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CONTROLLABILITY OF SEMILINEAR FUNCTIONAL INTEGRODIFFERENTIAL SYSTEMS IN BANACH SPACES

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Sufficient conditions for controllability of semilinear functional integrodifferential systems in a Banach space are established. The results are obtained by using the Schaefer fixed-point theorem.

1. INTRODUCTION

Controllability of nonlinear systems represented by ordinary differential equations in infinite-dimensional spaces has been extensively studied by several authors. Naito [12, 13] has studied the controllability of semilinear systems whereas Yamamoto and Park [19] discussed the same problem for parabolic equation with uniformly bounded nonlinear term. Chukwu and Lenhart [3] have studied the controllability of nonlinear systems in abstract spaces. Do [4] and Zhou [20] investigated the approximate controllability for a class of semilinear abstract equations. Kwun et al [7] established the approximate controllability for delay Volterra systems with bounded linear operators. Controllability for nonlinear Volterra integrodifferential systems has been studied by Naito [14]. Recently Balachandran et al [1, 2] studied the controllability and local null controllability of Sobolev-type integrodifferential systems and functional differential systems in Banach spaces by using Schauder's fixed-point theorem. The purpose of this paper is to study the controllability of semilinear functional integrodifferential systems in Banach spaces by using the Schaefer fixed-point theorem. The semilinear functional integrodifferential equation considered here serves as an abstract formulation of partial functional integrodifferential equations which arise in heat flow in material with memory [5, 6, 8, 9, 18].

2. PRELIMINARIES

Consider the semilinear functional integrodifferential system of the form

$$(Ex(t))' + Ax(t) \quad = \quad (Bu)(t) + \int_0^t f(s, x_s)\,\mathrm{d}s, \qquad t \in J = [0, b], \tag{1}$$

$$x(t) = \phi(t), \quad t \in [-r, 0],$$

where E and A are linear operators with domains contained in a Banach space X and ranges contained in a Banach space Y, the state $x(.)$ takes values in X and the control function $u(.)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space, B is a bounded linear operator from U into Y and the nonlinear operator $f : J \times C \to Y$ is a given function. Here $C = C([-r, 0], X)$ is the Banach space of all continuous functions $\phi : [-r, 0] \to X$ endowed with the norm $\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}$. Also for $x \in C([-r, b], X)$ we have $x_t \in C$ for $t \in [0, b]$, $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. The norm of X is denoted by $\|\cdot\|$ and Y by $|\cdot|$.

The operators $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$ satisfy the hypotheses [C_i] for $i = 1, \ldots, 4$:

[C_1] A and E are closed linear operators

[C_2] $D(E) \subset D(A)$ and E is bijective

[C_3] $E^{-1} : Y \to D(E)$ is continuous

[C_4] For each $t \in [0, b]$ and for some $\lambda \in \rho(-AE^{-1})$, the resolvent set of $-AE^{-1}$, the resolvent $R(\lambda, -AE^{-1})$ is a compact operator.

The hypotheses [C_1], [C_2] and the closed graph theorem imply the boundedness of the linear operator $AE^{-1} : Y \to Y$.

Lemma. [15] Let $S(t)$ be a uniformly continuous semigroup and let A be its infinitesimal generator. If the resolvent set $R(\lambda : A)$ of A is compact for every $\lambda \in \rho(A)$, then $S(t)$ is a compact semigroup.

From the above fact, $-AE^{-1}$ generates a compact semigroup $T(t)$, $t \geq 0$, on Y.

Definition. The system (1) is said to be controllable on the interval J if for every continuous initial function $\phi \in C$ and $x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the solution $x(t)$ of (1) satisfies $x(b) = x_1$.

We further assume the following hypotheses:

[C_5] $-AE^{-1}$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in Y satisfying

$$|T(t)| \leq M_1 e^{\omega t}, \quad t \geq 0 \text{ for some } M_1 \geq 1 \text{ and } \omega \geq 0.$$

[C_6] The linear operator $W : L^2(J, U) \to X$ defined by

$$Wu = \int_0^b E^{-1} T(b - s) B u(s) \, ds$$

has an inverse operator $\widetilde{W}^{-1} : X \to L^2(J, U)/\ker W$ and there exist positive constants M_2, M_3 such that $|B| \leq M_2$ and $|\widetilde{W}^{-1}| \leq M_3$ (See the remark for the construction of $\widetilde{W}^{-1}$).

[C$_7$] For each $t \in J$, the function $f(t,\cdot) : C \to Y$ is continuous and for each $x \in C$, the function $f(\cdot, x) : J \to Y$ is strongly measurable.

[C$_8$] There exists an integrable function $m : [0,b] \to [0,\infty)$ such that

$$|f(t,\phi)| \le m(t)\Omega(\|\phi\|), \quad 0 \le t \le b, \quad \phi \in C,$$

where $\Omega : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

[C$_9$]

$$\int_0^b \hat{m}(s)\,\mathrm{d}s < \int_c^\infty \frac{\mathrm{d}s}{1+s+\Omega(s)},$$

where

$$c = \|E^{-1}\|M_1|E\phi(0)|, \; \hat{m}(t) = \max\left\{\omega, \|E^{-1}\|M_1 N, \|E^{-1}\|M_1 \int_0^t m(s)\,\mathrm{d}s\right\}$$

and

$$\begin{aligned} N &= M_2M_3[\|x_1\| + \|E^{-1}\|M_1 e^{\omega b}\|\phi\| \\ &\quad + \|E^{-1}\|M_1 \int_0^b e^{\omega(b-s)} \int_0^s m(\tau)\Omega(\|x_\tau\|)\,\mathrm{d}\tau\,\mathrm{d}s]. \end{aligned}$$

We need the following fixed point theorem due to Schaefer [17].

Theorem 1. Let E be a normed space. Let $F : E \to E$ be a completely continuous operator, i.e., it is continuous and the image of any bounded set is contained in a compact set, and let

$$\zeta(F) = \{x \in E; x = \lambda Fx \ \text{ for some } \ 0 < \lambda < 1\}$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

Then the system (1) has a mild solution of the following form

$$\begin{aligned} x(t) &= E^{-1}T(t)E\phi(0) + \int_0^t E^{-1}T(t-s)\Big[(Bu)(s) \\ &\qquad + \int_0^s f(\tau, x_\tau)\,\mathrm{d}\tau\Big]\,\mathrm{d}s, \quad t \in J, \\ x(t) &= \phi(t), \quad t \in [-r, 0] \end{aligned}$$

and $Ex(t) \in C([0,b];Y) \cap C'((0,b);Y)$.

3. MAIN RESULT

Theorem 2. If the hypotheses $[C_1]-[C_9]$ are satisfied, then the system (1) is controllable on J.

Proof. Using the hypothesis $[C_6]$ for an arbitrary function $x(.)$, define the control

$$u(t) = \widetilde{W}^{-1}\left[x_1 - E^{-1}T(b)E\phi(0) - \int_0^b E^{-1}T(b-s)\int_0^s f(\tau, x_\tau)\,d\tau\,ds\right](t).$$

For $\phi \in C$, define $\hat{\phi} \in C_b$, $C_b = C([-r,b],X)$ by

$$\hat{\phi}(t) = \begin{cases} \phi(t), & -r \le t \le 0, \\ E^{-1}T(t)E\phi(0), & 0 \le t \le b. \end{cases}$$

If $x(t) = y(t) + \hat{\phi}(t)$, $t \in [-r,b]$, it is easy to see that y satisfies

$$\begin{aligned} y_0 &= 0, \\ y(t) &= \int_0^t E^{-1}T(t-\eta)B\widetilde{W}^{-1}[x_1 - E^{-1}T(b)E\phi(0) \\ &\quad - \int_0^b E^{-1}T(b-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\,d\tau\,ds](\eta)\,d\eta \\ &\quad + \int_0^t E^{-1}T(t-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\,d\tau\,ds, \quad 0 \le t \le b \end{aligned} \tag{2}$$

if and only if x satisfies

$$\begin{aligned} x(t) &= E^{-1}T(t)E\phi(0) + \int_0^t E^{-1}T(t-\eta)B\widetilde{W}^{-1}[x_1 - E^{-1}T(b)E\phi(0) \\ &\quad - \int_0^b E^{-1}T(b-s)\int_0^s f(\tau, x_\tau)\,d\tau\,ds](\eta)d\eta \\ &\quad + \int_0^t E^{-1}T(t-s)\int_0^s f(\tau, x_\tau)\,d\tau\,ds \end{aligned}$$

and $x(t) = \phi(t)$, $t \in [-r, 0]$.

Define $C_b^0 = \{y \in C_b : y_0 = 0\}$ and we now show that when using the control, the operator $F : C_b^0 \to C_b^0$, defined by

$$(Fy)(t) = \begin{cases} 0, & -r \le t \le 0, \\ \displaystyle\int_0^t E^{-1}T(t-\eta)B\widetilde{W}^{-1}[x_1 - E^{-1}T(b)E\phi(0) & \\ \displaystyle\quad - \int_0^b E^{-1}T(b-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\,d\tau\,ds](\eta)\,d\eta & \\ \displaystyle\quad + \int_0^t E^{-1}T(t-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\,d\tau\,ds, & 0 \le t \le b \end{cases}$$

has a fixed point. This fixed point is then a solution of equation (2).

Clearly $x(b) = x_1$ which means that the control u steers the system (1) from the initial function ϕ to x_1 in time b, provided we can obtain a fixed point of the nonlinear operator F.

In order to study the controllability problem of (1), we introduce a parameter $\lambda \in (0,1)$ and consider the following system

$$\begin{aligned} (Ex(t))' + Ax(t) &= \lambda(Bu)(t) + \lambda \int_0^t f(s, x_s)\,\mathrm{d}s, \quad t \in J = [0,b], \qquad (3)\\ x(t) &= \lambda\phi(t), \quad t \in [-r, 0]. \end{aligned}$$

First we obtain *a priori* bounds for the mild solution of the equation (3). Then from

$$\begin{aligned} x(t) &= \lambda E^{-1}T(t)E\phi(0) + \lambda \int_0^t E^{-1}T(t-\eta)B\widetilde{W}^{-1}\Big[x_1 - E^{-1}T(b)E\phi(0) \\ &\quad - \int_0^b E^{-1}T(b-s)\int_0^s f(\tau, x_\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Big](\eta)\,\mathrm{d}\eta \\ &\quad + \lambda \int_0^t E^{-1}T(t-s)\int_0^s f(\tau, x_\tau)\,\mathrm{d}\tau\,\mathrm{d}s, \end{aligned}$$

we have,

$$\begin{aligned} \|x(t)\| &\le \|E^{-1}\|M_1 e^{\omega t}|E\phi(0)| + \int_0^t \|E^{-1}\||T(t-\eta)|M_2M_3[\|x_1\| \\ &\quad + \|E^{-1}\|M_1 e^{\omega b}|E\phi(0)| + \int_0^b \|E^{-1}\|M_1 e^{\omega(b-s)}\int_0^s m(\tau)\Omega(\|x_\tau\|)\,\mathrm{d}\tau\mathrm{d}s]\,\mathrm{d}\eta \\ &\quad + \|E^{-1}\|M_1 e^{\omega t}\int_0^t e^{-\omega s}\int_0^s m(\tau)\Omega(\|x_\tau\|)\,\mathrm{d}\tau\,\mathrm{d}s, \quad t \in [0,b]. \end{aligned}$$

We consider the function μ given by

$$\mu(t) = \sup\{\|x(s)\| : \ -r \le s \le t\}, \ \ 0 \le t \le b.$$

Let $t^\star \in [-r, t]$ be such that $\mu(t) = \|x(t^\star)\|$. If $t^\star \in [0,b]$, by the previous inequality, we have

$$\begin{aligned} e^{-\omega t}\mu(t) &\le \|E^{-1}\|M_1|E\phi(0)| + \|E^{-1}\|M_1 N\int_0^{t^\star} e^{-\omega s}\,\mathrm{d}s \\ &\quad + \|E^{-1}\|M_1\int_0^{t^\star} e^{-\omega s}\int_0^s m(\tau)\Omega(\|x_\tau\|)\,\mathrm{d}\tau\,\mathrm{d}s \\ &\le \|E^{-1}\|M_1|E\phi(0)| \\ &\quad + \|E^{-1}\|M_1 N\int_0^t e^{-\omega s}\,\mathrm{d}s + \|E^{-1}\|M_1\int_0^t e^{-\omega s}\int_0^s m(\tau)\Omega(\mu(\tau))\,\mathrm{d}\tau\,\mathrm{d}s. \end{aligned}$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds since $M_1 \geq 1$.

Denoting by $v(t)$, the right-hand side of the above inequality, we have
$c = v(0) = \|E^{-1}\| M_1 |E\phi(0)|$, $\mu(t) \leq e^{\omega t} v(t)$, $0 \leq t \leq b$ and

$$\begin{aligned} v'(t) &= \|E^{-1}\| M_1 N e^{-\omega t} + \|E^{-1}\| M_1 e^{-\omega t} \int_0^t m(s)\Omega(\mu(s))\,\mathrm{d}s \\ &\leq \|E^{-1}\| M_1 N e^{-\omega t} + \|E^{-1}\| M_1 e^{-\omega t} \int_0^t m(s)\Omega(e^{\omega s} v(s))\,\mathrm{d}s. \end{aligned}$$

We remark that

$$\begin{aligned} (e^{\omega t} v(t))' &= \omega e^{\omega t} v(t) + e^{\omega t} v'(t) \\ &\leq \omega e^{\omega t} v(t) + \|E^{-1}\| M_1 N + \|E^{-1}\| M_1 \int_0^t m(s)\Omega(e^{\omega s} v(s))\,\mathrm{d}s \\ &\leq \omega e^{\omega t} v(t) + \|E^{-1}\| M_1 N + \|E^{-1}\| M_1 \Omega(e^{\omega t} v(t)) \int_0^t m(s)\,\mathrm{d}s \\ &\leq \hat{m}(t)[e^{\omega t} v(t) + 1 + \Omega(e^{\omega t} v(t))]. \end{aligned}$$

This implies

$$\int_{v(0)}^{e^{\omega t} v(t)} \frac{\mathrm{d}s}{1 + s + \Omega(s)} \leq \int_0^b \hat{m}(s)\,\mathrm{d}s < \int_c^\infty \frac{\mathrm{d}s}{1 + s + \Omega(s)}, \quad 0 \leq t \leq b.$$

This inequality implies that there is a constant K such that $v(t) \leq K$ and hence $\mu(t) \leq K$, $t \in [0, b]$. Since $\|x_t\| \leq \mu(t)$, $t \in [0, b]$, we have

$$\|x\|_1 = \sup\{\|x(t)\| : -r \leq t \leq b\} \leq K,$$

where K depends only on b and on the functions m and Ω.

Next we must prove that the operator F is a completely continuous operator.
Let $B_k = \{y \in C_b^0 : \|y\|_1 \leq k\}$ for some $k \geq 1$.

We first show that the set $\{Fy : y \in B_k\}$ is equicontinuous. Let $y \in B_k$ and $t_1, t_2 \in [0, b]$. Then if $0 < t_1 < t_2 \leq b$,

$$\begin{aligned} &\|(Fy)(t_1) - (Fy)(t_2)\| \\ &\leq \Big\| \int_0^{t_1} E^{-1}[T(t_1 - \eta) - T(t_2 - \eta)] B\widetilde{W}^{-1}[x_1 - E^{-1}T(b)E\phi(0) \\ &\qquad\qquad - \int_0^b E^{-1}T(b - s) \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\,\mathrm{d}\tau\,\mathrm{d}s](\eta)\,\mathrm{d}\eta \Big\| \\ &\quad + \Big\| \int_{t_1}^{t_2} E^{-1}T(t_2 - \eta) B\widetilde{W}^{-1}[x_1 - E^{-1}T(b)E\phi(0) \end{aligned}$$

$$
\begin{aligned}
& - \int_0^b E^{-1}T(b-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\,\mathrm{d}\tau\,\mathrm{d}s](\eta)\,\mathrm{d}\eta\Big\| \\
& + \Big\| \int_0^{t_1} E^{-1}[T(t_1-s) - T(t_2-s)] \int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\,\mathrm{d}\tau\mathrm{d}s \Big\| \\
& + \Big\| \int_{t_1}^{t_2} E^{-1}T(t_2-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\,\mathrm{d}\tau\,\mathrm{d}s \Big\| \\
\leq & \int_0^{t_1} \|E^{-1}\| |T(t_1-\eta) - T(t_2-\eta)| M_2M_3[\|x_1\| + \|E^{-1}\| M_1 e^{\omega b}|E\phi(0)| \\
& \qquad + \|E^{-1}\| M_1 \int_0^b e^{\omega(b-s)} \int_0^s m(\tau)\Omega(k')\,\mathrm{d}\tau\mathrm{d}s]\,\mathrm{d}\eta \\
& + \int_{t_1}^{t_2} \|E^{-1}\| |T(t_2-\eta)| M_2M_3[\|x_1\| + \|E^{-1}\| M_1 e^{\omega b}|E\phi(0)| \\
& \qquad + \|E^{-1}\| M_1 \int_0^b e^{\omega(b-s)} \int_0^s m(\tau)\Omega(k')\,\mathrm{d}\tau\mathrm{d}s]\,\mathrm{d}\eta \\
& + \int_0^{t_1} \|E^{-1}\| |T(t_1-s) - T(t_2-s)| \int_0^s m(\tau)\Omega(k')\,\mathrm{d}\tau\,\mathrm{d}s \\
& + \int_{t_1}^{t_2} \|E^{-1}\| |T(t_2-s)| \int_0^s m(\tau)\Omega(k')\,\mathrm{d}\tau\,\mathrm{d}s,
\end{aligned}
$$

where $k' = k + \|\hat{\phi}\|$. The right hand side is independent of $y \in B_k$ and tends to zero as $t_2 - t_1 \to 0$, since the compactness of $T(t)$, for $t > 0$, implies the continuity in the uniform operator topology.

Thus the set $\{Fy;\ y \in B_k\}$ is equicontinuous.

Notice that we considered here only the case $0 < t_1 < t_2$, since the other cases $t_1 < t_2 < 0$ or $t_1 < 0 < t_2$ are very simple.

It is easy to see that the family FB_k is uniformly bounded. Next we show that $\overline{FB_k}$ is compact. Since we have shown that FB_k is an equicontinuous collection, it suffices, by the Arzela–Ascoli theorem, to show that F maps B_k into a precompact set in X.

Let $0 < t \leq b$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $y \in B_k$, we define

$$
\begin{aligned}
(F_\epsilon y)(t) &= \int_0^{t-\epsilon} E^{-1}T(t-\eta)B\widetilde{W}^{-1}[x_1 - E^{-1}T(b)E\phi(0) \\
& \quad - \int_0^b E^{-1}T(b-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\,\mathrm{d}\tau\,\mathrm{d}s](\eta)\,\mathrm{d}\eta \\
& \quad + \int_0^{t-\epsilon} E^{-1}T(t-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\
&= T(\epsilon)\int_0^{t-\epsilon} E^{-1}T(t-\eta-\epsilon)B\widetilde{W}^{-1}[x_1 - E^{-1}T(b)E\phi(0) \\
& \quad - \int_0^b E^{-1}T(b-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\,\mathrm{d}\tau\,\mathrm{d}s](\eta)\,\mathrm{d}\eta
\end{aligned}
$$

$$+ T(\epsilon)\int_0^{t-\epsilon} E^{-1}T(t-s-\epsilon)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\,d\tau\,ds.$$

Since $T(t)$ is a compact operator, the set $Y_\epsilon(t) = \{(F_\epsilon y)(t) : y \in B_k\}$ is precompact in X for every ϵ, $0 < \epsilon < t$. Moreover for every $y \in B_k$ we have

$$\begin{aligned}
&\|(Fy)(t) - (F_\epsilon y)(t)\| \\
&\leq \int_{t-\epsilon}^{t} \|E^{-1}T(t-\eta)B\widetilde{W}^{-1}[x_1 - E^{-1}T(b)E\phi(0) \\
&\quad - \int_0^b E^{-1}T(b-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\,d\tau\,ds](\eta)\|\,d\eta \\
&\quad + \int_{t-\epsilon}^{t} \|E^{-1}T(t-s)\int_0^s f(\tau, y_\tau + \hat{\phi}_\tau)\|\,d\tau\,ds \\
&\leq \int_{t-\epsilon}^{t} \|E^{-1}\|\|T(t-\eta)\|M_2M_3[\|x_1\| + |E^{-1}|M_1e^{\omega b}|E\phi(0)| \\
&\quad + \|E^{-1}\|M_1\int_0^b e^{\omega(b-s)}\int_0^s m(\tau)\Omega(k')\,d\tau\,ds]\,d\eta \\
&\quad + \int_{t-\epsilon}^{t} \|E^{-1}\|\|T(t-s)\|\int_0^s m(\tau)\Omega(k')\,d\tau\,ds.
\end{aligned}$$

Therefore there are precompact sets arbitrarily close to the set $\{(Fy)(t) : y \in B_k\}$. Hence the set $\{(Fy)(t) : y \in B_k\}$ is precompact in X.

It remains to be shown that $F : C_b^0 \to C_b^0$ is continuous. Let $\{y_n\}_0^\infty \subseteq C_b^0$ with $y_n \to y$ in C_b^0. Then there is an integer r such that $\|y_n(t)\| \leq r$ for all n and $t \in J$, so $y_n \in B_r$ and $y \in B_r$. By [C$_7$], $f(t, y_n(t) + \hat{\phi}_t) \to f(t, y(t) + \hat{\phi}_t)$ for each $t \in J$ and since $|f(t, y_n(t) + \hat{\phi}_t) - f(t, y(t) + \hat{\phi}_t)| \leq 2g_{r'}(t)$, $r' = r + \|\hat{\phi}\|$, we have, by dominated convergence theorem,

$$\begin{aligned}
&\|Fy_n - Fy\| \\
&= \sup_{t\in J}\Big\| \int_0^t E^{-1}T(t-\eta)B\widetilde{W}^{-1}\Big[\int_0^b T(b-s) \\
&\qquad \int_0^s [f(\tau, y_n(\tau) + \hat{\phi}_\tau) - f(\tau, y(\tau) + \hat{\phi}_\tau)]\,d\tau\,ds\Big](\eta)\,d\eta \\
&\quad + \int_0^t E^{-1}T(t-s)\int_0^s [f(\tau, y_n(\tau) + \hat{\phi}_\tau) - f(\tau, y(\tau) + \hat{\phi}_\tau)]\,d\tau\,ds\Big\| \\
&\leq \int_0^b \|E^{-1}\|\|T(t-\eta)\|M_2M_3[M_1\int_0^b e^{\omega(b-s)} \\
&\qquad \int_0^s |f(\tau, y_n(\tau) + \hat{\phi}_\tau) - f(\tau, y(\tau) + \hat{\phi}_\tau)|\,d\tau\,ds]\,d\eta \\
&\quad + \int_0^b \|E^{-1}\|\|T(t-s)\|\int_0^s |f(\tau, y_n(\tau) + \hat{\phi}_\tau) - f(\tau, y(\tau) + \hat{\phi}_\tau)|\,d\tau\,ds \;\to 0 \\
&\qquad \text{as } n \to \infty.
\end{aligned}$$

Thus F is continuous. This completes the proof that F is completely continuous.

Finally the set $\zeta(F) = \{y \in C_b^0 : y = \lambda F y, \lambda \in (0,1)\}$ is bounded, since for every solution y in $\zeta(F)$, the function $x = y + \hat{\phi}$ is a mild solution of (3) for which we have proved that $\|x\|_1 \leq K$ and hence

$$\|y\|_1 \leq K + \|\hat{\phi}\|.$$

Consequently, by Schaefer's theorem, the operator F has a fixed point in C_b^0. This means that any fixed point of F is a mild solution of (1) on J satisfying $(Fx)(t) = x(t)$. Thus the system (1) is controllable on J. □

Example. Consider the following partial integrodifferential equation of the form

$$\begin{aligned} &\frac{\partial}{\partial t}(z(t,y) - z_{yy}(t,y)) - z_{yy}(t,y) \qquad (4) \\ &= \quad Bu(t) + \int_0^t p(s, z(y, s-r))\,\mathrm{d}s, \quad 0 < y < \pi,\ t \in J = [0,b] \end{aligned}$$

with

$$z(0,t) = z(\pi,t) = 0, \quad t > 0, \qquad z(t,y) = \phi(t,y), \quad -r \leq t \leq 0$$

where ϕ is continuous and $u \in L^2(J,U)$.
Assume that the following conditions hold with $X = Y = U = L^2(0,\pi)$.

[A$_1$] The operator $B : U \to Y$, is a bounded linear operator.

[A$_2$] The linear operator $W : L^2(J,U) \to X$, defined by

$$Wu = \int_0^b E^{-1}T(b-s)Bu(s)\,\mathrm{d}s$$

has bounded inverse operator $\widetilde{W}^{-1}$ which takes values in $L^2(J,U)/\ker W$.

[A$_3$] Further the function $p : J \times C \to Y$ is continuous in z and strongly measurable in t.

[A$_4$] Let $f(t, w_t)(y) = p(t, w(t-y)), \quad 0 < y < \pi$.

Define the operators $A : D(A) \subset X \to Y$, $E : D(E) \subset X \to Y$ by $Aw = -w_{yy}$, $Ew = w - w_{yy}$ respectively, where each domain $D(A), D(E)$ is given by
$\{w \in X, w, w_y$ are absolutely continuous, $w_{yy} \in X, w(0) = w(\pi) = 0\}$.
With this choice of E, A, B and f, (1) is an abstract formulation of (4) (see [8]).
Then A and E can be written respectively as

$$Aw = \sum_{n=1}^{\infty} n^2(w, w_n)w_n, \qquad w \in D(A),$$

$$Ew = \sum_{n=1}^{\infty} (1+n^2)(w, w_n)w_n, \qquad w \in D(E),$$

where $w_n(y) = \sqrt{2}\sin ny$, $n = 1, 2, 3, \ldots$, is the orthogonal set of eigenvectors of A. Furthermore for $w \in X$ we have

$$E^{-1}w = \sum_{n=1}^{\infty} \frac{1}{1+n^2}(w, w_n)w_n,$$

$$-AE^{-1}w = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2}(w, w_n)w_n,$$

$$T(t)w = \sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}t}(w, w_n)w_n,$$

It is easy to see that $-AE^{-1}$ generates a strongly continuous semigroup $T(t)$ on Y and $T(t)$ is compact such that $|T(t)| \leq e^{-t}$ for each $t > 0$.

[A$_5$] The function p satisfies the following conditions: There exists an integrable function $q : J \to [0, \infty)$ such that

$$|p(t, w(t-y))| \leq q(t)\Omega_1(\|w\|),$$

where $\Omega_1 : [0, \infty) \to (0, \infty)$ is continuous and nondecreasing. Also we have

$$\int_0^b \hat{n}(s)\,ds < \int_c^{\infty} \frac{ds}{1+s+\Omega_1(s)},$$

where $c = |E^{-1}|e^{-t}|E\phi(0)|$, and $\hat{n}(t) = \max\{-1, |E^{-1}|e^{-t}N, |E^{-1}|e^{-t}\int_0^t q(s)\,ds\}$. Here N depends on E, A, B, and p. Further all the conditions stated in the above theorem are satisfied. Hence the system (4) is controllable on J.

Remark. (See also [16].) *Construction of* $\widetilde{W}^{-1}$.

Let $Y = L^2[J, U]/\ker\ W$.

Since ker W is closed, Y is a Banach space under the norm

$$\|[u]\|_Y = \inf_{u\in[u]} \|u\|_{L^2[J,U]} = \inf_{W\hat{u}=0} \|u + \hat{u}\|_{L^2[J,U]}$$

where $[u]$ are the equivalence classes of u.

Define $\widetilde{W} : Y \to X$ by

$$\widetilde{W}[u] = Wu, \quad u \in [u].$$

Now $\widetilde{W}$ is one-to-one and

$$\|\widetilde{W}[u]\|_X \leq \|W\|\|[u]\|_Y.$$

We claim that $V =$ Range W is a Banach space with the norm

$$\|v\|_V = \|\widetilde{W}^{-1}v\|_Y.$$

This norm is equivalent to the graph norm on $D(\widetilde{W}^{-1}) =$ Range W, $\widetilde{W}$ is bounded and since $D(\widetilde{W}) = Y$ is closed, $\widetilde{W}^{-1}$ is closed and so the above norm makes Range $W = V$ a Banach space.

Moreover,

$$\begin{aligned}\|Wu\|_V &= \|\widetilde{W}^{-1}Wu\|_Y = \|\widetilde{W}^{-1}\widetilde{W}[u]\| \\ &= \|[u]\| = \inf_{u\in[u]} \|u\| \leq \|u\|,\end{aligned}$$

so

$$W \in \mathcal{L}(L^2[J,U], V).$$

Since $L^2[J,U]$ is reflexive and $\ker W$ is weakly closed, so that the infimum is actually attained. For any $v \in V$, we can therefore choose a control $u \in L^2[J,U]$ such that $u = \widetilde{W}^{-1}v$.

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