and refer the reader to the cited papers.

Theorem 2.3. Assume (N, V) satisfy $(A_1)-(A_5)$. Then, it admits a balanced w.u.c.s.

Proof. Note first that if (A_1), (A_2) and (A_3) are satisfied, then $b_i \in V(\{i\})$, for every $i \in N$.

Put, for every $C \subseteq N$, $\widetilde{V}(C) = \{u \in V(C) | u_i \geq b_i, \forall i \in C\}$, and define a new NTU game (N, V') by $V'(C) = \{x \in \boldsymbol{R}^C | x \leq u, \text{ for some } u \in \widetilde{V}(C)\}$, $C \subseteq N$. One can easily verify that (N, V') satisfies (A_2), (A_4) and

(A'_3) For each $C \subseteq N$, there exists some constant M_C such that $x_i \leq M_C$, for all $i \in C$, whenever $x \in V'(C)$.

Note that $V'(C)$ may be empty for some C, but $V'(\{i\}) \neq \emptyset$ for all $i \in N$.

Let $\mathcal{C}$ be any covering of N, and $w \in \boldsymbol{R}^N$. Show that the pair $(w, \mathcal{C})$ represents a w.u.c.s. of (N, V) if and only if

$$w_C \in V'(C), \quad \forall C \in \mathcal{C} \tag{4}$$

and

$$\text{There are no } S \subseteq N \text{ and } u \in V'(S), \text{ such that } u \gg w_S. \tag{5}$$

Assume first that $\mathcal{K}$ is a solution in (N, V) and let w be the ideal payoff vector. Then, $w_i \geq b_i$, for all $i \in N$. Hence, $w_C \in \widetilde{V}(C) \subset V'(C)$, for all $C \in \mathcal{C}$. If $u \gg w_S$ for some S and $u \in V'(S)$, one contradicts the definition of $\mathcal{K}$ since $u \in V(S)$, and by pseudo-monotonicity it follows that (u, S) is a proposal in (N, V).

For the converse implication observe that $w_C \in V(C) \subseteq \mathrm{pr}_C V(N)$, so that (x^C, C) is a proposal, where $x^C = w_C$. Then $\mathcal{K} = \{(x^C, C) | C \in \mathcal{C}\}$ is a u.c.s. Otherwise, $u \gg w_S$, for some proposal (u, S). But (5) says that $w_i \geq b_i$ for every $i \in N$, so that $u \in V'(S)$. One contradicts (5).

Now we can prove the theorem showing that there exist a balanced collection $\mathcal{C}$ and $w \in \boldsymbol{R}^N$ verifying (4) and (5).

For any subset B of an Euclidean space, $\mathrm{int}\, B$ and ∂B will designate the interior, respectively, the boundary of B.

We can assume that $0 \in \mathrm{int}\, V'(S)$, whenever $V'(S) \neq \emptyset$ (otherwise, one replaces V' by an appropriate "translation" which satisfies the same properties as V'.) Then, as a consequence of (A'_3) one can find a positive constant δ, such that

$$x_i < \delta, \text{ for all } i \in S, \text{ whenever } x \in V'(S), \text{ for some } S \subseteq N. \tag{6}$$

Consider the set

$$W = [\cup_{C \subseteq N}\{x \in \boldsymbol{R}^N | x_C \in V'(C)\}] \cap (-\infty, \delta]^n \tag{7}$$

The next six statements are proved in [1].

1. Let Δ the (n-1)-dimensional standard simplex. Then, for any $s \in \Delta$ there exists a unique $\alpha > 0$ such that $\alpha s \in \partial W$.

2. The function $f : \Delta \to \partial W \cap \boldsymbol{R}^n_+$, defined by $f(s) = \alpha s$ is continuous.

3. The correspondence $\Psi : \Delta \to 2^\Delta$, defined by

$$\Psi(s) = \left\{ \frac{1}{|C|} e^C | C \subseteq N, f(s) \in V'(C) \right\} .$$

where, $e^C \in \boldsymbol{R}^N$

$$e_i^C = \begin{cases} 1, & \text{if } i \in C \\ 0, & \text{if } i \notin C \end{cases}$$

is closed and non-empty valued.

4. The correspondence $s \to co\Psi(s)$, (coB stands for the convex hull of B) is closed with non-empty and convex values.

5. The correspondence $\Phi : \Delta \times \Delta \to 2^{\Delta \times \Delta}$, defined by $\Phi(s,t) = \{g(s,t)\} \times co\Psi(s)$, where

$$g(s,t) = \left(\frac{s_i + \max\{0, t_i - 1/n\}}{1 + \sum_{j \in N} \max\{0, t_j - 1/n\}} \right)_{i \in N},$$

admits fixed points (it satisfies all assumptions of the Kakutani's fixed point theorem).

6. Let (s,t) any fixed point of Φ. Denote by $w = f(s)$. Then, the collection of coalitions $\mathcal{C}(w) = \{C \subseteq N | w_C \in V'(C)\}$ is balanced. Particularly, $(w, \mathcal{C}(w))$ satisfies (4).

Now we can easily show that w also satisfies (5). To the contrary, assume that there exists $S \subseteq N$ and $u \in V'(S)$ such that $u \gg w_S$. Extend u up to $\overline{u} \in \boldsymbol{R}^N$, by

$$\overline{u}_i = \begin{cases} u_i, & \text{if } i \in S \\ \delta, & \text{if } i \in \overline{S}. \end{cases}$$

By (4) and (6) one has $w_i < \delta$ for all $i \in N$. Hence, $w \ll \overline{u}$. Since $\overline{u} \in W$, then $w \in \operatorname{int} W$, in contradiction with $f(s) \in \partial W$. □

The existence of a balanced u.c.s. can be also proved.

Theorem 2.4. Assume (N, V) satisfy $(\mathrm{A}_1)-(\mathrm{A}_6)$. Then, it admits a balanced u.c.s.

Proof. We can show that (A_6) implies that any w.u.c.s. is a u.c.s. too.

Let us assume that w is the ideal payoff vector associated to a w.u.c.s. $\mathcal{K}$, but there exists a proposal (u, S), such that $u > w_S$. Suppose that $u_i - w_i = \varepsilon > 0$, for some $i \in S$. Then, by (A_6) there exists $y \in V(S)$ such that $y_i = u_i - \frac{\varepsilon}{2} > w_i$, and $y_j > u_j \geq w_j$, for $j \in S \backslash \{i\}$. Thus, $y \gg w_S$, a contradiction. □

3. THE CORE AND THE STRONG CORE OF A BALANCED GAME

By its classical definition, the core $C(N, V)$ of the game (N, V) consists of all non-dominated feasible utility vectors, i. e.

$$C(N,V) = \{u \in V(N) | C \in 2^N, u' \in \mathbb{R}^C, u' \gg u_C \Rightarrow u' \notin V(C)\}.$$

Strengthening the non-domination condition we define the strong core:

$$SC(N,V) = \{u \in V(N) | C \in 2^N, u' \in \mathbb{R}^C, u' > u_C \Rightarrow u' \notin V(C)\}.$$

The general relationships between the core and uniform competitive solutions have been established in [5].

The proofs of the next two propositions are straightforward.

Proposition 3.1. If $u \in C(N, V)$, then $\{(u, N)\}$ is a w.u.c.s.

Proposition 3.2. If $u \in SC(N, V)$, then $\{(u, N)\}$ is a u.c.s.

Now, we can show that if the game is *balanced* (i. e. $\{u \in \boldsymbol{R}^N | u_C \in V(C), \forall C \in \mathcal{C}\} \subseteq V(N)$ whenever $\mathcal{C}$ is balanced), then the ideal payoff vector of any balanced w.u.c.s. (u.c.s) belongs to the core (strong core). Thus we have,

Theorem 3.3. Let (N, V) a balanced game satisfying $(\mathrm{A}_1)-(\mathrm{A}_4)$. Then, the core $C(N, V)$ is nonempty and consists of all ideal payoff vectors associated to the balanced w.u.c.s.

Proof. Note that the pseudo-monotonicity of V is used in the proof of Theorem 2.3 only to show that if $(w, \mathcal{C})$ verifies (4), then each pair (x^C, C), where $x^C = w_C$ for $C \in \mathcal{C}$, is a proposal of the game (N, V). If the game is *balanced*, this conclusion trivially follows if $\mathcal{C}$ is balanced. Or, by statement 6 of the proof it is claimed that there exists w such that $(w, \mathcal{C}(w))$ represents a w.u.c.s. and $\mathcal{C}(w)$ is balanced. Hence, by Theorem 2.3 any balanced game satisfying $(\mathrm{A}_1)-(\mathrm{A}_4)$ admits balanced w.u.c.s.

Moreover, if w is the ideal payoff vector associated to any balanced w.u.c.s., then $w \in V(N)$. Since it satisfies (5) with respect to V, it results that $w \in C(N, V)$. Then the proof is completed by Proposition 3.1. □

Theorem 3.4. Let (N, V) a balanced game satisfying $(A_1)-(A_4)$ and (A_6). Then, the strong core $SC(N, V)$ is nonempty and consists of all ideal payoff vectors associated to the balanced u.c.s.

Proof. As in the above, using Theorem 2.4 and Proposition 3.2. □

The above results highlight the structure of the core (strong core) of a balanced game. It appears that every utility vector of the core is supported by a balanced configuration of coalitions. Moreover, this configuration is stable in the sense of the definitions of uniform competitive solutions.

On the other hand, by the existence theorems of the previous section, (weakly) uniform competitive solutions exists for a wide category of games. Therefore, the set of ideal payoff vectors associated to all balanced w.u.c.s. (u.c.s.) can be considered as an extension of the concept of core.

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