

## A NEW INDIRECT ADAPTIVE POLE PLACER FOR POSSIBLY NON-MINIMUM PHASE MIMO LINEAR SYSTEMS

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The use of generalized sampled-data hold functions, in order to synthesize adaptive pole placers for linear multiple-input, multiple-output systems with unknown parameters, is investigated in this paper, for the first time. Such a control scheme relies on a periodically varying controller, which suitably modulates the sampled outputs of the controlled plant. The proposed control strategy allows us to assign the poles of the sampled closed-loop system arbitrarily in desired locations, and does not make assumptions on the plant other than controllability and observability of the continuous and the sampled system, and the knowledge of a set of structural indices, namely the locally minimum controllability indices of the continuous-time plant. The indirect adaptive control scheme presented here, estimates the unknown plant parameters (and hence the parameters of the desired modulating matrix function) on line, from sequential data of the inputs and the outputs of the plant, which are recursively updated within the time limit imposed by a fundamental sampling period  $T_0$ . The controller determination is based on the transformation of the discrete analogue of the system under control to a phase-variable canonical form, prior to the application of the control design procedure. The solution of the problem can, then, be obtained by a quite simple utilization of the concept of state similarity transformation, whereas known indirect adaptive pole placement techniques require the solution of matrix polynomial Diophantine equations. Moreover, in many cases, the solution of the Diophantine equation for a desired set of closed-loop eigenvalues might yield an unstable controller, and the overall adaptive pole placement scheme is then unstable with unstable compensators because their outputs are unbounded. The proposed strategy avoids these problems, since here gain controllers are essentially needed to be designed. Moreover, persistency of excitation and, therefore, parameter convergence, of the continuous-time plant is provided without making assumptions either on the existence of specific convex sets in which the estimated parameters belong or on the coprimeness of the polynomials describing the ARMA model, or finally on the richness of the reference signals, as compared to known adaptive pole placement schemes.

### 1. INTRODUCTION

In the last decade, many pieces of work treating design issues of linear time-invariant systems by periodically time-varying and/or multirate sampled-data controllers have been reported in the literature [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 25, 27, 29, 30, 32, 38, 40]. The interest for such a type of control strategies is warranted by the new

dimensions of flexibility of the design process offered by these control schemes, which also provide a series of remarkable advantages over ordinary time-invariant feedback strategies, such as state feedback, dynamic compensation or state observers (for an overview of these advantages see [2, 6, 7, 8, 9, 10, 11, 12, 27, 29, 30, 40]). Among the most interesting control strategies of this type, is feedback control based on generalized sampled-data hold functions (GSHF). GSHF control has been proposed first in [29], and subsequently has successfully been applied in solving a variety of important control problems (see for example, [2, 29, 30, 40]).

In his excellent work [29], Kabamba proposes a GSHF based periodic controller, which suitably modulates the sampled output of the system under control and discrete reference signals by periodically varying functions, in order to solve, among other important control problems, the sampled pole placement problem for linear time-invariant continuous-time systems. Under certain conditions, the modulating functions can be tailored to a given system in such a way that for the sampled closed loop system a desired set of prespecified eigenvalues can be reached. A main feature of the approach reported in [29] is that the pole placement problem is obtained without the requirement of pole-zero cancellation.

The aim of the present paper is to explore the possibility to extend the GSHF technique proposed in [29], to the control of linear time-invariant multi-input, multi-output systems with unknown parameters. To this end, the certainty equivalence principle is used to combine the identification method with a control structure derived for the pole placement problem. Adaptive pole placement control has long been the focus of interest by many control designers, for obvious reasons. Several techniques based on either direct or indirect adaptive control schemes were presented to treat the problem and a very large number of papers were reported on the subject (see for example [1, 3, 13, 14, 16, 17, 19, 20, 21, 22, 23, 34, 36, 39, 43], and the references therein). The feedback strategies proposed to solve the adaptive pole placement problem, are hitherto based on dynamic output feedback, thus introducing high order exogenous dynamics in the control loop. On the other hand, a common feature of these techniques is that they reduce the solution of the problem to the solution of a polynomial Diophantine equation. This approach, however, does not ensure that the compensators obtained from the solution of the Diophantine equation are necessarily stable. In the case of unstable solutions, the control scheme composed by feedforward and feedback compensators is not stable and thus is not useful. The control signals are calculated from two sets of unbounded signals that are the outputs of the compensators. In a short time the system becomes unstable. It is worth noticed at this point, that unstable solutions of the Diophantine equation, can occur even though, the system under control possesses the parity interlacing property (p.i.p.) [44] (is strongly stabilizable). A plant is said that it possesses the p.i.p. if the number of its real poles between each pair of zeros in the unstable domain is even. In this case, it is possible to obtain a stable controller from these unstable solutions by using the approach presented in [33], which is based on an interpolation procedure. Unfortunately, as mentioned above, this approach can be applied only in cases where the system under control is strongly stabilizable. When the system under control contains unknown parameters (as in the case of adaptive

pole placement control), this information of crucial importance is not available to the designer. Thus, up to now, the design of a stable and useful adaptive pole placement compensator cannot be guaranteed.

In the present paper the adaptive pole placement problem of linear multi-input, multi-output systems is solved using GSHF control, for the first time. Using the proposed approach, the determination of the GSHF based pole placers sought is mainly based on the transformation of the discrete analogue of the continuous-time system under control to a phase variable canonical form, prior to the application of the control design procedure. As a consequence of this fact, the solution of the problem can be obtained by a quite simple utilization of the concept of state similarity transformation. No Diophantine equations are needed to be solved here, in contrast to known techniques. Moreover, no exogenous dynamics is introduced in the control loop, and the designed GSHF based adaptive pole placers are always stable, since gain controllers are needed to be designed here, as compared to dynamic compensators or state observers needed by known techniques. As a consequence of this design philosophy, a useful globally stable indirect adaptive control scheme is derived, which estimates the unknown plant parameters (and consequently the controller parameters) on-line, from sequential data of the inputs and the outputs of the plant, which are recursively updated within the time limit imposed by a fundamental sampling period  $T_0$ . This adaptive scheme is applicable to plants with arbitrary poles and zeros and relative degree, including plants with non-minimum phase zeros and plants which do not possess the p.i.p. It is remarked that, the a priori knowledge needed in order to implement the proposed adaptive pole placers, is controllability and observability of the continuous and the discretized plant under control, its order, and a set of structural indices, namely the locally minimum controllability indices of the continuous-time plant. Finally, persistency of excitation of the controlled plant is assured without making any assumption either on the existence of special convex sets in which the estimated parameters belong or on the coprimeness of the polynomials describing the ARMA model or, finally, on the richness of the reference signals, as in known indirect adaptive pole placement techniques.

## 2. PRELIMINARIES AND PROBLEM FORMULATION

Consider the continuous-time, linear time-invariant multi-input, multi-output system having the following state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \quad (2.1)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $u(t) \in \mathbb{R}^m$  is the input vector and  $y(t) \in \mathbb{R}^p$  is the output vector and where the matrices  $A$ ,  $B$  and  $C$  have appropriate dimensions.

With regard to system (2.1), we make the following two assumptions:

**Assumption 2.1.** (a) System (2.1) is controllable and observable and of known order  $n$ . (b) There are known integers  $n_i$ ,  $i \in J_m$ ,  $J_m = \{1, 2, \dots, m\}$ , which comprise a set of locally minimum controllability indices of the pair  $(A, B)$ .

**Assumption 2.2.** Let  $N_i, i \in J_m$  be positive integers. Also let  $N = lcm\{N_1, \dots, N_m\}$ , where  $lcm\{*, \dots, *\}$  denotes the least common multiplier of the arguments quoted in the braces. Then, there is a sampling period  $T_0 \in \mathbb{R}^+$ , such that the discretized systems, obtained by sampling (2.1) with periods  $T_0$  and  $\tau = T_0/(6n - 1)N$  and having the following matrix triplets

$$\begin{aligned} (\Phi, \tilde{B}, C) &\equiv \left( \exp(AT_0), \int_0^{T_0} \exp(A\lambda) B \, d\lambda, C \right), \\ (\Phi_\tau, B_\tau, C) &\equiv \left( \exp(A\tau), \int_0^\tau \exp(A\lambda) B \, d\lambda, C \right) \end{aligned}$$

respectively, are controllable and observable.

Except for this prior information, the matrix triplet  $(A, B, C)$  is arbitrary and unknown. It is mentioned that, no assumption is made here on the relative degree of the plant or its stable invertibility.

For a controllable matrix pair  $(A, B)$ , with  $B \hat{=} [b_1 \ b_2 \ \dots \ b_m]$ , its locally minimum controllability indices (LMCI) are a collection of  $m$  integers  $\{n_1, n_2, \dots, n_m\}$ , for which the following relationships simultaneously hold

$$\sum_{i=1}^m n_i = n \quad \text{and} \quad \text{rank} [b_1 \ \dots \ A^{n_1-1} b_1 \ \dots \ b_m \ \dots \ A^{n_m-1} b_m] = n.$$

Note that, LMCI defined as above are also known as the "Kronecker invariants" or "Kronecker indexes" of the pair  $(A, B)$ .

Consider now applying to system (2.1) the following control law

$$u(t) = F(t) y(kT_0) + w(kT_0), \quad t \in [kT_0, (k+1)T_0], \quad k \geq 0 \quad (2.2)$$

where  $y(kT_0) \in \mathbb{R}^p$  is a discrete measurement vector, obtained by sampling  $y(t)$  with sampling period  $T_0$ , and  $w(kT_0) \in \mathbb{R}^m$  is a vector of uniformly bounded reference signals. The modulating matrix function  $F(t) \in \mathbb{R}^{m \times p}$ , known as generalized sampled-data hold function (GSHF) [29], is assumed to be bounded, integrable and  $T_0$ -periodic, i. e.  $F(t+T_0) = F(t)$ , for  $t \in [kT_0, (k+1)T_0]$ . The overall control strategy is depicted in Figure 1, wherein the hold circuit  $H_0$  is the zero order hold with holding time  $T_0$ . The resulting closed-loop system is described by the following state-space equations

$$x[(k+1)T_0] = (\Phi + K_f C) x(kT_0) + \tilde{B} w(kT_0), \quad y(kT_0) = C x(kT_0), \quad k \geq 0$$

where  $x(kT_0) \in \mathbb{R}^n$  is a discrete measurement vector obtained by sampling  $x(t)$  with sampling period  $T_0$  and where the matrix  $K_f \in \mathbb{R}^{n \times p}$  is defined as

$$K_f = \int_0^{T_0} \exp[A(T_0 - \lambda)] B F(\lambda) \, d\lambda. \quad (2.3)$$

The adaptive pole placement problem treated in the present paper is as follows: Find a periodic controller  $F(t)$ , which when applied to system (2.1), drives the

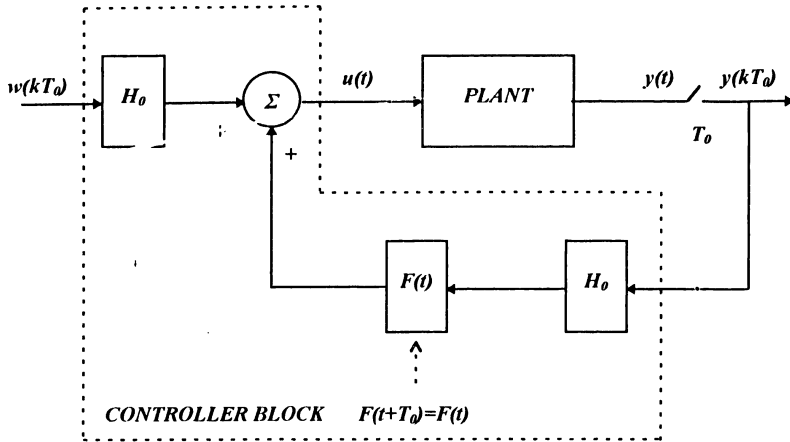


Fig. 1. Control strategy in the nonadaptive case.

poles of the resulting closed-loop system (also called the closed-loop monodromy eigenvalues), to new desired values  $\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_n$ , where complex poles appear in conjugate pairs.

To solve the above problem, an indirect adaptive control scheme is exhibited in the sequel. In particular, we first solve the pole placement problem, namely, the assignment of the poles of the sampled system to the prespecified values  $\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_n$ , using GSHF, for known systems. This is done in Section 3. Next, using these results, the pole placement problem is solved for the configuration of Figure 2, wherein the periodic controller  $F(t)$  is with prespecified periodic behavior and persistent excitation signals are introduced in the control loop for future identification purposes. This is done in Section 4. It is remarked that the motivation for modifying the control strategy as in Figure 2, is that it facilitates the derivation of the indirect adaptive control scheme sought, which is presented in Section 5. In Section 5, the global stability of the proposed scheme is also studied.

### 3. SOLUTION OF THE POLE PLACEMENT PROBLEM VIA GSHF CONTROL FOR KNOWN SYSTEMS

The procedure for stabilization through pole placement using the GSHF control depicted in Figure 1, consists in finding a periodic controller  $F(t)$ , such that

$$\det(zI - \Phi - K_f C) \equiv \hat{p}(z) \tag{3.1a}$$

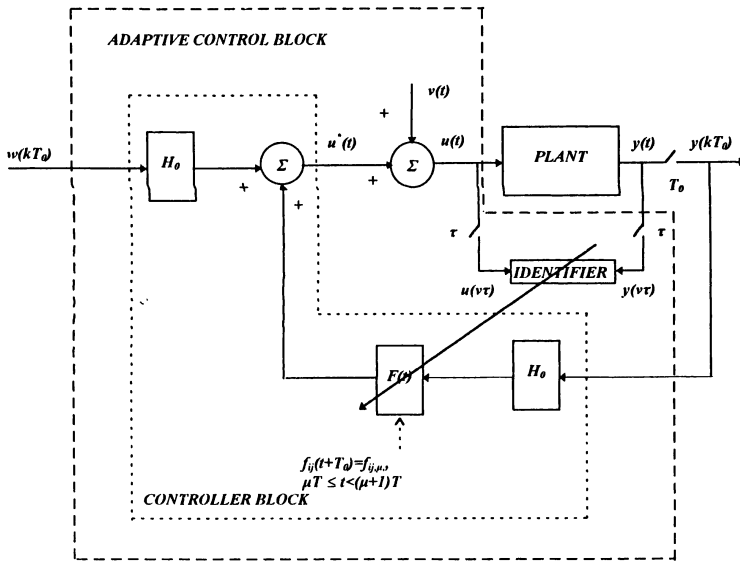


Fig. 2. The structure of the adaptive control system.

where

$$\hat{p}(z) = \prod_{i=1}^n (z - \hat{\lambda}_i) \hat{=} z^n + \hat{a}_1 z^{n-1} + \dots + \hat{a}_{n-1} z + \hat{a}_n. \tag{3.1b}$$

Since,  $\det(zI - \Phi - K_f C) \equiv \det(zI - \Phi^T - C^T K_f^T)$ , relation (3.1a), is equivalent to the relation

$$\det(zI - \Phi^T - C^T K_f^T) = \hat{p}(z). \tag{3.2}$$

Consider now the following fictitious discrete time system

$$\tilde{x}([k + 1] T_0) = \Phi^T \tilde{x}(k T_0) + C^T \tilde{u}(k T_0), \quad \tilde{y}(k T_0) = \tilde{B}^T \tilde{x}(k T_0). \tag{3.3}$$

Clearly, the pole placement problem via GSHF control, defined in relation (3.2), is equivalent to the problem of choosing the matrix  $K_f^T$  in the state feedback control law

$$\tilde{u}(k T_0) = K_f^T \tilde{x}(k T_0) + \tilde{w}(k T_0) \tag{3.4}$$

such that (3.2) to be satisfied.

We start our analysis to this equivalent state feedback pole placement problem by first transforming system (3.3), to its equivalent input Luenberger canonical form. To this end, let  $\delta_i, i = 1, 2, \dots, m$  be the controllability indices of the pair  $(\Phi^T, C^T)$  (which obviously are the observability indices of the pair  $(\Phi, C)$ ) and let  $P \in \mathbb{R}^{n \times n}$  be the following matrix

$$P = [c_1^T \dots (\Phi^T)^{\delta_1-1} c_1^T \dots c_p^T \dots (\Phi^T)^{\delta_p-1} c_p^T]$$

where  $c_i^T$ ,  $i = 1, 2, \dots, p$  are the ordered columns of  $C^T$ . Setting

$$\gamma_j = \sum_{\rho=1}^j \delta_\rho, \quad j = 1, 2, \dots, p$$

and defining  $h_j^T$  as the  $\gamma_j$ th row of  $P^{-1}$ , it can be shown that under the transformation  $\tilde{z}(kT_0) = Q \tilde{x}(kT_0)$ , where  $Q \in \mathbb{R}^{n \times n}$  is the columnar stack of  $\delta_1 + \dots + \delta_p$  ( $= n$ ) rows, defined by

$$Q = \begin{bmatrix} h_1^T \\ \vdots \\ h_1^T (\Phi^T)^{\delta_1-1} \\ \vdots \\ h_p^T \\ \vdots \\ h_p^T (\Phi^T)^{\delta_p-1} \end{bmatrix}$$

the system (3.3) can be written as

$$\tilde{z}[(k+1)T_0] = \Phi^* \tilde{z}(kT_0) + C^* \tilde{u}(kT_0), \quad \tilde{y}(kT_0) = \tilde{B}^* \tilde{z}(kT_0) \tag{3.5}$$

where

$$\Phi^* = Q\Phi^T Q^{-1}, \quad C^* = QC^T, \quad \tilde{B}^* = \tilde{B}^T Q^{-1}$$

and where the matrices  $\Phi^*$  and  $C^*$  have the following respective forms

$$\Phi^* = \begin{bmatrix} \Phi_{11}^* & \cdots & \Phi_{1p}^* \\ \vdots & \ddots & \vdots \\ \Phi_{p1}^* & \cdots & \Phi_{pp}^* \end{bmatrix}, \quad C^* = \begin{bmatrix} C_1^* \\ \vdots \\ C_p^* \end{bmatrix}$$

where

$$\Phi_{ii}^* = \begin{bmatrix} 0_{\delta_i-1} & I_{\delta_i-1} \\ -a_{ii}^T & \end{bmatrix} \in \mathbb{R}^{\delta_i \times \delta_i}, \quad \Phi_{ij}^* = \begin{bmatrix} O_{(\delta_i-1) \times \delta_j} \\ -a_{ij}^T \end{bmatrix} \in \mathbb{R}^{\delta_i \times \delta_j}, \quad (i \neq j),$$

$$C_i^* = \begin{bmatrix} O_{(\delta_i-1) \times p} \\ -q_i^T \end{bmatrix} \in \mathbb{R}^{\delta_i \times p}$$

where

$$a_{ii}^T = [(a_{ii})_0 \ (a_{ii})_1 \ \cdots \ (a_{ii})_{\delta_i-1}], \quad a_{ij}^T = [(a_{ij})_0 \ (a_{ij})_1 \ \cdots \ (a_{ij})_{\delta_j-1}] \quad (i \neq j)$$

$$q_i^T = [0_{i-1}^T \ 1 \ \tilde{q}_i^T], \quad \tilde{q}_i^T = [(c_{ii})_{i+1} \ (c_{ii})_{i+2} \ \cdots \ (c_{ii})_p].$$

Here,  $0_r$ ,  $O_{r \times q}$  and  $I_r$  represent a zero  $r$ -dimensional vector a zero  $r \times q$  matrix and an  $r$ -dimensional identity matrix respectively (empty if  $r$  or  $q$  is zero). Now, let  $\tilde{v}(kT_0)$  be the set of inputs defined as follows

$$\tilde{u}(kT_0) = L\tilde{v}(kT_0)$$

where  $L$  is the following upper triangular nonsingular matrix

$$L = \begin{bmatrix} 1 & \tilde{q}_1^T & & & \\ & 0 & 1 & \tilde{q}_2^T & \\ & & & \ddots & \\ & & & & 1 & \tilde{q}_{p-1}^T \\ 0 & 0 & \cdots & 0 & 1 & \\ & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

It is now obvious that

$$C^*L = QC^TL \equiv \tilde{C}^* = \begin{bmatrix} \tilde{C}_1^* \\ \vdots \\ \tilde{C}_m^* \end{bmatrix}$$

where

$$\tilde{C}_i^* = \begin{bmatrix} O_{(\delta_i-1) \times p} \\ 0_{i-1}^T & 1 & 0_{p-i}^T \end{bmatrix}$$

and that system (3.5) can be transformed to the following form

$$\tilde{z}[(k+1)T_0] = \Phi^* \tilde{z}(kT_0) + C^* \tilde{v}(kT_0) \quad (3.6)$$

which is the input Luenberger canonical form corresponding to system (3.3). In what follows, to system (3.6), we apply the following state feedback law

$$\tilde{v}(kT_0) = F^* \tilde{z}(kT_0) + L^{-1} \tilde{w}(kT_0) \quad (3.7)$$

in order to drive the eigenvalues of system (3.6) to desired positions  $\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_n$ . Clearly, this is equivalent to the application of a state feedback law of the form (3.4), with

$$K_f^T = LF^*Q \quad (3.8)$$

to system (3.3), in order to drive its eigenvalues to the desired positions  $\hat{\lambda}_i$ ,  $i = 1, 2, \dots, n$ .

From the above analysis, it is clear that in order to solve the pole placement problem for system (3.3), under the control law (3.4), one can equivalently solve the pole placement problem for system (3.6), under the control law (3.7). The solution of this later problem can be obtained as follows: Observe first that the solution of this problem is equivalent to the problem of selecting  $F^*$  and a nonsingular transformation matrix  $T$  such that

$$\Phi^* + \tilde{C}^* F^* = T\Pi T^{-1} \quad (3.9)$$

where

$$\Pi = \begin{cases} \text{diag}(\hat{\lambda}_1, \dots, \hat{\lambda}_n) & \text{if the desired eigenvalues are distinct} \\ \text{blockdiag}(J_1, \dots, J_s) & \text{if the desired eigenvalues are repeated} \end{cases} \quad (3.10)$$



with

$$J_q = \begin{bmatrix} \hat{\lambda}_q & 1 & 0 & \cdots & 0 \\ 0 & \hat{\lambda}_q & 1 & \cdots & 0 \\ 0 & 0 & \hat{\lambda}_q & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \hat{\lambda}_q \end{bmatrix} \in \mathbb{R}^{r_q \times r_q} \tag{3.11}$$

and where the order  $r_q$  of the  $q$ th Jordan block  $J_q$ , is the multiplicity of the eigenvalue  $\hat{\lambda}_q$ .

To solve (3.9) for  $F^*$  and  $T$ , partition the matrices  $\Phi_{ii}^*$  and  $\Phi_{ij}^*$  as

$$\Phi_{ii}^* = \begin{bmatrix} \hat{\Phi}_{ii}^* \\ -a_{ii}^T \end{bmatrix}, \quad \Phi_{ij}^* = \begin{bmatrix} \hat{\Phi}_{ij}^* \\ -a_{ij}^T \end{bmatrix} \quad (i \neq j)$$

where

$$\hat{\Phi}_{ii}^* = [0_{\delta_i-1} \ I_{\delta_i-1}], \quad \hat{\Phi}_{ij}^* = O_{(\delta_i-1) \times \delta_j} \quad (i \neq j)$$

define the matrices

$$T = \begin{bmatrix} T_{11} & \cdots & T_{1p} \\ \vdots & \ddots & \vdots \\ T_{p1} & \vdots & T_{pp} \end{bmatrix}, \quad T_{ij} = \begin{bmatrix} \hat{T}_{ij} \\ \tilde{t}_{ij}^T \end{bmatrix}, \quad \hat{T}_{ij} = \begin{bmatrix} t_{ij1}^T \\ \vdots \\ t_{ij,\delta_i-1}^T \end{bmatrix}, \quad \tilde{t}_{ij}^T = t_{ij,\delta_i}^T$$

partition  $\tilde{C}_i^*$  as

$$\tilde{C}_i^* = \begin{bmatrix} \tilde{C}_i^+ \\ \tilde{c}_i^T \end{bmatrix}$$

where

$$\tilde{C}_i^+ = O_{(\delta_i-1) \times p}, \quad \tilde{c}_i^T = [0_{i-1}^T \ 1 \ 0_{p-1}^T]$$

and define

$$\Pi = \text{blockdiag}\{\Pi_1 \ \cdots \ \Pi_p\}$$

where each matrix  $T_{ij}$  has the dimensionality of  $\Phi_{ij}^*$ , and each matrix  $\Pi_i$  has the dimensionality of  $\Phi_{ii}^*$ , and may have one of the forms given in (3.10) (or its combination).

Next, define  $\hat{\Phi}^*$ ,  $\tilde{C}^+$ ,  $\hat{T}$ ,  $\tilde{\Phi}^*$ ,  $\tilde{C}$  and  $\tilde{T}$  as

$$\hat{\Phi}^* = [(\hat{\Phi}_{ij}^*)_{i=1,\dots,p}]_{j=1,\dots,p}, \quad \tilde{C}^+ = (\tilde{C}_i^+)_{i=1,\dots,p}, \quad \hat{T} = [(\hat{T}_{ij})_{i=1,\dots,p}]_{j=1,\dots,p} \tag{3.12a}$$

$$\tilde{\Phi}^* = [(-a_{ij}^T)_{i=1,\dots,p}]_{j=1,\dots,p}, \quad \tilde{C} = (\tilde{c}_i^T)_{i=1,\dots,p}, \quad \tilde{T} = [(\tilde{t}_{ij})_{i=1,\dots,p}]_{j=1,\dots,p} \tag{3.12b}$$

where parentheses define a column of blocks, and brackets  $[\ ]$  define a row of block columns, and apply a linear transformation upon (3.9) to obtain

$$\begin{bmatrix} \hat{\Phi}^* \\ \tilde{\Phi}^* \end{bmatrix} + \begin{bmatrix} \tilde{C}^+ \\ \tilde{C} \end{bmatrix} F^* = \begin{bmatrix} \hat{T} \\ \tilde{T} \end{bmatrix} \Pi T^{-1} \tag{3.13}$$

where it is noted that  $\tilde{C}^+ = O_{(n-p) \times p}$  and  $\tilde{C} = I_p$ . From (3.13) we obtain

$$\hat{\Phi}^* T = \hat{T} \Pi \tag{3.14}$$

$$F^* = -\tilde{\Phi}^* + \tilde{T} \Pi T^{-1}. \tag{3.15}$$

Equations (3.14) and (3.15) show that the problem of determining  $F^*$  and  $T$  has been decoupled, i. e. one first finds  $T$  from (3.14) and then  $F^*$  from (3.15). To find  $T$ , observe that

$$\hat{\Phi}^* T = \begin{bmatrix} \hat{\Phi}_{11}^* T_{11} & \hat{\Phi}_{11}^* T_{12} & \cdots & \hat{\Phi}_{11}^* T_{1p} \\ \hat{\Phi}_{22}^* T_{21} & \hat{\Phi}_{22}^* T_{22} & \cdots & \hat{\Phi}_{22}^* T_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\Phi}_{pp}^* T_{p1} & \hat{\Phi}_{pp}^* T_{p2} & \cdots & \hat{\Phi}_{pp}^* T_{pp} \end{bmatrix} \tag{3.16}$$

$$\hat{T} \Pi = \begin{bmatrix} \hat{T}_{11} \Pi_1 & \hat{T}_{12} \Pi_2 & \cdots & \hat{T}_{1p} \Pi_p \\ \hat{T}_{21} \Pi_1 & \hat{T}_{22} \Pi_2 & \cdots & \hat{T}_{2p} \Pi_p \\ \vdots & \vdots & \ddots & \vdots \\ \hat{T}_{p1} \Pi_1 & \hat{T}_{p2} \Pi_2 & \cdots & \hat{T}_{pp} \Pi_p \end{bmatrix}.$$

Hence, equation (3.14) reduces to

$$\hat{\Phi}_{ij}^* T_{ij} = \hat{T}_{ij} \Pi_j \quad (i, j = 1, \dots, p). \tag{3.17}$$

As it can be shown, the solution of (3.17) with regard to  $T_{ij}$  has the form

$$T_{ij} = \begin{bmatrix} \rho_{ij}^T \\ \rho_{ij}^T \Pi_j \\ \vdots \\ \rho_{ij}^T \Pi_j^{\delta_i-1} \end{bmatrix} \hat{=} (\rho_{ij}^T \Pi_j^k)_{k=0, \dots, \delta_i-1} \tag{3.18}$$

where  $\rho_{ij}^T$  is a  $\delta_j$ -dimensional row vector with arbitrary elements for all  $i, j = 1, 2, \dots, p$ . The general form of  $T$  will be

$$T = \begin{bmatrix} (\rho_{11}^T \Pi_1^k)_{k=0, \dots, \delta_1-1} & (\rho_{12}^T \Pi_2^k)_{k=0, \dots, \delta_1-1} & \cdots & (\rho_{1p}^T \Pi_p^k)_{k=0, \dots, \delta_1-1} \\ (\rho_{21}^T \Pi_1^k)_{k=0, \dots, \delta_2-1} & (\rho_{22}^T \Pi_2^k)_{k=0, \dots, \delta_2-1} & \cdots & (\rho_{2p}^T \Pi_p^k)_{k=0, \dots, \delta_2-1} \\ \vdots & \vdots & \ddots & \vdots \\ (\rho_{p1}^T \Pi_1^k)_{k=0, \dots, \delta_p-1} & (\rho_{p2}^T \Pi_2^k)_{k=0, \dots, \delta_p-1} & \cdots & (\rho_{pp}^T \Pi_p^k)_{k=0, \dots, \delta_p-1} \end{bmatrix}. \tag{3.19}$$

In (3.19), all elements of the first row of each block of  $T$  are arbitrary and hence we have a total number of arbitrary elements in  $T$  equal to  $n \times p$ . Note also that this arbitrariness is constrained by the requirement that  $T$  must be invertible, i. e.  $\det T \neq 0$ .

To find  $F^*$ , observe that relations (3.12), (3.18) and (3.19) yield

$$\tilde{T} = \begin{bmatrix} \rho_{11}^T \Pi_1^{\delta_1-1} & \rho_{12}^T \Pi_2^{\delta_1-1} & \cdots & \rho_{1p}^T \Pi_p^{\delta_1-1} \\ \rho_{21}^T \Pi_1^{\delta_2-1} & \rho_{22}^T \Pi_2^{\delta_2-1} & \cdots & \rho_{2p}^T \Pi_p^{\delta_2-1} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p1}^T \Pi_1^{\delta_p-1} & \rho_{p2}^T \Pi_2^{\delta_p-1} & \cdots & \rho_{pp}^T \Pi_p^{\delta_p-1} \end{bmatrix} \quad (3.20)$$

On the basis of (3.20), relation (3.15) yields

$$F^* = -\tilde{\Phi}^* + R\Pi^*T^{-1} \quad (3.21)$$

where

$$R = \text{blockdiag}\{\rho_1^T, \dots, \rho_p^T\}, \quad \Pi^* = \begin{bmatrix} \Pi^{\delta_1} \\ \vdots \\ \Pi^{\delta_p} \end{bmatrix} \quad (3.22)$$

with  $\rho_i^T = [\rho_{i1}^T \cdots \rho_{ip}^T]$ . Note that, when  $\Pi_j$  is in Jordan form,  $T_{ij}$  can take the form

$$T_{ij} = \begin{bmatrix} (\rho_{ij}^T)_1 & (\rho_{ij}^T)_2 & \cdots & (\rho_{ij}^T)_{s_j} \\ (\rho_{ij}^T)_1(J_j)_1 & (\rho_{ij}^T)_2(J_j)_2 & \cdots & (\rho_{ij}^T)_{s_j}(J_j)_{s_j} \\ \vdots & \vdots & \ddots & \vdots \\ (\rho_{ij}^T)_1(J_j)_1^{\delta_i-1} & (\rho_{ij}^T)_2(J_j)_2^{\delta_i-1} & \cdots & (\rho_{ij}^T)_{s_j}(J_j)_{s_j}^{\delta_i-1} \end{bmatrix} \quad (3.23)$$

where

$$\Pi_j = \text{blockdiag}\{(J_j)_1, \dots, (J_j)_{s_j}\}, \quad \rho_{ij}^T = [(\rho_{ij}^T)_1 \cdots (\rho_{ij}^T)_{s_j}]$$

in which  $(\rho_{ij}^T)_q$  is a row vector of dimensionality equal to that of  $(J_j)_q$ . In particular, if  $(J_j)_q = (\hat{\lambda}_j)_q$ , then  $s_j = \delta_j$ ,  $(\rho_{ij}^T)_q = \rho_{ijq}$

$$T_{ij} = \begin{bmatrix} \rho_{ij1} & \rho_{ij2} & \cdots & \rho_{ij\delta_j} \\ \rho_{ij1}(\hat{\lambda}_j)_1 & \rho_{ij2}(\hat{\lambda}_j)_2 & \cdots & \rho_{ij\delta_j}(\hat{\lambda}_j)_{s_j} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{ij1}(\hat{\lambda}_j)_1^{\delta_i-1} & \rho_{ij2}(\hat{\lambda}_j)_2^{\delta_i-1} & \cdots & \rho_{ij\delta_j}(\hat{\lambda}_j)_{\delta_j}^{\delta_i-1} \end{bmatrix} \quad (3.24)$$

It is remarked that, if we choose all the arbitrary elements  $\rho_{ij}$ , for  $i \neq j$  equal to 0, and all elements of each  $\rho_{ii}$  equal to 1, relation (3.21) can be written as

$$F^* = -\tilde{\Phi}^* + R_{sp}\Pi^*T_{sp}^{-1}. \quad (3.25)$$

In this case, the open-loop poles contained in the subsystem determined by  $\Phi_{ii}^*$ , when closing the feedback, are shifted to the desired poles involved in the corresponding

block  $\Pi_i$ . In order to determine the matrix  $K_f$ , substitute relation (3.21) in (3.8) to yield

$$K_f = Q^T(-\tilde{\Phi}^* + R_{sp}\Pi^*T_{sp}^{-1})^T L^T. \tag{3.26}$$

Using the matrix  $K_f$  as specified by (3.26), we can readily determine the controller matrix  $F(t)$ , by solving (2.3). Under Assumption 2.1, on the controllability of the pair  $(A, B)$ , a solution of (2.3) is the following [29]

$$F(t) = B^T \exp[A^T(T_0 - t)] W^{-1}(A, B, T_0) K_f \tag{3.27}$$

where,  $W(A, B, T_0)$  is the controllability Grammian on  $[0, T_0]$  of the pair  $(A, B)$ , which has the form

$$W(A, B, T_0) = \int_0^{T_0} \exp[A(T_0 - \lambda)] B B^T \exp[A^T(T_0 - \lambda)] d\lambda.$$

Note that, the controllability Grammian  $W(A, B, T_0)$  is nonsingular and hence a solution of (2.3) of the form (3.27) exists if the pair  $(A, B)$  is controllable.

On the basis of (3.26) and (3.27), a solution of the pole placement problem using GSHF control is given by

$$F(t) = B^T \exp[A^T(T_0 - t)] W^{-1}(A, B, T_0) Q^T(-\tilde{\Phi}^* + R_{sp}\Pi^*T_{sp}^{-1})^T L^T. \tag{3.28}$$

#### 4. A SOLUTION OF THE POLE PLACEMENT PROBLEM APPROPRIATE FOR THE ADAPTIVE CASE

In order to obtain a solution of the pole placement problem which will be more appropriate for application in the case of systems with unknown parameters, we slightly modify in the sequel the control strategy of Figure 1 as it is depicted in Figure 2. In particular, we focus our attention on the special class of the time-varying  $T_0$ -periodic matrix functions  $F(t)$ , for which every element of  $F(t)$ , denoted by  $f_{ij}(t)$ , is piecewise constant over intervals of length  $T_i = T_0/N_i$ , i. e.

$$f_{ij}(t) = f_{ij,\mu}, \quad \forall t \in [\mu T_i, (\mu + 1) T_i], \quad \mu = 0, 1, \dots, N_i - 1. \tag{4.1}$$

Moreover, the persistent excitation signals  $v_i(t)$ ,  $\forall i \in J_m$ , are defined as

$$v_i(t) = d_i^T(t) v_i, \quad d_i^T(t) = [(d_i)_0(t) \cdots (d_i)_{N_i-1}(t)]. \tag{4.2a}$$

Here,  $d_i(t)$  is the  $T_i$ -periodic vector function with elements having the form

$$(d_i)_q(t) = (d_i)_{q,\mu}, \quad \text{for } t \in [\mu T_i, (\mu + 1) T_i], \quad q = 0, 1, \dots, N_i - 1, \mu = 0, 1, \dots, N_i - 1 \tag{4.2b}$$

where  $(d_i)_{q,\mu}$  are constant taking the following values

$$(d_i)_{q,\mu} = \begin{cases} 1, & \text{for } \mu = q \\ 0, & \text{for } \mu \neq q \end{cases} \tag{4.3}$$

and where  $v_i$  is as yet unknown. It is worth noticed that the additive term  $v_i(t) = d_i^T(t) v_i, \forall i \in J_m$ , in each one of the inputs of the continuous-time system, are used only for identification purposes and as it will be shown later, they are selected so that they will not influence the pole placement problem.

We are now able to establish the following Lemma.

**Lemma 4.1.** Consider the controllable and observable system of the form (2.1), controlled by GSHF of the form (4.1). Furthermore, consider that persistent excitation signals of the form (4.2), (4.3) are introduced in each input of the system. Then, the sampled closed-loop system takes the form

$$x[(k+1)T_0] = (\Phi + \widehat{B}\widehat{F}C)x(kT_0) + \widetilde{B}w(kT_0) + B^*v, \quad y(kT_0) = Cx(kT_0), \quad \text{for } k \geq 0 \tag{4.4}$$

where

$$\widehat{B} \hat{=} [\widehat{b}_1 \cdots \widehat{A}_1^{N_1-1}\widehat{b}_1 \cdots \widehat{b}_m \cdots \widehat{A}_m^{N_m-1}\widehat{b}_m] \tag{4.5a}$$

$$\widehat{A} \hat{=} \exp(AT_i) \equiv \exp(A1_iT_N),$$

$$\widehat{b}_i \hat{=} \int_0^{T_i} \exp(A\lambda) b_i d\lambda \equiv \int_0^{1_iT_N} \exp(A\lambda) b_i d\lambda, \tag{4.5b}$$

$$1_i = N/N_i, \quad T_N = T_0/N$$

$$B^* = \widehat{B}M, \quad M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_m \end{bmatrix}, \quad M_j = \begin{bmatrix} e_{\sigma_j} \\ e_{\sigma_j-1} \\ \vdots \\ e_{\sigma_j-N_j+1} \end{bmatrix} \tag{4.5c}$$

and where the  $m \times p$  block matrix  $\widehat{F}$  and the column vector  $v \in \mathbb{R}^{N^*}, N^* = \sum_{i=1}^m N_i$ , have the forms

$$\widehat{F} = \begin{bmatrix} \widehat{f}_{11} & \cdots & \widehat{f}_{1p} \\ \vdots & \ddots & \vdots \\ \widehat{f}_{m1} & \cdots & \widehat{f}_{mp} \end{bmatrix}, \quad \widehat{f}_{ij} = \begin{bmatrix} f_{ij,N_i-1} \\ \vdots \\ f_{ij,0} \end{bmatrix} \tag{4.6a}$$

$$v = [v_1^T \quad v_2^T \quad \cdots \quad v_m^T]^T \tag{4.6a}$$

while  $\sigma_j = \sum_{\kappa=1}^j N_\kappa$ , where in general, the vector  $e_i \in \mathbb{R}^{N^*}$  is the row vector whose elements are zeros except for a unity appearing in the  $i$ th position.

**Proof.** To show that the sampled closed-loop system takes the form (4.4), we start by discretizing system (2.1) with sampling period  $T_0$ . This operation yields

$$x[(k+1)T_0] = \Phi x(kT_0) + \int_{kT_0}^{(k+1)T_0} \exp\{A[(k+1)T_0 - \lambda]\} Bu(\lambda) d\lambda. \tag{4.7}$$

Observing that  $u_i(t) = r_i(t) + d_i^T(t) v_i$  and taking into account the structure of the control system in Figure 2, we obtain

$$u_i(t) = f_i^T(t) y(kT_0) + w_i(kT_0) + d_i^T(t) v_i, \quad \text{for } t \in [\mu T_i, (\mu + 1) T_i] \quad (4.8)$$

where  $f_i^T(t)$  is the  $i$ th row of the controller matrix  $F(t)$  and  $w_i(kT_0)$  is the  $i$ th element of  $w(kT_0)$ . Combining relations (4.7) and (4.8), we obtain the following relationship

$$x[(k + 1) T_0] = (\Phi + K_f C) x(kT_0) + \tilde{B} w(kT_0) + \Gamma v \quad (4.9)$$

where

$$\Gamma = \int_{kT_0}^{(k+1)T_0} \exp\{A[(k + 1) T_0 - \lambda]\} B D(\lambda) d\lambda, \quad D(t) \hat{=} \text{blockdiag} \{d_i^T(t)\}_{i \in J_m}.$$

Now, partition  $\Gamma$  as follows

$$\Gamma = [\Gamma_1 \ \Gamma_2 \ \cdots \ \Gamma_m].$$

Then, the  $(q + 1)$ th column of the matrix  $\Gamma_i$ , for  $i \in J_m$ , denoted by  $(\Gamma_i)_{q+1}$ , for  $q = 0, 1, \dots, N_i - 1$ , can be expressed as

$$(\Gamma_i)_{q+1} = \int_0^{T_0} \exp[A(T_0 - \lambda)] b_i(d_i)_q(\lambda) d\lambda, \quad \text{for } q = 0, 1, \dots, N_i - 1. \quad (4.10)$$

Introducing relations (4.2a) and (4.3) in (4.10), we obtain

$$(\Gamma_i)_{q+1} = \sum_{\mu=0}^{N_i-1} \int_{\mu T_i}^{(\mu+1)T_i} \exp[A(T_0 - \lambda)] b_i(d_i)_{q,\mu} d\lambda, \quad \text{for } q = 0, 1, \dots, N_i - 1. \quad (4.11)$$

Relation (4.11) may further be written as

$$\begin{aligned} (\Gamma_i)_{q+1} &= \sum_{\mu=0}^{N_i-1} (d_i)_{q,\mu} \exp[A(N_i - \mu - 1)] T_i \int_0^{T_i} \exp[A(T_i - \lambda)] b_i d\lambda \\ &= \left( \sum_{\rho=1}^{N_i} (d_i)_{q, N_i - \rho} \hat{A}_i^{\rho-1} \right) \hat{b}_i. \end{aligned}$$

Making use of relation (4.3), we arrive at the following relationship

$$(\Gamma_i)_{q+1} = \hat{A}_i^{N_i - q - 1} \hat{b}_i.$$

Clearly  $\Gamma \equiv B^*$ . Application of the above algorithm to the first term of (4.9) yields  $K_f \equiv \hat{B} \hat{F}$  (see [4] for details). This completes the proof of the Lemma.  $\square$

Thus far, we have established that the pole placement controller matrix  $K_f$  is related to the matrix  $\hat{F}$  via the relation  $K_f = \hat{B} \hat{F}$ . It remains to determine  $\hat{F}$ . To this end, we need the following result, whose proof is given in [4].

**Lemma 4.2.** Let  $(A, B)$  be a controllable pair. Let also  $n_i, i \in J_m$  be a set of locally minimum controllability indices of the pair  $(A, B)$ . Define an analytic function  $\psi(T_N)$  by

$$\psi(T_N) = \det \left[ \widehat{b}_1 \cdots \widehat{A}_1^{n_1-1} \widehat{b}_1 \cdots \widehat{b}_m \cdots \widehat{A}_m^{n_m-1} \widehat{b}_m \right].$$

Then the set of zeros of  $\psi(T_N)$  does not have any limiting points except infinity, and therefore,  $\psi(T_N)$  is not equal to zero for almost all  $T_N$  (i.e., in a finite interval  $[T_N^1, T_N^2]$ , there are at most a finite number of points such that  $\psi(T_N) = 0$ ).

Applying Lemma 4.2, we can conclude that the matrix of the form

$$\widehat{S} = \left[ \widehat{b}_1 \cdots \widehat{A}_1^{n_1-1} \widehat{b}_1 \cdots \widehat{b}_m \cdots \widehat{A}_m^{n_m-1} \widehat{b}_m \right] \tag{4.12}$$

is nonsingular for almost all  $T_N \in [T_N^1, T_N^2]$ . Furthermore, if the input multiplicities of the sampling  $N_i$  are chosen such that  $N_i \geq n_i, i \in J_m$  then, the matrices  $\widehat{B}$  and  $B^*$  have full row rank  $n$  for almost all  $T_N \in [T_N^1, T_N^2]$ .

Now, let  $E \in \mathbb{R}^{N^* \times N^*}$  be the nonsingular permutation matrix with the property  $E^{-1} \equiv E^T$ , having the form

$$E = [E_1 \ E_2]^T$$

where

$$E_1 = [\varepsilon_1 \ \varepsilon_2 \cdots \varepsilon_{n_1} \ \varepsilon_{N_1+1} \ \varepsilon_{N_1+2} \cdots \varepsilon_{N_1+n_2} \cdots \varepsilon_{N^*-N_m+1} \ \varepsilon_{N^*-N_m+2} \cdots \varepsilon_{N^*-N_m+n_m}]$$

and

$$E_2 = [\varepsilon_{n_1+1} \cdots \varepsilon_{N_1} \ \varepsilon_{N_1+n_2+1} \cdots \varepsilon_{N_1+N_2} \cdots \varepsilon_{N^*-N_m+n_m+1} \cdots \varepsilon_{N^*}]$$

where, in general,  $\varepsilon_j \in \mathbb{R}^{N^*}$  is the column vector whose elements are zeros except for a unity appearing in the  $j$ th position. Also, let

$$B^\# \triangleq \widehat{B}E^{-1} = [\widehat{S} \ \widehat{Q}]$$

where the matrix  $\widehat{S}$  is defined by (4.13) and matrix  $\widehat{Q}$  is given by

$$\widehat{Q} = \left[ \widehat{A}_1^{n_1} \widehat{b}_1 \cdots \widehat{A}_1^{N_1-1} \widehat{b}_1 \cdots \widehat{A}_m^{n_m} \widehat{b}_m \cdots \widehat{A}_m^{N_m-1} \widehat{b}_m \right].$$

Furthermore, let  $\Delta \in \mathbb{R}^{N^* \times N^*}$  be the nonsingular permutation matrix with the property  $\Delta^{-1} \equiv \Delta^T$ , having the form

$$\Delta = [\Delta_1 \ \Delta_2 \ \Delta_3]^T$$

where

$$\Delta_1 = [\varepsilon_{1-n_1+1} \cdots \varepsilon_{N_1} \ \varepsilon_{N_1+N_2-n_2+1} \cdots \varepsilon_{N_1+N_2} \cdots \varepsilon_{N^*-n_m+1} \cdots \varepsilon_{N^*}]$$

$$\Delta_2 = [\varepsilon_{N_1-n_1} \ \varepsilon_{N_1+N_2-n_2} \cdots \varepsilon_{N^*-n_m}]$$

$$\Delta_3 = [\varepsilon_1 \cdots \varepsilon_{N_1-n_1-1} \ \varepsilon_{N_1+1} \cdots \varepsilon_{N_1+N_2-n_2-1} \cdots \varepsilon_{N^*-N_m+1} \cdots \varepsilon_{N^*-n_m-1}].$$

Finally, let

$$\tilde{B}\# \triangleq B^* \Delta^{-1} \equiv \left[ \hat{S}^* \quad \hat{A}_1^{n_1} \hat{b}_1 \quad \dots \quad \hat{A}_m^{n_m} \hat{b}_m \quad \hat{Q}^* \right]$$

where

$$\begin{aligned} \hat{S}^* &= \left[ \hat{A}_1^{n_1-1} \hat{b}_1 \quad \dots \quad \hat{b}_1 \quad \dots \quad \hat{A}_m^{n_m-1} \hat{b}_m \quad \dots \quad \hat{b}_m \right] \\ \hat{Q}^* &= \left[ \hat{A}_1^{N_1-1} \hat{b}_1 \quad \dots \quad \hat{A}_1^{n_1+1} \hat{b}_1 \quad \dots \quad \hat{A}_m^{N_m-1} \hat{b}_m \quad \dots \quad \hat{A}_m^{n_m+1} \hat{b}_m \right]. \end{aligned} \tag{4.13}$$

Using these definitions, it is plausible to determine  $\hat{F}$  by mere inspection, as

$$\hat{F} \triangleq E^T \left[ \hat{S}^{-1} Q^T \begin{pmatrix} -\tilde{\Phi}^* + R_{sp} \Pi^* T_{sp}^{-1} \\ 0 \end{pmatrix}^T \quad L^T \right]. \tag{4.14}$$

It only remains to determine the appropriate vector  $v$  which guarantees that the pole placement problem will not be dependent on the vector  $v$ . In other words

$$v \in \ker B^* \quad \text{or} \quad B^* v = 0.$$

An obvious selection of such  $v$  obtained also by inspection is the following

$$v = \Delta^T \begin{bmatrix} -\hat{S}^{*-1} (\hat{A}_1^{n_1} \hat{b}_1 + \dots + \hat{A}_m^{n_m} \hat{b}_m) \\ \zeta \\ O_{N^*-n-m} \end{bmatrix} \tag{4.15}$$

where  $\zeta \in \mathbb{R}^m$  is the column vector whose elements are all equal to 1.

It is noted that the  $N^*$ -dimensional column vector  $v$ , eventhough does not affects the discrete pole placement problem, it provides persistent excitation useful for the consistent identification of the system, as will be shown in the following section.

Clearly, the modulating matrix function  $F(t)$  of Figure 2 can readily be determined by making use of relations (4.1), (4.6a), (4.14) and (3.28). More precisely, the  $i$ th row  $f_i^T(t)$  of the matrix  $F(t)$  and the  $i$ th block row of the matrix  $\hat{F}$  are interrelated as

$$f_i^T(t) = [f_{i1}(t) \dots f_{ip}(t)] = e_{N_i-\mu} \left[ \hat{f}_{i1} \dots \hat{f}_{ip} \right], \quad \forall \frac{\mu T_0}{N_i} \leq t < \frac{(\mu+1)T_0}{N_i} \tag{4.16}$$

for  $i \in J_m$  and for  $\mu = 0, 1, \dots, N_i - 1$ , where  $e_{N_i-\mu} \in \mathbb{R}^{N_i}$  is the row vector defined as  $e_{N_i-\mu} = \varepsilon_{N_i-\mu}^T$ . Note that, the controller matrix  $F(t)$ , as specified by (4.16), is largely affected by the multirate mechanism, while the controller matrix  $F(t)$  as specified by relation (3.28) is not. Furthermore, the introduction of the excitation signals  $v_i(t)$  in the control loop, greatly facilitates the consistent estimation of the plant parameters in the case of unknown systems. For these reasons, the control strategy of Figure 2 is more appropriate than the control strategy of Figure 1 for the development of the indirect adaptive control scheme presented in the following section.



### 5. CONTROL STRATEGY FOR THE ADAPTIVE CASE

The control scheme presented in Section 4 has a corresponding scheme in the case where the system is unknown. For this case, the control strategy is largely based on the computation of the matrix  $\widehat{F}$  and of the vector  $v$  from estimates of the plant parameters, and results in a globally stable closed-loop system whose poles are located to the prespecified values  $\widehat{\lambda}_1, \widehat{\lambda}_2, \dots, \widehat{\lambda}_n$ .

#### 5.1. Plant parameters estimation algorithm

The algorithm proposed here for estimating the unknown plant parameters is as follows: System (2.1), discretized with sampling period  $\tau = T_0/(6n - 1)N$ , takes the form

$$x[(v + 1)\tau] = \Phi_\tau x(v\tau) + B_\tau u(v\tau), \quad y(v\tau) = Cx(v\tau), \quad v \geq 0 \quad (5.1)$$

where

$$\Phi_\tau = \exp(A\tau), \quad B_\tau = \int_0^\tau \exp(A\lambda) B \, d\lambda.$$

Clearly,  $u(v\tau)$  takes constant values for  $v\tau \in [\rho T_N, (\rho + 1)T_N]$ ,  $\rho \geq 0$ . This can be easily shown by taking into account the action of the proposed controller. Hence, iterating relation (5.1)  $6n - 1$  times, we obtain

$$x[(m + 1)T_N] = \Phi_{T_N} x(mT_N) + B_{T_N} u(mT_N), \quad m \geq 0$$

where

$$\Phi_{T_N} = (\Phi_\tau)^{6n-1}, \quad B_{T_N} = \sum_{\rho=0}^{6n-2} \Phi_\tau^\rho B_\tau. \quad (5.2)$$

Using the same argument, we can easily conclude that

$$\widehat{A}_i = \Phi_{T_N}^{1_i}, \quad \widehat{b}_i = \sum_{\rho=0}^{1_i-1} \Phi_{T_N}^\rho (B_{T_N})_i \quad (5.3)$$

where  $(B_{T_N})_i$  is the  $i$ th column of the matrix  $B_{T_N}$ . Introducing relation (5.2) in (5.3), yields

$$\widehat{A}_i = (\Phi_\tau)^{(6n-1)1_i}, \quad \widehat{b}_i = \sum_{j=0}^{1_i-1} (\Phi_\tau)^{(6n-1)j} \left( \sum_{\rho=0}^{(6n-2)} \Phi_\tau^\rho B_\tau \right)_i. \quad (5.4)$$

Moreover, the matrix  $\Phi$  can be written as

$$\Phi = \widehat{A}_i^{N_i} = \Phi_{T_N}^N = (\Phi_\tau)^{(6n-1)N}. \quad (5.5)$$

Therefore,  $\Phi$ ,  $\widehat{A}$  and  $\widehat{b}_i$  (which are the only matrices involved in computing  $\widehat{F}$  and  $v$ ) can be computed on the basis of  $\Phi_\tau$  and  $B_\tau$ . For this reason, in what follows

our aim will be the estimation of the matrix triplet  $(\Phi_\tau, B_\tau, C)$ . To this end, let the matrix  $\Omega$  be defined as

$$\Omega = \{\Omega_{ij}\}_{i=1,2,\dots,n}^{j=1,2,\dots,n}, \quad \Omega_{ij} = C\Phi_\tau^{i+j-2}B_\tau. \tag{5.6}$$

Clearly, if one establishes estimates of the matrix  $\Omega$ , then one may easily compute the desired matrix triplet  $(\Phi_\tau, B_\tau, C)$ , using anyone of the minimal realization algorithms reported in the literature (see for example those reported in [18, 28, 42]). To estimate matrix  $\Omega$ , one must resort to an input-output representation (also called ARMA representation) of system (5.1). This representation is summarized in the following Theorem:

**Theorem 5.1.** Suppose that there is a sampling period  $T_0 \in \mathbb{R}^+$  and input multiplicities of the sampling  $N_i, i \in J_m$ , such that system (5.1), obtained by sampling the controllable and observable system (2.1), is also controllable and observable. Then, an alternative representation of system (5.1), is given by

$$\Psi(v\tau) = J_1\Psi[(v - 2n)\tau] + J_2W(v\tau) + VW[(v - n)\tau] + V^*W[(v - 2n)\tau] \tag{5.7}$$

where

$$\Psi(v\tau) = \begin{bmatrix} y[(v - n + 1)\tau] \\ y[(v - n + 2)\tau] \\ \vdots \\ y(v\tau) \end{bmatrix}, \quad \Psi[(v - 2n)\tau] = \begin{bmatrix} y[(v - 3n + 1)\tau] \\ y[(v - 3n + 2)\tau] \\ \vdots \\ y[(v - 2n)\tau] \end{bmatrix}, \tag{5.8a}$$

$$W(v\tau) = \begin{bmatrix} u[(v - n + 1)\tau] \\ u[(v - n + 2)\tau] \\ \vdots \\ u(v\tau) \end{bmatrix}$$

$$W[(v - n)\tau] = \begin{bmatrix} u[(v - 2n + 1)\tau] \\ u[(v - 2n + 2)\tau] \\ \vdots \\ u[(v - n)\tau] \end{bmatrix}, \tag{5.8b}$$

$$W[(v - 2n)\tau] = \begin{bmatrix} u[(v - 3n + 1)\tau] \\ u[(v - 3n + 2)\tau] \\ \vdots \\ u[(v - 2n)\tau] \end{bmatrix}$$

$$J_1 = \Xi^{*-1} \begin{bmatrix} \hat{J} & 0 \\ 0 & 0 \end{bmatrix} \Xi^*, \quad J_2 = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ CB_\tau & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ C\Phi_\tau^{n-2}B_\tau & \cdots & CB_\tau & 0 \end{bmatrix}, \tag{5.8c}$$

$$V = P^*\Sigma \quad \text{and} \quad V^* = \Xi^{*-1} \begin{bmatrix} V^+ \\ 0 \end{bmatrix}$$

and where

$$\hat{J} = P_1^* \Phi_\tau^{2n} P_1^{*-1}, \quad P^* = \begin{bmatrix} C \\ C\Phi_\tau \\ \vdots \\ C\Phi_\tau^{n-1} \end{bmatrix}, \quad \Sigma = [\Phi_\tau^{n-1} B_\tau \cdots \Phi_\tau B_\tau \ B_\tau] \quad (5.9a)$$

$$V^+ = P_1^* \Phi_\tau^n [\Sigma - \Phi_\tau^n P_1^{*-1} U_1] \quad (5.9b)$$

while the nonsingular permutation matrix  $\Xi^* \in \mathbb{R}^{np \times np}$ , is such that  $\Xi^* \in \mathbb{R}^{np \times np}$ , is such that

$$\Xi^* P^* = \begin{bmatrix} P_1^* \\ 0 \end{bmatrix} \quad (5.10)$$

where  $P_1^* \in \mathbb{R}^{n \times n}$  is the nonsingular matrix whose rows are the linearly independent rows of the matrix  $P^*$ . Finally,  $U_1 \in \mathbb{R}^{n \times np}$  is the matrix containing the first  $n$  rows of the matrix

$$U = \Xi^* J_2. \quad (5.11)$$

*Proof.* In order to prove relation (5.7), we next generalize the approach presented in [35], to the multivariable case. More precisely, from relations (5.1) we have

$$\begin{aligned} y[(v-n+1)\tau] &= Cx[(v-n+1)\tau] \\ y[(v-n+2)\tau] &= C\Phi_\tau x[(v-n+1)\tau] + CB_\tau u[(v-n+1)\tau] \\ &\vdots \\ y(v\tau) &= C\Phi_\tau^{n-1} x[(v-n+1)\tau] + \sum_{\rho=0}^{n-2} C\Phi_\tau^\rho B_\tau u[(v-\rho-1)\tau] \end{aligned}$$

or more compactly,

$$\Psi(v\tau) = P^* x[(v-n+1)\tau] + J_2 W(v\tau) \quad (5.12)$$

where,  $\Psi(v\tau)$  and  $W(v\tau)$  are defined by (5.8a) and  $P^*$  and  $J_2$  are defined by (5.9a) and (5.8c), respectively.

Since, by Assumption 2.2, the pair  $(\Phi_\tau, C)$  is observable, the matrix  $P^*$  has full column rank. Hence, there exists a nonsingular permutation matrix  $\Xi^* \in \mathbb{R}^{np \times np}$ , such that relation (5.10) to hold, where, as already mentioned,  $P_1^* \in \mathbb{R}^{n \times n}$  is the nonsingular matrix whose rows are the linearly independent rows of the matrix  $P^*$ . It is pointed out that matrix  $\Xi^*$  can be defined as a product of two nonsingular

matrices  $\tilde{\Xi} \in \mathbb{R}^{np \times np}$  and  $\hat{\Xi} \in \mathbb{R}^{np \times np}$  via the following chain of definitions

$$\Xi^* = \tilde{\Xi} \hat{\Xi}, \quad \tilde{\Xi} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \\ \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{np-n} \end{bmatrix}, \quad \hat{\Xi} = \begin{bmatrix} e_{j_1} \\ e_{j_2} \\ \vdots \\ e_{j_n} \\ \Xi_1^+ \end{bmatrix}$$

where  $\Xi_1^+ \in \mathbb{R}^{(np-n) \times np}$  is the matrix produced by the nonsingular matrix  $\Xi^+ \in \mathbb{R}^{np \times np}$  of the form

$$\Xi^+ = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{np} \end{bmatrix}$$

by dropping the row vectors  $e_i$ ,  $i = j_1, j_2, \dots, j_n$ , where  $j_1, j_2, \dots, j_n$  are the indices of the  $n$  linearly independent rows of  $P^*$  defined as  $p_{j_\rho}^{*T}$ ,  $\rho = 1, 2, \dots, n$ . Note also that  $\omega_k \in \mathbb{R}^{np}$ ,  $k = 1, 2, \dots, np - n$  is the column vector of the form

$$\omega_k = \begin{bmatrix} (\lambda_{j_1})_k & (\lambda_{j_2})_k & \cdots & (\lambda_{j_n})_k & 0 & \cdots & 0 & \underbrace{-1}_{(n+k)\text{th position}} & 0 \cdots 0 \end{bmatrix}$$

where  $(\lambda_{j_\rho})_k$ ,  $\rho = 1, 2, \dots, n$ ,  $k = 1, 2, \dots, np - n$  are the coefficients of the following dependence relation holding for the rows of the matrix  $P^*$

$$\sum_{\rho=1}^n (\lambda_{j_\rho})_k p_{j_\rho}^{*T} - p_k^{*T} = 0, \quad k \notin \{j_1, j_2, \dots, j_n\}$$

where,  $p_k^{*T}$ ,  $k \notin \{j_1, j_2, \dots, j_n\}$  is the  $k$ th row of the matrix  $P^*$ .

Now, multiplying (5.12) from the left by  $\Xi^*$ , yields

$$Z^*(v\tau) = \begin{bmatrix} P_1^* \\ 0 \end{bmatrix} x[(v - n + 1)\tau] + UW(v\tau)$$

where

$$Z^*(v\tau) = \Xi^* \Psi(v\tau) \tag{5.13}$$

and where  $U$  is defined by (5.11). Next, decompose  $Z^*(v\tau)$  and  $U$  as follows

$$Z^*(v\tau) = \begin{bmatrix} Z_1^*(v\tau) \\ Z_2^*(v\tau) \end{bmatrix}, \quad U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{5.14}$$

where  $Z_1^*(v\tau) \in \mathbb{R}^n$ ,  $Z_2^*(v\tau) \in \mathbb{R}^{n(p-1)}$ ,  $U_1 \in \mathbb{R}^{n \times np}$  and  $U_2 \in \mathbb{R}^{n(p-1) \times np}$ . Clearly,

$$Z_1^*(v\tau) = P_1^* \xi[(v - n + 1)\tau] + U_1 W(v\tau) \quad \text{and} \quad Z_2^*(v\tau) = U_2 W(v\tau). \quad (5.15)$$

From (5.15), one may easily obtain the following relation

$$x[(v - n + 1)\tau] + P_1^{*-1}[Z_1^*(v\tau) - U_1 W(v\tau)]. \quad (5.16)$$

Furthermore, as it can be easily shown, the following relationship holds

$$x[(v - n + 1)\tau] = \Phi_\tau^{2n} x[(v - 3n + 1)\tau] + \Phi_\tau^n \Sigma W[(v - 2n)\tau] + \Sigma W[(v - n)\tau] \quad (5.17)$$

where  $W[(v - n)\tau]$  and  $W[(v - 2n)\tau]$  are given by (5.8b), and where  $\Sigma$  is defined by (5.9a). Introducing appropriately relation (5.16) in relation (5.17), after some algebraic manipulations, yields

$$Z_1^*(v\tau) = U_1 W(v\tau) + \hat{J} Z_1^*[(v - 2n)\tau] + V^+ W[(v - 2n)\tau] + P_1^* \Sigma W[(v - n)\tau] \quad (5.18)$$

where  $\hat{J}$  and  $V^+$ , are defined as by (5.9a) and (5.9b), respectively. Combining relations (5.11), (5.13)–(5.15) and (5.18), we readily obtain (5.7). This completes the proof of the Theorem.  $\square$

It is remarked at this point that matrix  $V$  and matrix  $\Omega$  are related through the following relationship

$$\Omega = V\Lambda, \quad \Lambda = \begin{bmatrix} 0 & \dots & 0 & I \\ 0 & \dots & I & 0 \\ \vdots & \ddots & \vdots & \vdots \\ I & \dots & 0 & 0 \end{bmatrix}. \quad (5.19)$$

Relation (5.7) will be used in the sequel for the identification of the unknown matrices  $J_1, J_2, V$  and  $V^*$ . To this end, relation (5.7) is next written in the linear regression form

$$\Psi(v\tau) = \Theta \phi(v\tau)$$

where

$$\Theta = [J_1 \quad J_2 \quad V \quad V^*]$$

is the true value of the plant parameter matrix, and where

$$\phi^T(v\tau) = [\Psi^T[(v - 2n)\tau] \quad W^T(v\tau) \quad W[(v - n)\tau] \quad W^T[(v - 2n)\tau]].$$

Next, define

$$\begin{aligned} Z(kT_0) &= [\phi(kT_0) \quad \phi(kT_0 - \tau) \quad \dots \quad \phi[(k - 1)T_0]] \\ Y(kT_0) &= [\Psi(kT_0) \quad \Psi(kT_0 - \tau) \quad \dots \quad \Psi[(k - 1)T_0]] \\ \hat{\Theta}(kT_0) &= [J_1(kT_0) \quad J_2(kT_0 - \tau) \quad V(kT_0) \quad V^*(kT_0)] \end{aligned}$$

where  $J_1(kT_0)$ ,  $J_2(kT_0)$ ,  $V(kT_0)$  and  $V^*(kT_0)$ , are the matrices  $J_1, J_2, V$  and  $V^*$  evaluated at  $kT_0$ , through the identification procedure. Clearly, the following relation holds

$$Y(kT_0) = \Theta Z(kT_0).$$

We now choose the recursive algorithm for the estimation of  $\hat{\Theta}(kT_0)$  as

$$\begin{aligned} \hat{\Theta}(kT_0) = & \hat{\Theta}[(k-1)T_0] - \left[ \hat{\Theta}[(k-1)T_0] Z[(k-1)T_0] - Y[(k-1)T_0] \right] \\ & \times Z^T[(k-1)T_0] [\alpha I + Z[(k-1)T_0] Z^T[(k-1)T_0]]^{-1} \quad (5.20) \end{aligned}$$

where  $\alpha \in \mathbb{R}^+$  is arbitrary,  $\hat{\Theta}(kT_0)$  is estimated parameter matrix  $\hat{\Theta}$  at time  $t = kT_0$  and  $\hat{\Theta}_0 = \hat{\Theta}(kT_0)|_{k=0}$  is arbitrarily specified. It is pointed out that the term  $\alpha I$  in (5.20), is added in order to avoid numerical ill conditioning, arising in the identification procedure based on the usual least-squares algorithm, when the determinant of the matrix  $Z [(k-1)T_0] Z^T(k-1)$  takes small values.

Commenting on the nature of the adaptive law (5.20), we point out that, it describes an on-line estimation procedure which deals with sequential data and in which the parameter estimates are recursively updated within the time-limit imposed by the sampling period  $T_0$ . It is worth noticed, at this point that, in the present case, it is presumed that, a complete block of information needed for the estimation of the plant parameters, is *not* available prior to analysis and control, as in several off-line estimation procedures. Therefore, in our case, identification and control of the plant are performed concurrently. In order to calculate the parameters of the desired GSHF based pole placement controller, it is necessary here to update the plant parameter estimates using (5.20) and then solve the canonical equations of Sections 3 and 4 for every time step  $k$  (see the following subsection for details). This is in contrast, to the standard policy followed in cases where identification and control of the plant are performed separately, in which we solve equations for the plant and the controller parameters once, after an appropriate minimum number of observations on the basis of which, a fixed model for the controlled plant is available for further analysis (see [24, 41] for a comparative study of the two approaches).

It is worth of noticing, at this point, that, although exact solutions to the equation schemes of the paper are possible, the convergence of the identification procedure is crucial for our analysis. This is due to the fact that the adaptive law (5.20) is chosen so that  $\hat{\Theta}(kT_0)$  will satisfy equation  $Y(kT_0) = \Theta Z(kT_0)$  ( $k \geq 0$ ) asymptotically with time, i. e., for  $k \rightarrow \infty$ , rather than at every time instant. In other words, in the early stages of the on-line identification procedure, the estimated parameter matrix  $\hat{\Theta}(kT_0)$ , obtained by (5.20), is usually far from its true value  $\Theta$  and it is expected that the plant parameter estimates (and consequently the controller parameter estimates) converge to their true values, only as  $k \rightarrow \infty$ . Therefore, exact determination of the desired GSHF based pole placement controller through the procedures presented in Sections 3 and 4, is expected here, only after a certain step of the overall control procedure. Before this step, the calculated controllers are far from being those, which guarantee the desired performance of the closed-loop system. However, it is a standard fact in all adaptive control schemes that, convergence

of the parameter estimates to their true values, depends on the specific properties of the particular identification procedure used and crucially affects the adaptation, since, in cases where convergence of the estimated parameters to their true values, is not guaranteed, either the calculated controllers are not the admissible ones or they cannot be computed (for instance, if  $\hat{\Theta}(kT_0)$ , as obtained by the identification, is unbounded). So the effectiveness of our method, depends on the convergence and the boundedness properties of the of the proposed identification procedure. These properties are summarized in the following Proposition.

**Proposition 5.1.** Let  $\tilde{\Theta}(kT_0)$  be the parameter estimation error, defined as

$$\tilde{\Theta}(kT_0) = \hat{\Theta}(kT_0) - \Theta^T. \tag{5.21}$$

Then, for the parameter estimation algorithm of the form (5.20), the following properties hold

- (a)  $\|\hat{\Theta}(kT_0)\| \leq \mu$ , for some finite  $\mu \in \mathbb{R}^+$ .
- (b) If  $\lim_{k \rightarrow \infty} \sum_{\rho=0}^k \lambda_{\min}(Z(\rho T_0) Z^T(\rho T_0)) = \infty$  then  $\lim_{k \rightarrow \infty} \hat{\Theta}(kT_0) = \Theta$

where  $\lambda_{\min}(\cdot)$  denotes the minimum eigenvalue of a matrix.

*Proof.* (a) Taking the transpose of both sides in (5.20), introducing (5.21) in the resulting relation and taking into account the fact that  $Z^T(kT_0)\Theta^T - Y^T(kT_0) = 0$ , we readily obtain

$$\begin{aligned} \tilde{\Theta}(kT_0) &= \{I - [\alpha I + Z[(k-1)T_0] Z^T[(k-1)T_0]]^{-1} Z[(k-1)T_0] Z^T[(k-1)T_0]\} \\ &\quad \tilde{\Theta}[(k-1)T_0]. \end{aligned} \tag{5.22}$$

On the basis of the Matrix Inversion Lemma, relation (5.22) may further be written as

$$\tilde{\Theta}(kT_0) = \left\{ I + \frac{1}{\alpha} Z[(k-1)T_0] Z^T[(k-1)T_0] \right\}^{-1} \tilde{\Theta}[(k-1)T_0]. \tag{5.23}$$

Therefore,

$$\begin{aligned} &\tilde{\Theta}^T(kT_0) \tilde{\Theta}(kT_0) \\ &= \tilde{\Theta}^T[(k-1)T_0] \left\{ I + \frac{1}{\alpha} Z[(k-1)T_0] Z^T[(k-1)T_0] \right\}^{-2} \tilde{\Theta}[(k-1)T_0] \\ &\leq \left( 1 + \frac{\lambda_{\min}(Z[(k-1)T_0] Z^T[(k-1)T_0])}{\alpha} \right)^{-2} \tilde{\Theta}^T[(k-1)T_0] \tilde{\Theta}[(k-1)T_0]. \end{aligned} \tag{5.24}$$

By repeatedly using the above inequality, we obtain

$$\begin{aligned} \tilde{\Theta}^T(kT_0) \tilde{\Theta}(kT_0) &\leq \left[ \prod_{\rho=0}^{k-1} \left( 1 + \frac{\lambda_{\min}(Z(\rho T_0) Z^T(\rho T_0))}{\alpha} \right) \right]^{-2} \tilde{\Theta}_0^T \tilde{\Theta}_0 \\ &\leq \left[ 1 + \frac{1}{\alpha} \sum_{\rho=0}^{k-1} \lambda_{\min}(Z(\rho T_0) Z^T(\rho T_0)) \right]^{-2} \tilde{\Theta}_0^T \tilde{\Theta}_0 \end{aligned} \tag{5.25}$$

where  $\tilde{\Theta}_0 = \hat{\Theta}_0^T - \Theta^T$ . Hence,  $\|\tilde{\Theta}(kT_0)\|$  is uniformly bounded by  $\|\tilde{\Theta}_0\|$ , and since  $\Theta$  is finite,  $\hat{\Theta}(kT_0)$  is also uniformly bounded by some finite  $\mu \in \mathbb{R}^+$ .

(b) If  $\lim_{k \rightarrow \infty} \sum_{\rho=0}^k \lambda_{\min}(Z(\rho T_0) Z^T(\rho T_0)) = \infty$  then, from (5.24), it follows that  $\lim_{k \rightarrow \infty} \tilde{\Theta}(kT_0) = 0$ , and therefore,  $\lim_{k \rightarrow \infty} \hat{\Theta}(kT_0) = \Theta$ .  $\square$

Clearly, Proposition 5.1 states that for the convergence of the plant parameters estimates  $\hat{\Theta}(kT_0)$  to their true values  $\Theta$  it is sufficient that the regression vector  $Z(kT_0)$  is persistently exciting to the amount that

$$\lim_{k \rightarrow \infty} \sum_{\rho=0}^k \lambda_{\min}(Z(\rho T_0) Z^T(\rho T_0)) = \infty.$$

Therefore, since adaptation and stability of the adaptive scheme depend on the convergence of the parameter estimates to their true values, it is necessary to prove excitation of  $Z(kT_0)$ . This done in Subsection 5.3, that follows (see Theorem 5.2, therein).

**Remark 5.1.** It is pointed out that, although controllability and observability of the sampled system (5.1) is instrumental for our analysis, no assumption is made, in the present paper, on the canonical structure of the triplet  $(\Phi_\tau, B_\tau, C)$ . This is in contrast to the standard policy of many known adaptive schemes, in which controllability or observability canonical forms are assumed for the matrix triplet involved in the estimation procedure (see for example [26, 37]). The reason for avoiding here an assumption on the canonical structure for the triplet  $(\Phi_\tau, B_\tau, C)$ , is mainly due to the fact that canonical forms for multivariable systems are interwoven with the knowledge of a set of controllability or observability indices of the matrix triplet sought (for example, in [26, 37] a set of observability indices is needed to be known). As a consequence, when identification procedures based on canonical structures are used, much more prior knowledge relative to the structure of the controlled plant is necessary, as compared to our approach.

## 5.2. Algorithm for the synthesis of the adaptive controller

On the basis of the estimated parameter matrix  $\hat{\Theta}(kT_0)$  obtained by (5.20), as well as on the basis of the relations (5.4)–(5.6) and (5.19) and of anyone of the algorithms reported in the literature for the construction of a minimal realization, one can obtain the estimates which are necessary for the computation of the unknown matrices  $\hat{A}_i \equiv \hat{A}_i(kT_0)$ ,  $\hat{\Phi}(kT_0)$  and the unknown vector  $\hat{b}_i \equiv \hat{b}_i(kT_0)$  involved in the algorithms presented in the previous sections. Moreover, since the matrices  $Q$ ,  $\hat{\Phi}^*$ ,  $R_{sp}$ ,  $T_{sp}$ ,  $L$ ,  $\hat{S}$  and  $\hat{S}^*$  are constructed on the basis of  $\hat{A}_i(kT_0)$ ,  $\hat{\Phi}(kT_0)$  and  $\hat{b}_i(kT_0)$ , then provided that the matrix triplet  $(\hat{\Phi}(kT_0), \hat{B}(kT_0), C(kT_0))$  is controllable and observable for any possible value of  $\hat{\Theta}(kT_0)$ , we can obtain the following results sought:

$$\hat{F} \equiv \hat{F}(\hat{\Theta}(kT_0)), \quad v \equiv v(\hat{\Theta}(kT_0)) \quad (5.26)$$



whereas no update is taken otherwise.

Overall, the procedure for the synthesis of the adaptive GSHF based adaptive pole placer, consists of the ten steps given below:

*Step 1.* Choose the input multiplicities of the sampling  $N_i$  such that  $N_i \geq n_i$  and the sampling period  $\tau$  such that  $\tau = T_0 / (6n - 1)N$ .

*Step 2.* Update the estimates of the matrix  $V$  using relation (5.20).

*Step 3.* Find the matrix  $\Omega$  using relation (5.19).

*Step 4.* Obtain a minimal realization for the matrix triplet  $(\Phi_\tau, B_\tau, C)$  using anyone of the minimal realization algorithms reported in the literature (see e.g. the algorithms in [18, 28, 42]).

*Step 5.* Find the matrices  $\hat{A}_i$  and the vectors  $\hat{b}_i$ , as well as the matrix  $\Phi$  using relations (5.4) and (5.5), respectively.

*Step 6.* Use the algorithm presented in Section 3 to compute the controllability indices  $\delta_i$  of the pair  $(\Phi^T, C^T)$ , as well as the values of the matrices  $L, Q, \tilde{\Phi}^*, R_{sp}$  and  $T_{sp}$ .

*Step 7.* Use (3.26) to compute the controller matrix  $K_f$ .

*Step 8.* Find the matrices  $\hat{S}$  and  $\hat{S}^*$  using relation (4.12) and (4.13), respectively.

*Step 9.* Find the matrix  $\hat{F}$  and the vector  $v$  using relations (4.14) and (4.15), respectively.

*Step 10.* Find the matrix  $F(t)$  of the GSHF based controller sought and the persistent excitation signals  $v_i(t)$  using relations (4.16) and (4.2a), (4.2b), (4.3), respectively.

### 5.3. Stability analysis of the adaptive control scheme

We now investigate the stability of the closed-loop system for arbitrary initial conditions on the plant. To this end, the following fundamental result, can be established.

**Theorem 5.2.** In the closed-loop adaptive control system the regressor sequence  $\phi(v\tau)$  is persistently exciting, i.e. there is a  $\delta > 0$ , such that

$$Z(kT_0)Z^T(kT_0) = \sum_{v=0}^{(6n-1)N} \phi(kT_0 - v\tau)\phi^T(kT_0 - v\tau) \geq \delta I. \quad (5.27)$$

*Proof.* In order to prove relation (5.27), we work as follows: Set  $u_i(t) = d_i^T(t)v_i$ . Then, relation (5.7), yields

$$y_i(\tau) = \sum_{\rho=0}^{n-1} (J_1)_{(n-1)p+i, (n-\rho-1)p+i} y_i[(v - 2n - \rho)\tau]$$

$$\begin{aligned}
 & + \sum_{\substack{\kappa=1 \\ \kappa \neq i}}^p \sum_{\rho=0}^{n-1} (J_1)_{(n-1)p+i, (n-\rho-1)p+\kappa} y_{\kappa} [(v-2n-\rho)\tau] \\
 & + \sum_{j=1}^m \sum_{\rho=0}^{n-2} (J_2)_{(n-1)p+i, (n-\rho-2)m+j} u_j [(v-\rho-1)\tau] \\
 & + \sum_{j=1}^m \sum_{\rho=0}^{n-1} (V)_{(n-1)p+i, (n-\rho-1)m+j} u_j [(v-n-\rho)\tau] \\
 & + \sum_{j=1}^m \sum_{\rho=0}^{n-1} (V^*)_{(n-1)p+i, (n-\rho-1)m+j} u_j [(v-2n-\rho)\tau] \quad (5.28)
 \end{aligned}$$

where in general  $(J_1)_{r,q}$ ,  $(V)_{r,q}$  and  $(V^*)_{r,q}$ , are the  $r - q$  elements of the matrices  $J_1$ ,  $J_2$ ,  $V$  and  $V^*$ , respectively. Introducing the pseudovariables  $\beta_{i,u_j}(v\tau)$ ,  $j \in J_m$  and  $\beta_{i,y_{\kappa}}(v\tau)$ ,  $\kappa = 1, 2, \dots, p$ ,  $\kappa \neq i$ , relation (5.28), can be decomposed as follows

$$\beta_{i,u_j}(v\tau) - \sum_{\rho=0}^{n-1} (J_1)_{(n-1)p+i, (n-\rho-1)p+i} \beta_{i,u_j} [(v-2n-\rho)\tau] = u_j(v\tau) \quad (5.29a)$$

$$\begin{aligned}
 y_{i,u_j}(v\tau) & = \sum_{\rho=0}^{n-2} (J_2)_{(n-1)p+i, (n-\rho-2)m+j} \beta_{i,u_j} [(v-\rho-1)\tau] \\
 & + \sum_{\rho=0}^{n-1} (V)_{(n-1)p+i, (n-\rho-1)m+j} \beta_{i,u_j} [(v-n-\rho)\tau]
 \end{aligned}$$

$$+ \sum_{\rho=0}^{n-1} (V^*)_{(n-1)p+i, (n-\rho-1)m+j} \beta_{i,u_j} [(v-2n-\rho)\tau], \quad \text{for } j \in J_m \quad (5.29b)$$

$$\beta_{i,y_{\kappa}} - \sum_{\rho=0}^{n-1} (J_1)_{(n-1)p+i, (n-\rho-1)p+i} \beta_{i,y_{\kappa}} [(v-2n-\rho)\tau] = y_{\kappa}(v\tau), \quad (5.29c)$$

$$y_{i,y_{\kappa}}(v\tau) = \sum_{\rho=0}^{n-1} (J_1)_{(n-1)p+i, (n-\rho-1)p+\kappa} \beta_{i,y_{\kappa}} [(v-2n-\rho)\tau], \quad \text{for } \kappa = 1, 2, \dots, p, \kappa \neq i \quad (5.29d)$$

while

$$y_i(v\tau) = \sum_{j=1}^m y_{i,u_j}(v\tau) + \sum_{\substack{\kappa=1 \\ \kappa \neq i}}^p y_{i,y_{\kappa}}(v\tau). \quad (5.29e)$$

From relations (5.29b) - (5.29e), we obtain

$$y_i(v\tau) = \left(\frac{1}{p}\right) \left\{ \sum_{j=1}^m \left\{ \sum_{\rho=0}^{n-2} (J_2)_{(n-1)p+i, (n-\rho-2)m+j} \beta_{i,u_j} [(v-\rho-1)\tau] \right. \right.$$

$$\begin{aligned}
 & + \sum_{\rho=0}^{n-1} (V)_{(n-1)p+i, (n-\rho-1)m+j} \beta_{i,u_j} [(v-n-\rho)\tau] \\
 & + \sum_{\rho=0}^{n-1} (V^*)_{(n-1)p+i, (n-\rho-1)m+j} \beta_{i,u_j} [(v-2n-\rho)\tau] \Big\} \\
 & + \sum_{\substack{\kappa=1 \\ \kappa \neq i}}^p \sum_{\rho=0}^{n-1} (J_1)_{(n-1)p+i, (n-\rho-1)p+\kappa} \beta_{i,y_\kappa} [(v-2n-\rho)\tau] \tag{5.30} \\
 & + \sum_{\substack{\kappa=1 \\ \kappa \neq i}}^p \left\{ \beta_{\kappa,y_i}(v\tau) - \sum_{\rho=0}^{n-1} (J_1)_{(n-1)p+\kappa, (n-\rho-1)p+\kappa} \beta_{\kappa,y_i} [(v-2n-\rho)\tau] \right\} \Big\}
 \end{aligned}$$

whereas relation (5.29a), yields

$$u_j(v\tau) = \left(\frac{1}{p}\right) \sum_{i=1}^p \left\{ \beta_{i,u_j}(v\tau) - \sum_{\rho=0}^{n-1} (J_1)_{(n-1)p+i, (n-\rho-1)p+i} \beta_{i,u_j} [(v-2n-\rho)\tau] \right\}. \tag{5.31}$$

On the basis of relations (5.7), (5.30) and (5.31), the regressor vector  $\phi(v\tau)$ , can also be expressed as

$$\phi(v\tau) = \widehat{\Sigma} \widehat{\beta}(v\tau)$$

where

$$\begin{aligned}
 \widehat{\beta}^T(v\tau) &= [\widetilde{\beta}(v\tau) \dots \widetilde{\beta}[(v-6n-2)\tau]] \\
 \widetilde{\beta}(\rho\tau) &= [\widetilde{\beta}_{u_1}(\rho\tau) \dots \widetilde{\beta}_{u_m}(\rho\tau) \widetilde{\beta}_{y_1}(\rho\tau) \dots \widetilde{\beta}_{y_p}(\rho\tau)], \quad \rho = v-6n+2, \dots, v \\
 \widetilde{\beta}_{u_j}(\rho\tau) &= [\beta_{1,u_j}(\rho\tau) \dots \beta_{p,u_j}(\rho\tau)], \quad \rho = v-6n+2, \dots, v, \quad j \in J_m \\
 \widetilde{\beta}_{y_1}(\rho\tau) &= [\beta_{2,y_1}(\rho\tau) \dots \beta_{p,y_1}(\rho\tau)], \quad \rho = v-6n+2, \dots, v \\
 \widetilde{\beta}_{y_\kappa}(\rho\tau) &= [\beta_{1,y_\kappa}(\rho\tau) \dots \beta_{p-1,y_\kappa}(\rho\tau)] \quad \rho = v-6n+2, \dots, v, \quad \kappa = 2, 3, \dots, p
 \end{aligned}$$

and where  $\widehat{\Sigma} \in \mathbb{R}^{(3nm+np) \times (6n-1)p(p+m-1)}$  is a full row rank matrix. Clearly, the vector  $\phi(v\tau)$  is persistently exciting if  $\widehat{\beta}(v\tau)$  is also persistently exciting. So, in what follows, it suffices to investigate excitation of  $\widehat{\beta}(v\tau)$ . To this end, observe that (5.31), can be written as

$$u_j(v\tau) = \psi_j^T \widehat{\beta}(v\tau) \tag{5.32}$$

where  $\psi_j^T \in \mathbb{R}^{(6n-1)m(p+m-1)}$  is a row vector whose elements are known. In order to prove excitation of  $\widehat{\beta}(v\tau)$ , it suffices to prove that the following relationship holds

$$\sum_{v=1}^{T_0/\tau} \widehat{\beta}(kT_0 + v\tau) \widehat{\beta}^T(kT_0 + v\tau) \geq \varepsilon I \tag{5.33}$$

for some  $\varepsilon > 0$ . To this end, observe that from relation (5.32), we can easily obtain

$$\sum_{v=1}^{T_0/\tau} u_j^2(kT_0 + v\tau) = \psi_j^T \left\{ \sum_{v=1}^{T_0/\tau} \widehat{\beta}(kT_0 + v\tau) \widehat{\beta}^T(kT_0 + v\tau) \right\} \psi_j. \tag{5.34}$$

Observe also that the following relation holds

$$u_j(kT_0 + v\tau) = \begin{cases} 0, & \text{if } v = 1, 2, \dots, (6n - 1)(N_j - n_j - 1)1_j - 1 \\ 1, & \text{if } v = (6n - 1)(N_j - n_j - 1)1_j, \dots, (6n - 1)(N_j - n_j)1_j - 1 \end{cases}$$

Hence, relation (5.34), can also be written as

$$(6n - 1)1_{j+} \sum_{v=(6n-1)(N_j-n_j)1_j}^{T_0/\tau} u_j^2(kT_0 + v\tau) = \psi_j^T \left\{ \sum_{v=1}^{T_0/\tau} \widehat{\beta}(kT_0 + v\tau) \widehat{\beta}^T(kT_0 + v\tau) \right\} \psi_j.$$

We can then conclude that

$$\psi_j^T \left\{ \sum_{v=1}^{T_0/\tau} \widehat{\beta}(kT_0 + v\tau) \widehat{\beta}^T(kT_0 + v\tau) \right\} \psi_j \geq (6n - 1)1_j$$

and that

$$\left\{ \frac{\psi_j^T}{\|\psi_j\|} \right\} \left\{ \sum_{v=1}^{T_0/\tau} \widehat{\beta}(kT_0 + v\tau) \widehat{\beta}^T(kT_0 + v\tau) \right\} \left\{ \frac{\psi_j}{\|\psi_j\|} \right\} \geq \frac{(6n - 1)1_j}{\|\psi_j\|^2}.$$

It is now clear that, the vector  $\frac{\psi_j}{\|\psi_j\|}$ , is a vector whose norm equals to unity. Hence there is a unity norm vector such that

$$\chi^T \left\{ \sum_{v=1}^{T_0/\tau} \widehat{\beta}(kT_0 + v\tau) \widehat{\beta}^T(kT_0 + v\tau) \right\} \chi - \frac{(6n - 1)1_j}{\|\psi_j\|^2} \geq 0.$$

In conclusion, relation (5.33) holds. As a consequence, the vector  $\widehat{\beta}(v\tau)$  is persistently exciting. Therefore,  $\phi(v\tau)$  is also persistently exciting and hence there is a  $\delta > 0$  (which, in general, depends on the matrix  $\widehat{\Sigma}$ ), such that relation (5.27) holds. This completes the proof of the Theorem.  $\square$

We are now able to establish the stability of the adaptive control system.

**Proposition 5.2.** The closed-loop adaptive control system presented above is globally stable, i. e. for arbitrary finite initial conditions all states are uniformly bounded, and pole placement control is asymptotically attained. Furthermore, the proposed adaptive scheme provides exponential convergence of the estimated parameters.

**Proof.** Since, according to Theorem 5.2, the regressor sequence is persistently exciting, then the difference  $\hat{\Theta}(kT_0) - \Theta$  converges to zero. That is, the plant parameter estimates converge to their true values. As a consequence of this and of the fact that  $\hat{\Theta}(kT_0)$  uniformly bounded, the controller parameter estimates (5.26) also converge to their true values. Therefore, at the sampling instants uniform boundedness of all states and discrete pole placement follow on the basis of (4.4). Uniform boundedness of  $u(t)$  and  $x(t)$  then follows from (2.1), (4.8) and (4.16) and from the fact that  $w(kT_0)$  is bounded by assumption. Finally, exponential convergence of the plant parameter estimates follows from (5.23), which together with (5.27), ensures that  $\hat{\Theta}(kT_0) \rightarrow \Theta$  exponentially as  $k \rightarrow \infty$ .  $\square$

**Remark 5.2.** Commenting on the assumptions needed here, in order to implement the GSHF based adaptive pole placer presented above, we point out the following:

Assumption 2.1a, on the controllability and observability of the continuous-time plant as well as on the knowledge of its order is a standard assumption in the area of adaptive control. It is worth noticed that here, controllability of the pair  $(A, B)$  is also necessary for obtaining a solution of the integral equation (3.2), with respect to the controller matrix  $F(t)$ . Note also that, uncontrollability (and/or unobservability) of the pair  $(A, B)$  implies uncontrollability (and/or unobservability) of the plants obtained from (2.1), by discretizing with sampling periods  $T_0, T_N$ , and  $\tau$ . From the previous analysis, however, it becomes clear that for the implementation of the adaptive control scheme, these discretized plants must be controllable and observable.

Assumption 2.1b, on the knowledge of a set of LMCI indices of the pair  $(A, B)$ , is instrumental for the implementation of the proposed adaptive scheme, since, on the one hand, the forms of the multirate GSHF based controller (4.1) and the persistent excitation signals (4.2a), (4.2b), (4.3) depend on the LMCI used, and on the other hand, the control strategy in the case of unknown systems is based on the fundamental sampling period  $\tau$ , which also depends of the knowledge of a set of LMCI. Note also that, whenever Assumption 2.1b is not fulfilled, one can readily compute a set of LMCI by estimating the continuous-time system matrices  $A$  and  $B$ . This can be done either using a continuous-time counterpart of the identification procedure presented in Section 5.1 or following the structural identification approach proposed in [42]. For the sake of simplicity, we assume here that the initial information about a set of LMCI of the pair  $(A, B)$  is available.

Assumption 2.2 on the existence of a sampling period, for which controllability and observability of the matrix triplets  $(\Phi, \bar{B}, C)$  and  $(\Phi_\tau, B_\tau, C)$  are guaranteed, is also instrumental for our analysis. In particular, observability of the pair  $(\Phi, C)$  must be guaranteed, for being able to transform the pair  $(\Phi^T, C^T)$  in its input Luenberger canonical form and for obtaining a solution of the pole placement control problem, in the case of known systems. On the other hand, controllability and observability of the matrix triplet  $(\Phi_\tau, B_\tau, C)$  is necessary for resorting to the equivalent input-output representation (5.7), for the state space system of the form (5.1), as well as for being able to apply anyone of the minimal realization algorithms presented in [18, 28, 42], which are needed here to obtain the estimates of the triplet  $(\Phi_\tau, B_\tau, C)$ .

Note that, for ensuring controllability and observability of the triplets  $(\Phi, \tilde{B}, C)$  and  $(\Phi_\tau, B_\tau, C)$ , the fundamental sampling period  $T_0$  must be selected such that simultaneously

$$(a) \frac{2\rho\pi j}{T_0}, \rho = 0, 1, \dots (j = \sqrt{-1}) \text{ is not the difference} \\ \text{of any two eigenvalues of the matrix } A. \tag{5.35a}$$

$$(b) \frac{2(6n - 1)N\rho\pi j}{T_0}, \rho = 0, 1, \dots \text{ is not the difference} \\ \text{of any two eigenvalues of the matrix } A. \tag{5.35b}$$

$$(c) \psi(T_N) \neq 0. \tag{5.35c}$$

This implies that, in the multirate adaptive case treated here, certain sampling frequencies must avoided, as compared to the non-adaptive non-multirate case. It is pointed out that, conditions (5.35a) and (5.35b), are standard conditions for the selection of a regular sampling period, in order to avoid loss of controllability and observability under sampling (see [31], for a detailed analysis of this issue).

**Remark 5.3.** The results of the present paper hold also in the special case where  $N_1 = N_2 = \dots = N_m \equiv N_0$ , taking into account several modifications needed in the previous analysis, in order to fit this particular case. It is important to note that in this case, less prior information is needed for the implementation of the adaptive control scheme presented above, since there is no need of the prior knowledge of a set of LMCI of the pair  $(A, B)$ . We can simply take  $N_0 \geq n$ . With this choice, the matrices  $\hat{S}$  and  $\hat{S}^*$  have full row rank for almost all  $T_{N_0} \in [T_{N_0}^1, T_{N_0}^2]$ . Then, the matrix  $\hat{F}$  and the vector  $v$  have the following forms

$$\hat{F} = E^T \begin{bmatrix} \hat{S}(\hat{S}\hat{S}^T)^{-1} Q^T(-\tilde{\Phi}^* + R_{s,p}\Pi^*T_{s,p}^{-1})^T L^T \\ 0 \end{bmatrix}$$

$$v = \Delta^T \begin{bmatrix} -\hat{S}^*(\hat{S}^*\hat{S}^{*T})^{-1}(\hat{A}^n\hat{b}_1 + \dots + \hat{A}^n\hat{b}_m) \\ \zeta \\ 0_{m(N_0-m-1)} \end{bmatrix}, \quad \hat{A} = \exp(AT_0/N_0).$$

### 6. CONCLUSIONS

The adaptive pole placement problem of linear time-invariant continuous-time multi-input, multi-output systems has been investigated and an indirect adaptive control scheme based on generalized sampled-data hold functions has been presented, for the first time. The proposed control strategy has, as compared to known related techniques, the following main advantages:

(a) It is readily applicable to nostably invertible systems having arbitrary poles and zeros and relative degree. This is due to the fact that the approach used here to solve the adaptive pole placement problem does not rely on pole-zero cancellations.

(b) Following the proposed technique a gain controller is essentially needed to be designed, as compared to dynamic compensators or state observers needed by known indirect adaptive pole placement techniques. Consequently, the present approach avoids the problems of known adaptive pole placement techniques, interwoven with the possibly unstable solutions of the Diophantine equation. Moreover, no exogenous dynamics are introduced in the control loop by our technique, whereas in many known techniques the dynamics introduced are of high order. This fact improves the computational aspect of the problem, since the proposed technique does not require many on-line computations and its practical implementation requires computer memory only for storing the modulating matrix function  $F(t)$  over one period of time.

(c) It offers a solution to the problem of ensuring persistency of excitation of the continuous-time plant under control, without imposing any special requirement on the reference signal  $w(kT_0)$  (except boundedness) and without making any assumption concerning either the existence of specific convex sets in which the estimated parameters belong or the coprimeness of the polynomials describing the ARMA model.

The present paper gives some new insights to the adaptive pole placement problem of linear systems. The present results can be extended to solve other related adaptive control problems, as for example the problems of model reference adaptive control and adaptive decoupling using multirate sampled-data hold functions. Adaptive control schemes based on alternative parameter estimation algorithms (as, for example, the algorithm proposed in [37]) and without the need of persistent excitation signals are currently under investigation.

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