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GENERATED TRIANGULAR NORMS

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An overview of generated triangular norms and their applications is presented. Several properties of generated t-norms are investigated by means of the corresponding generators, including convergence properties. Some applications are given. An exhaustive list of relevant references is included.

1. INTRODUCTION

Triangular norms were introduced in 1942 by Menger [43] in the framework of probabilistic (statistical) metric spaces. Nowadays axiomatics of triangular norms (t-norms in short) is due to Schweizer and Sklar [54].

Definition 1. A mapping $T : [0,1]^2 \to [0,1]$ is called a triangular norm if it is commutative, associative, non-decreasing and 1 is its neutral element, i.e.,

$$T(x,1) = T(1,x) = x \quad \text{for all } x \in [0,1].$$

Note that the dual operation S to T, $S : [0,1]^2 \to [0,1]$,

$$S(x,y) = 1 - T(1-x,\, 1-y),$$

is called a triangular conorm (t-conorm). It is evident that dual operation to S is again the starting operation T. Because of this duality, all properties of t-conorms can be directly derived from the correspondent properties of t-norms. Further, the structure $([0,1],T)$ with T a t-norm is an Abelian fully ordered semigroup with neutral element 1 and annihilator 0. Another such structure is, for example, $([0,\infty],+)$ with neutral element 0 and annihilator ∞. Hence any decreasing bijection $f : [0,1] \to [0,\infty]$ defines a t-norm T_f via semigroup isomorphism,

$$T_f(x,y) = f^{-1}(f(x) + f(y)). \tag{1}$$

The function f is then called an additive generator of T_f. Similarly, we can look for the relationship of t-norms and the semigroup $([0,1],\cdot)$ where $\cdot$ is the standard product and then the corresponding isomorphism is called a multiplicative generator. However, as far as the semigroups $([0,\infty],+)$ and $([0,1],\cdot)$ are isomorphic, the concept of additive generators is equivalent with the concept of multiplicative generators and therefore we will deal in this paper with additive generators only.

We have proposed an approach related to (1) to build up a binary operation on $[0,1]$ in [48].

Definition 2. Let $f:[0,1]\to[0,\infty]$ be a strictly decreasing mapping with $f(1)=0$. Then f is called a conjunctive additive generator and the mapping $T_f:[0,1]^2\to[0,1]$ defined by

$$T_f(x,y)=f^{(-1)}(f(x)+f(y)) \tag{2}$$

is called a generated conjunctor. Here the mapping $f^{(-1)}:[0,\infty]\to[0,1]$ is so called pseudo-inverse of f, see [27], defined by

$$f^{(-1)}(u)=\sup(x\in[0,1];f(x)>z). \tag{3}$$

Note that if a conjunctive additive generator f is continuous and $f(0)=\infty$ then $f^{(-1)}=f^{-1}$ and formulas (1) and (2) are identical.

It is evident that a generated conjunctor T_f is commutative, non-decreasing and 1 is its neutral element, i. e., T_f is a t-norm if and only if it is associative. The problem of the associativity of T_f (based on the properties of the corresponding conjunctive additive generator f) is still an open problem!

Definition 3. Let $f:[0,1]\to[0,\infty]$ be a conjunctive additive generator. If the corresponding generated conjunctor T_f is associative, i. e., it is a t-norm, then f will be called an additive generator (of the t-norm T_f).

Note that the strongest t-norm T_M (i. e., operator min) cannot be generated by means of additive generators, see [5, 63], while the weakest t-norm T_D (the drastic product vanishing on $[0,1[^2)$ is generated by any conjunctive additive generator such that for all x, $y\in]0,1[$ the inequality $2f(x)>f(y)$ holds [25].

Triangular norms are two-place functions and in general their processing may be rather complicated. Note that there exist even (Borel) non-measurable t-norms, see [23]. However, if a t-norm is generated by means of an additive generator, i. e., by means of a one-place function, the situation is much more simpler (and all such t-norms are measurable). Moreover, the complete information about a generated t-norm T_f is hidden in its additive generator f. There are some sufficient conditions and some necessary conditions on f corresponding to the associativity of T_f, see [21, 27, 61, 65]. We recall one sufficient condition from [27].

Proposition 1. Let $f:[0,1]\to[0,\infty]$ be a conjunctive additive generator with the range relatively closed under addition, i. e., for all x, $y\in[0,1]$,

$$f(x)+f(y)\in Ran\, f\cup[f(0^+),\infty], \tag{4}$$

where $f(0^+) = \sup(f(x);\ x \in]0,1])$. Then T_f is a t-norm, i.e., f is an additive generator.

Evidently, the continuity of f on $]0,1[$ is enough to ensure the validity of (4). Note that any t-norm T_f generated by means of an additive generator f with the range relatively closed under addition is Archimedean, see [27, 64], i.e., for all x, $y \in]0,1[$ there is $n \in N$ so that $x^{(n)} < y$, where $x^{(1)} = x$ and for $n = 2,3,\ldots, x^{(n)} = T(x, x^{(n-1)})$. This is not more true in the case of a general additive generators. There are generated t-norms T_f which have non-trivial idempotent elements and hence are not Archimedean [62, 63].

In the framework of probabilistic metric spaces, fuzzy logic, and several other domains where t-norms are applied, the corresponding t-norms are required to be left-continuous (sup-preserving) mappings. An important result concerning the left-continuity of generated t-norms is due to Viceník [66].

Proposition 2. Let T_f be a t-norm generated by means of an additive generator f. Then the following are equivalent:

(i) T_f is left-continuous;

(ii) T_f is continuous;

(iii) f is left-continuous;

(iv) f is continuous.

The importance of the left-continuity for t-norms is the reason why we will deal throughout this paper only with such t-norms, and consequently (because of Proposition 2) with continuous generated t-norms and continuous additive generators only. The paper is organized as follows. The next section gives the famous Ling's representation theorem for continuous Archimedean t-norms, recalls the approximation theorems for continuous t-norms by means of generated t-norms and several convergence results. Section 3 is devoted to the ordering of generated t-norms and the domination relation. Finite systems of generated t-norms are shown to be bounded in the class of generated t-norms. Section 4 discusses several applications of generated t-norms and shows the role of corresponding additive generators. Finally, in conclusions we discuss some other problems in the domain of generated t-norms and recall similar results in some related domains.

2. REPRESENTATION THEOREM AND CONVERGENCE OF GENERATED t-NORMS

One of the basic t-norms is the product t-norm T_P. Its additive fenerator f_P is given by $f_P(x) = -\log x$. Another typical t-norm is generated by means of linear additive generators f_c given by $f_c(x) = c(1-x)$, $c \in]0,\infty[$. For any such c we obtain the same t-norm generated by means of f_c which is usually called the Lukasiewicz

t-norm with notation T_L (also bold product, Giles intersection etc.), $T_L(x, y) = \max(0, x + y - 1)$. Both T_P and T_L are continuous Archimedean t-norms. The structure of all continuous Archimedean t-norms was first explicitly described by Ling [36] based on the previous results of Aczél [2] and Mostert and Shields ([51]).

Theorem 1. A t-norm T is continuous and Archimedean if and only if there is a continuous additive generator f so that $T = T_f$.

Therefore, the class of continuous Archimedean t-norms coincides with the class of (left-)continuous generated t-norms. It is a matter of simple computation that for a continuous additive generator f it is

$$T_f(x, y) = f^{-1}(\min(f(0),\, f(x) + f(y))) \tag{5}$$

and that for two continuous additive generators f and g we have $T_f = T_g$ if and only if $g = cf$ for some positive multiplicative constant c. Consequently, if we fix the values of additive generators in some point x_0 from $]0, 1[$, e. g., we will require that $f(0.5) = 0.5$, then there is a one-to-one correspondence between continuous Archimedean t-norms and continuous additive generators (with that prescribed value). Recall that this is not more true in general case when we admit also non-continuous additive generators, see e. g. [25, 61, 62].

There are two principal subclasses of continuous Archimedean t-norms distinguished by the value of their respective additive generators in the point 0. Namely, all continuous generated t-norms whose additive generators are unbounded, i. e., $f(0) = \infty$, are isomorphic with the product t-norm T_P and they are called strict t-norms. They are strictly increasing on the half-open square $]0, 1]^2$, or, equivalently, they fit the cancellation law ($T(x, y) = T(x, z)$ if and only if $x = 0$ or $y = z$). Continuous generated t-norms with bounded additive generators ($f(0) < \infty$) are isomorphic with the Lukasiewicz t-norm T_L and they are called nilpotent. Each element $x \in]0, 1[$ is a nilpotent element of such t-norm, i. e., for some $n \in N$ we have $x^{(n)} = 0$.

The important role of continuous generated t-norms in the framework of continuous t-norms is stressed not only by the general representation theorem for continuous t-norms [36, 28] showing that each continuous t-norms can be built by means of the strongest t-norm T_M (minimum) and by means of continuous generated t-norms, but also by the next approximation theorem, which is due to [22, 46, 52].

Theorem 2. Let T be a given continuous t-norm. Then for any $\varepsilon > 0$ there is a continuous generated t-norm T_ε which is ε-close to T, i. e., for all x, $y \in [0, 1]$ we have

$$|T(x, y) - T_\varepsilon(x, y)| \leq \varepsilon.$$

Note that the t-norm T_ε in the previous theorem can be chosen either strict or nilpotent. From the topological point of view, Theorem 2 means that continuous generated t-norms are dense in the class of continuous t-norms.

Let f be a given continuous additive generator of a t-norm T. For $\lambda \in]0, \infty[$, we can introduce two new functions $f_{(\lambda)}, f^{(\lambda)} : [0,1] \to [0, \infty]$ defined by

$$f_{(\lambda)}(x) = (f(x))^{\lambda}$$

and

$$f^{(\lambda)}(x) = f(x^{\lambda}).$$

It is evident that then both $f_{(\lambda)}$ and $f^{(\lambda)}$ are continuous additive generarots and we denote the corresponding continuous generated t-norms by $T_{(\lambda)}$ and $T^{(\lambda)}$, respectively. In [28], we have investigated the limit properties of these t-norms when parameter λ converges to 0 and ∞, respectively.

Theorem 3. Let T be any continuous generated t-norm. Then

$$\lim_{\lambda \to \infty} T_{(\lambda)} = T_M \quad \text{(uniformly)}$$

and

$$\lim_{\lambda \to 0+} T_{\lambda} = T_D \quad \text{(pointwisely)}.$$

It is interesting to see that the limit properties of $T_{(\lambda)}$ are not dependent on the t-norm T (what is not true e.g. in the case of generated aggregation operators, see [32]).

Theorem 4. Let T be a continuous generated t-norm generated by a continuous additive generator f and let $f'(1^-) = \lim_{x \to 1^-} f(x)/(x-1) \in] -\infty, 0[$. Then

$$\lim_{\lambda \to 0+} T^{(\lambda)} = T_P \quad \text{(uniformly)}.$$

If T is nilpotent then (independently of T)

$$\lim_{\lambda \to \infty} T^{(\lambda)} = T_D \quad \text{(pointwisely)}.$$

The convergence in the class of continuous generated t-norms (which is necessarily uniform) have their counterparts in the convergence in the class of continuous additive generators (not necessarily uniform!). The next results were first shown in [21, 22] and then simplified in [28, 47].

Theorem 5. Let $T, T_1, T_2, \ldots,$ be generated continuous t-norms. Then the following are equivalent:

(i) $\lim_{n \to \infty} T_n = T$ (pointwisely);

(ii) $\lim_{n \to \infty} T_n = T$ (uniformly);

(iii) There is a continuous additive generator f of T and a sequence $\{f_n\}$ of continuous additive generators generating t-norms T_n, respectively, such that $\lim_{n\to\infty} f(x) = f(x)$ for all $x \in]0,1]$.

Apply, e. g., Theorem 4 to the Lukasiewicz t-norm T_L and its additive generator $f_L(x) = 1 - x$. Then the generators $f_L^{(\lambda)}$ are defined by

$$f_L^{(\lambda)}(x) = 1 - x^\lambda$$

and the corresponding t-norms $T_L^{(\lambda)}$ converge uniformly to the product t-norm T_P for $\lambda \to 0^+$. Now, we can put $T = T_P$ and $T_n = T_L^{(1/n)}$. Evidently, for all $x \in]0,1]$ we have $\lim_{n\to\infty} f_L^{(1/n)}(x) = 0$, i. e., the limit function is not an additive generator. However, also the functions $f_n = n\, f_L^{(1/n)}$ are continuous additive generators of T_n, respectively, and $\lim_{n\to\infty} f_n(x) = -\log x$ for all $x \in [0,1]$. Recall that above mentioned t-norms are the t-norms introduced by Schweizer and Sklar [55]. As an example where the pointwise convergence of additive generators on whole interval $[0,1]$ fails (in point 0) though the convergence of corresponding t-norms holds, recall the family of Frank t-norms [15] in which the strict t-norms converge to a nilpotent t-norm (namely, to the Lukasiewicz t-norm T_L). However, then in point 0 the pointwise convergence of additive generators is necessarily violated.

3. ORDERING AND DOMINATION OF CONTINUOUS GENERATED t-NORMS

The partial order in the class of t-norms is inherited from the standard order on the unit interval. We say that the t-norm T is weaker than the t-norm T^*, i. e., $T \le T^*$, if for all $(x,y) \in [0,1]^2$ we get $T(x,y) \le T^*(x,y)$. Recall that for any t-norm T it holds $T_D \le T \le T_M$. In the case of comparing continuous generated t-norms, we are able to exploit their respective additive generators. The next result was first shown in [55], see also [26, 28, 57].

Theorem 6. Let T_i be a continuous generated t-norm with continuous additive generator f_i, $i = 1, 2$. Then $T_1 \le T_2$ if and only if the composite function $h = f_1 \circ f_2^{-1} : [0, f_2(0)] \to [0,\infty]$ is subadditive, i. e., for all $x,\ y,\ x+y \in [0, f_2(0)]$ it holds $h(x+y) \le h(x) + h(y)$.

Note that the subadditivity of composite h is equivalent with the superadditivity of its inverse $h^{-1} = f_2 \circ f_1^{-1}$. As a straightforward corollary of Theorems 3, 5 and 6 we have the next result.

Corollary 1. Let T be a continuous generated t-norm and let $T_{(\lambda)}$ be defined as in the previous section, $\lambda \in]0,\infty[$. Let $T_0 = T_D$ and $T_\infty = T_M$. Then the family $(T_{(\lambda)})_{\lambda\in[0,\infty]}$ is strictly monotone and continuous with respect to the parameter λ.

Recall that several well-known families of t-norms are based on the construction described in Corollary 1. So, e. g., the Yager family [67] is just $((T_L)_{(\lambda)})_{\lambda\in[0,\infty]}$, the Aczél–Alsina family [4] is just $((T_P)_{(\lambda)})_{\lambda\in[0,\infty]}$, the Dombi family [11] is $((T_H)_{(\lambda)})_{\lambda\in[0,\infty]}$, where T_H is the Hamacher product [19] generated by additive generator $f_H(x) = 1 - 1/x$, $T_H(x,y) = xy/(x+y-xy)$ whenever $(x,y) \neq (0,0)$.

It is clear that a strict t-norm T can never be weaker than a nilpotent t-norm T^* (note that then $T(x,y) > 0$ whenever $x > 0,\ y > 0$, while $T^*(x,y) = 0$ for some $x > 0,\ y > 0$). However, strict and nilpotent t-norms may be incomparable. So, e. g., strict t-norm T_P is incomparable with the nilpotent Yager t-norm $(T_L)_{(2)} = T_2^y$ with parameter $\lambda = 2$. For an infinite system of continuous generated t-norms, there need not exist an upper and a lower bound in the class of continuous generated t-norms. By Theorem 3, there are such systems with only upper bound T_M (which is continuous but not Archimedean neither generated) and with the only lower bound T_D (which is Archimedean and generated but not continuous). However, in the case of finite systems we can always ensure the existence of lower and upper bounds (in the class of continuous generated t-norms), see [38, 39].

Theorem 7. Let $T_1,\ldots,T_n$ be a finite system of continuous generated t-norms. Then there exist continuous generated t-norms T (which is nilpotent) and T^* (which is strict) so that

$$T \leq T_i \leq T^* \quad \text{for all } i = 1,\ldots,n.$$

Moreover, if all $T_i,\ i = 1,\ldots,n$, are nilpotent (strict), then also the upper bound T^* can be chosen to be nilpotent (the lower bound T can be chosen to be strict).

Recall that the proof of Theorem 7 in [39] is constructive. However, it need not give the best upper (lower) bound neither in the case of two comparable t-norms and hence the problem whether the class of continuous generated t-norms forms a lattice is still open.

Another important relationship between two t-norms is the domination. Domination plays a key role in several domains where t-norms are applied, mostly when we discuss the Cartesian product of structures as Menger spaces, fuzzy equivalences or equalities, etc.

Definition 4. Let T and T^* be t-norms. We say that T dominates T^*, $T \gg T^*$, if for all $x,\ y,\ u,\ v \in [0,1]$ it holds

$$T(T^*(x,y),\ T^*(u,v)) \geq T^*(T(x,u),\ T(y,v)).$$

Again for any t-norm T we have $T_D \ll T \ll T_M$. Moreover, $T \gg T^*$ implies $T \geq T^*$ but the opposite need not be true, i. e., there are t-norms T and T^* such that $T \geq T^*$ but not $T \gg T^*$. Note that the relation of domination is reflexive, antisymmetric, but it is an open problem whether it is also transitive, i. e., whether $\gg$ is a partial order. For more information about the domination we recommend [28, 57], where also the next result first shown in [60] can be found.

Theorem 8. Let T_i be a strict t-norm with continuous additive generator f_i, $i = 1, 2$. Then $T_1 \gg T_2$ if and only if the composite function $h = f_1 \circ f_2^{-1}$: $[0, f_2(0)] \to [0, \infty]$ fulfills (for all a, b, c, $d \in [0, 1]$)

$$h^{-1}(h(a+c) + h(b+d)) \leq h^{-1}(h(a) + h(b)) + h^{-1}(h(c) + h(d)). \tag{6}$$

Again as in the case of ordering it can be shown that starting from any strict T, the family $(T_{(\lambda)})_{\lambda \in [0,\infty]}$ is monotone with respect to domination, i. e., $T_\lambda \gg T_\mu$ if and only if $\lambda \geq \mu$. Recall that the inequality (6) forces the convexity of the composite function h (which is in any case continuous, strictly increasing, and $h(0) = 0$). However, the convexity of a continuous strictly increasing mapping $h : [0, u] \to [0, \infty]$ with $h(0) = 0$ is sufficient for the superadditivity of h but not sufficient for h to fulfill (6). For interested readers note that several interesting results concerning the domination of continuous generated t-norms can be found e. g. in [60].

4. APPLICATIONS OF CONTINUOUS GENERATED t-NORMS

To stress the importance of continuous generated t-norms and their corresponding additive generators, we give now some applications. In the framework of probability theory, an important issue is the determination of the joint distribution from given marginal distributions. Because of the measurability of random variables, it is enough to determine the joint distribution on the product of intervals by means of the values of marginal distributions of the underlying intervals. This processing is done by means of copulas [14, 58], which need not be associative (neither commutative) in general. However, if we require the non-dependence of the result on the order of marginal distributions we work with (applying the obvious rearranging of arguments whenever necessary), then the associativity of the applied copula is necessary. Recall, e. g., the case of independent random variables, when the joint distribution is constructed from the marginale ones by means of the product, i. e., by means of the associative copula T_P.

Definition 5. A non-decreasing mapping $C : [0, 1]^2 \to [0, 1]$ is called a (binary) copula if 1 is its neutral element and for all x, y, u, $v \in [0, 1]$ such that $x \leq u$, $y \leq v$ it holds

$$C(x, y) + C(u, v) \geq C(x, v) + C(u, y). \tag{7}$$

Note that any copula is continuous (more, it is a Lipschitz function with constant 1) and for any copula C we have $T_L \leq C \leq T_M$. As mentioned above, a copula C need not be neither commutative nor associative. However, if C is an associative copula, then it is automatically also commutative (see e. g. [28, 51, 57]) and hence a t-norm.

Proposition 3. A copula C is associative if and only if it is a t-norm fulfilling (7).

Coming back to the domination of t-norms, it is evident that each t-norm T which dominates the weakest copula T_L, $T \gg T_L$, is an associative copula (and the opposite is not true, i.e., there are associative copulas not dominating T_L). Each associative copula C is constructed from the strongest copula T_M by means of generated copulas, i.e., by means of continuous generated t-norms which fulfill (7). However, if a t-norm T is generated by means of a continuous additive generator f then (7) can be rewritten to a more transparent requirement [57].

Theorem 9. A continuous generated t-norm T is a copula if and only if any continuous additive generator f of T is convex.

In the fuzzy set theory, an important role is played by fuzzy equivalence relations. The transitivity here is essentially based on some given t-norm T which plays the role of logical conjunction [6, 8, 18, 20, 68].

Definition 6. Let X be a given (non-empty) universe and let T be a given t-norm. A fuzzy subset E of the Cartesian product X^2, i.e., a mapping $E : X^2 \rightarrow [0,1]$ is called a T-fuzzy equivalence if it is reflexive, symmetric and T-transitive, i.e., for all x, y, $z \in X$ we have

$$\begin{aligned} &E(x,x) = 1, \\ &E(x,y) = E(y,x) \quad \text{and} \\ &T(E(x,y),\, E(y,z)) \leq E(x,z). \end{aligned}$$

Moreover, if $E(x,y) = 1$ only if $x = y$ then a T-fuzzy equivalence E is called a T-fuzzy equality.

It can be easily shown that if a t-norm T is weaker then another t-norm T^*, $T \leq T^*$, then each T^*-fuzzy equivalence relation (equality) is also a T-fuzzy equivalence relation (equality). In the case of the Lukasiewicz t-norm T_L, Bezdek and Harris [6] gave an interesting relationship between T_L-fuzzy equivalence relations and pseudo-metrics on X. We have generalized this result for continuous generated t-norms in [8].

Theorem 10. Let T be a continuous generated t-norm with a continuous additive generator f. Let E be a T^*-fuzzy equivalence relation (equality) on a given universe X, where T^* is an arbitrary t-norm such that $T^* \geq T$. Then the mapping $d : X^2 \rightarrow [0,\infty]$ given by $d = f \circ E$, i.e. $d(x,y) = f(E(x,y))$ is a pseudo-metric (metric) on X.

Vice-versa, let $d : X^2 \rightarrow [0,\infty]$ be a given pseudo-metric (metric) on X. Then the mapping $E : X^2 \rightarrow [0,1]$ given by $E = f^{(-1)} \circ d$, i.e., $E(x,y) = f^{-1}(\min(f(0),\, d(x,y)))$ is a T_*-fuzzy equivalence relation (equality) for any t-norm $T_* \leq T$.

Recall that T-fuzzy equivalence relations are closely related to fuzzy partitions [9] and that the additive generators play an important role when defining fuzzy partitions based on generated continuous t-norms [20]. So, e. g., when working on real line, the shapes of fuzzy points (elements of fuzzy partitions) corresponds to pseudo-inverses of additive generators of applied general continuous t-norms.

As the last field of this small overview of application of additive generators of continuous generated t-norms we mention the field of fuzzy arithmetics. The standard addition of real numbers is extended to the addition of fuzzy quantities (fuzzy subsets of the real line R) by means of the generalized Zadeh extension principle [69] based on some t-norm T.

Definition 7. Let A, $B : R \to [0,1]$ be two given fuzzy quantities and let T be a given t-norm. Then the T-based sum of A and B is the fuzzy quantity $A \oplus_T B : R \to [0,1]$ defined by

$$A \oplus_T B(x) = \sup(T(A(u),\, B(v)) \,|\, u + v = x). \tag{8}$$

The fuzzy arithmetics based on the strongest t-norm T_M is exhaustively discussed in [37]. In general, the processing with formula (8) is rather complicated and time consuming. However, in the case of generated continuous t-norms and some special types of fuzzy quantities (related to the respective additive generators), we have several interesting (and simple) results, see [12, 16, 29, 30, 41, 45]. An exhaustive overview of fuzzy intervals theory can be found in [13].

Theorem 11. Let T be a continuous generated t-norm and let f be a continuous additive generator of T. Let A, $B : R \to [0,1]$ be two unimodal continuous fuzzy intervals, i. e., there are peaks a, $b \in R$ so that $A(x) = 1$ if and only if $x = a$, $B(y) = 1$ if and only if $y = b$, and the partial mappings $A|]-\infty, a]$, $A|[a, \infty[$, $B|]-\infty, b]$ and $B|[b, \infty[$ are monotone, Then:

(i) if all composite mappings $f \circ A|]-\infty, a]$, $f \circ A|[a, \infty[$, $f \circ B|]-\infty, b]$ and $f \circ B|[b, \infty[$ are concave then $A \oplus_T B$ is a unimodal continuous fuzzy interval with peak $a + b$ and

$$A \oplus_T B(x) = \max(A(x-b),\, B(x-a)),\ x \in R;$$

(ii) if there is a convex function $K : [0, u[\to [0, \infty[$, $u \in]0, \infty]$, $K(0) = 0$ so that there are constants v, w, r, $s \in]0, \infty[$ such that

$$\begin{aligned} A(x) &= f^{(-1)}(vK((x-a)/v)) && \text{if } x \in [a,\, a+uv] \\ &= f^{(-1)}(wK((a-x)/w)) && \text{if } x \in [a-wu,\, a] \\ &= 0 && \text{otherwise,} \end{aligned}$$

and

$$
\begin{aligned}
B(x) &= f^{(-1)}(rK((x-b)/r)) && \text{if } x \in [b,\, b+ru] \\
&= f^{(-1)}(sK((b-x)/s)) && \text{if } x \in [b-su,\, b] \\
&= 0 && \text{otherwise,}
\end{aligned}
$$

then

$$
\begin{aligned}
A \oplus_T B(x) &= f^{-1}(tK((x-a-b)/t)) && \text{if } x \in [a+b,\, a+b+tu] \\
&= f^{-1}(qK((a+b-x)/q)) && \text{if } x \in [a+b-qu,\, a+b] \\
&= 0 && \text{otherwise,}
\end{aligned}
$$

where $t = v + r$ and $q = w + s$.

The rather complicated general formulation of Theorem 11 is much simplified in several special cases. Note only that its direct application to the addition of triangular (trapezoidal with necessary modification concerning the relevant peaks) fuzzy numbers results to the preserving of the linear shape of the final output whenever we work with t-norm $T \le T_L$ or any of Yager's t-norm T^Y_λ. More, it allows to reduce the computation with fuzzy numbers in mentioned case to the processing with relevant parameters only. Similarly, the T_P based addition of Gaussian fuzzy numbers can be derived straightforwardly (if $A_i = G(\mu_i, \sigma_i^2)$ then $A_1 \oplus_{T_P} \cdots \oplus_{T_P} A_n = G(\sum \mu_i, \sum \sigma_i^2)$, where $A(x) = \exp((x-\mu)^2/\sigma^2)$ whenever $A = G(\mu, \sigma^2)$). For more details we recommend [13, 30, 45].

5. CONCLUDING REMARKS

We have given an overview of older and recent results on continuous generated triangular norms and their continuous additive generators, including some applications. Several problems were not mentioned. So, for examples, the majority of specific requirements on continuous generated t-norms can be rewritten for their additive generators and hence we get some specific functional equations (or inequalities, etc.). So, for example, the famous Frank family of t-norms [15] gives the Archimedean solutions of the functional equation

$$F(x, y) + G(x, y) = x + y,$$

where F is some associative copula (i.e., a continuous t-norm) and G is its dual copula which is supposed to be associative too (and hence a t-conorm).

Several other families already mentioned in this overview have their origine in the solution of some special functional equations extensively treated in [3]. An interesting problem is the determination of a t-norm when we know its values on some subset of the unit square. So, e.g., in the case of generated continuous t-norms, if the values on diagonal are known, we have to solve the Schroeder functional equation [3] and the relevant solutions are completely described in [50]. An interesting consequence of results from Section 3 (comparison of t-norms) and results of [50] is the fact, that two different continuous t-norms coinciding on the diagonal of the unit equare are always incomparable (what is not true in the case of general t-norms).

Another interesting field of applications of continuous generated t-norms (and t-conorms) is the field of general measure and integration theory. We recall here, for example, the characterization of fuzzy σ-algebras based on Frank t-norms and subsequent characterization of measure on these σ-algebras in [49]. Closely related are also results concerning so-called g-integral and g-derivatives [40, 53].

The concept of generators has been applied also in several other domains. Recall, for examples, generated conjunctors for many-valued logic introduced in [48] and applied in [59]. Interesting and promising is the concept of additive generators of aggregation operators introduced in [34] and further developed in [35]. Note that the full characterization of continuous associative generated aggregation operators is given in [7]. Several other properties of special generated aggregation operators are discussed in [31, 32].

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