KYBERNETIKA — VOLUME 36 (2000), NUMBER 2, PAGES 255-262

DESIGN OF AN ADAPTIVE CONTROLLER OF LQG TYPE: SPLINE-BASED APPROACH

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The paper presents an alternative approach to the design of a hybrid adaptive controller of Linear Quadratic Gaussian (LQG) type for linear stochastic controlled system. The approach is based on the combination standard building blocks of discrete LQG adaptive controller with the non-standard technique of modelling of a controlled system and spline approximation of involved signals. The method could be of interest for control of systems with complex models, in particular distributed parameter systems.

1. INTRODUCTION

Most industrial controlled processes work in continuous time. At the same time, adaptive controllers are mainly based upon discrete-time models of a controlled process. These models are popular because they are relatively simple, well elaborated theoretically and easy to implement on digital computers.

The success/failure of discrete adaptive control of a continuous dynamic system depends heavily on the sampling (control) period used. Discrete controllers can be designed using low order model if the sampling rate is low enough. The inter-sampling behaviour may be rather bad if the process dynamics is fast. Naturally, more information on the intersample behaviour can be gained by increasing the sampling frequency. The growing sampling frequency makes, at the same time, whole design sensitive to mismodelling and numeric errors.

A well chosen filtering may help. However, its success depends heavily on the prior information fed into the filter and its design is far from being easy.

The search for an alternative technique for the design of an adaptive controller reported in the paper is mainly stimulated by the above-mentioned reasons. The main aim is to create a high-quality adaptive controller that respects the continuous nature of a controlled process and discrete data handling.

There is a strong additional reason for the work. The system dynamics is often described by an infinite-dimensional model with unknown parameters. The model is usually represented by a set of partial differential equations. Modelling and control design for such systems are rather complex and success has been achieved for special types of equations only.

The design advocated here is substantially simpler. It is based on a general convolution description of a continuous time controlled process and spline approximation of involved signals [2, 6]. Spline models are hybrid models in a sense that they model continuous variable of a system at arbitrary time instant. And, at the same time, they have the form suitable for digital handling.

The original idea of spline-based self-tuning controller was described in [5] and connected with application in paper industry. The key point was simultaneous spline approximation of the signals as well as operator kernels, applied to the convolution model of a controlled system.

Compare to [5], the unnecessary approximation of the operator kernels is omitted in the present article, which significantly simplifies the whole algorithm.

The adopted idea allows us to construct the hybrid adaptive controller of LQG type by a simple modification of standard building blocks of discrete-time LQG adaptive controllers.

2. PRELIMINARIES

The section summarises key assumptions and basic mathematical tools used in the paper.

2.1. Basic assumptions

The considered design of the hybrid adaptive controller of a continuous system is performed under the following assumptions:

- the system is stable, linear (linearisable) and time-invariant or at most slowly time-varying;
- the relations of the system output and the process noise to the system input are searched in terms of convolution operators;
- the signals are smooth enough, thus they can be approximated by properly chosen splines;
- the only limited past history of the controlled process has a time effect on the future process behaviour;
- the single-input single-output (SISO) controlled system is considered (for simplicity).

2.2. Spline functions

Let a grid on the time interval $[0,T]$ be specified by nodes $\{t_i\}$, $i = 1,\ldots,N$,

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T < \infty. \tag{1}$$

A function of time $S_{m,d}(t)$ is called *spline of the degree m and the defect $d \in [0, m+1]$* with nodes on the grid (1) iff [1]

- $S_{m,d}(t)$ is a polynomial in t of the degree m on every subinterval $[t_i, t_{i+1})$;
- $S_{m,d}(t)$ possesses continuous derivatives up to the order $m - d$ on the interval $[0, T]$.

For the fixed grid (1), splines form a linear space with dim $= (N-1)\,d + m + 1$. Thus, any approximating polynomial spline $S_{m,d}(t)$ can be expressed as a linear combination of *basis functions* $\{\phi_i\}_{i=0}^{\text{dim}-1}$

$$S_{m,d}(t) = \sum_{i=0}^{\text{dim}-1} w_i \phi_i(t) = \boldsymbol{w}' \, \boldsymbol{\phi}(t), \tag{2}$$

where $\boldsymbol{w}'$ denotes transposition of the weighting vector $\boldsymbol{w} = [w_0, w_1, \ldots, w_{\text{dim}-1}]'$. Similarly, the basis functions are grouped into the vector $\boldsymbol{\phi}(t) = [\phi_0, \phi_1, \ldots, \phi_{\text{dim}-1}]'$.

2.3. System model

The considered continuous-time controlled system is assumed to be described by the linear operator equation relating the system input $u(t)$, the output $y(t)$ and the noise $\bar{e}(t)$ signals

$$\mathcal{A}\, y + \mathcal{B}\, u + \mathcal{C}\, \bar{e} = 0. \tag{3}$$

$\mathcal{A}, \mathcal{B}, \mathcal{C}$ are time-invariant, casual operators acting of the output y, the input u and the noise $\bar{e}$, respectively.

Let us assume that there is a "practically whitening" operator $\mathcal{C}^*$, i.e. the transformed noise signal $e(\cdot) \equiv \mathcal{C}^* \, \mathcal{C}\, \bar{e}(\cdot)$ becomes white discrete process after sampling with the shortest technically feasible period.

Multiplication of (3) by $\mathcal{C}^*$ provides an alternative form of the system description

$$\boldsymbol{A}\, y + \boldsymbol{B}\, u + e = 0. \tag{4}$$

Here, $\boldsymbol{A} = \mathcal{C}^* \, \mathcal{A}$ and $\boldsymbol{B} = \mathcal{C}^* \, \mathcal{B}$ are time-invariant casual linear operators.

Using the convolution theorem [7], the system model can be rewritten in the form

$$\int_0^t k_A(\tau) y(t-\tau) \, \mathrm{d}\tau + \int_0^t k_B(\tau) u(t-\tau) \, \mathrm{d}\tau + e(t) + O_t = 0, \tag{5}$$

where $O_t = O_t^A + O_t^B$ is the total offset reflecting the initial conditions (tends to zero for a stable system). $k_A(\tau)$ and $k_B(\tau)$ are casual ($k_A(\tau) = 0$ for $\tau \le 0$) kernels of the operators $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively.

According to the finite-memory assumption, both the kernels k_A and k_B vanish for $t > L_\bullet > 0$ ($\bullet = \{A, B\}$), where $L_\bullet$ denotes the length of the operator kernel in terms of signal sampling unit. If the influence of initial conditions is neglected (the inspected system is stable), *final form of the considered system model* is gained

$$\int_0^{\min(t, L_A)} k_A(\tau) y(t-\tau) \, \mathrm{d}\tau + \int_0^{\min(t, L_B)} k_B(\tau) u(t-\tau) \, \mathrm{d}\tau + e(t) = 0. \tag{6}$$

Adaptive controller to be designed has to respect that kernels are unknown functions.

2.4. Spline approximation of signals

The continuous time input/output signals $u(t)$ and $y(t)$ will be interpolated using splines. Grid points of the splines are taken as sampling points of the corresponding signals (a common sampling rate for all signals is supposed for simplicity). That is, a regular sampling with a sampling period t_s is supposed:

$$\begin{aligned} u(t_i^u) &\equiv u(t_i^u) \quad \text{for } t_i^u = i\, t_s, \\ y(t_i^y) &\equiv y(t_i^y) \quad \text{for } t_i^y = i\, t_s + \textit{shift of I/O sampling}. \end{aligned} \tag{7}$$

To simplify the weights determination (2), the interpolation spline basis ϕ^u and ϕ^y (for the input and the output, respectively) with the finite supports satisfying the condition:

$$\phi_i^u(t_j^u) = \phi_i^y(t_j^y) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

have been supposed. Then, the weights of approximated signals (2) will coincide with the values of signals u and y at the sampling instants. Hence, the approximate values $\tilde{u}(t-\tau)$ and $\tilde{y}(t-\tau)$ of the input and the output, respectively, can be expressed using the interpolation splines:

$$\begin{aligned} u(t-\tau) &\approx \tilde{u}(t-\tau) = \sum_{j=0}^{\mathrm{dim}-1} u(t_j^u)\, \phi_j^u(t-\tau), \\ y(t-\tau) &\approx \tilde{y}(t-\tau) = \sum_{j=0}^{\mathrm{dim}-1} y(t_j^y)\, \phi_j^y(t-\tau). \end{aligned} \tag{9}$$

To be more brief, the present discussion considers the approximation by the interpolating splines only. It should be stressed, however, that in the case of smoothing splines, the expression for the approximated signals induces filtration of the measured data.

3. ADAPTIVE CONTROLLER

The conversion of the final form of the system model to the regression type and design for LQ criterion are performed in this section.

3.1. Spline–based regression model

Substitution the approximated values of the input/output signals (9) into the system model (6) converts the last one to the form (L_A and L_B are the length of support of the operator kernels k_A, k_B, dim is a dimension of the chosen spline space):

$$\sum_{j=0}^{\dim-1} y(t_j^y) \int_0^{\min(t,L_A)} k_A(\tau)\, \phi_j^y(t-\tau)\, \mathrm{d}\tau$$
$$+ \sum_{j=0}^{\dim-1} u(t_j^u) \int_0^{\min(t,L_B)} k_B(\tau)\, \phi_j^u(t-\tau)\, \mathrm{d}\tau$$
$$+ e(t) = 0. \tag{10}$$

In the case of unbounded control horizon, dim tends to the infinity too (because the number of nodes is proportional to the assumed horizon).

Due to the finite memory assumption, the operator kernels $k_A(\cdot)$ and $k_B(\cdot)$ have finite length. At the same time, the basis functions are non-zero for the finite number of supports. Thus, the only limited number of basis functions have non-empty intersection with the operator kernels. That is, the product in the element of integration gets the only limited number of the non-zero values that become unknown coefficients of the standard regression model.

Let the vectors $\Phi^A = [\Phi_0^A, \Phi_1^A, \ldots, \Phi_j^A \ldots]'$, $\Phi^B = [\Phi_0^B, \Phi_1^B, \ldots, \Phi_j^B \ldots]'$ with the following elements:

$$\Phi_j^A(t) = \int_0^{\min(t,L_A)} k_A(\tau)\, \phi_j^y(t-\tau)\, \mathrm{d}\tau,$$
$$\Phi_j^B(t) = \int_0^{\min(t,L_B)} k_B(\tau)\, \phi_j^u(t-\tau)\, \mathrm{d}\tau \tag{11}$$

are introduced. These vectors are potentially infinite dimensional with the fixed finite number of the non-zero members (denoted m_A and m_B). Under the notations (11), the model (10) can be rewritten in the form:

$$\sum_{j=0}^{m_A-1} y(t_j^y)\ \Phi_j^A(t) + \sum_{j=0}^{m_B-1} u(t_j^u)\ \Phi_j^B(t) + e(t) = 0. \tag{12}$$

To perform the identification, let us choose the time sequence $\{t_k\}_{k=0}^{\infty}$ such that the sampled noise $\{e_k\}_{k=0}^{\infty}$ is a white discrete process and $e_k = e(t_k)$.

One can see, that, in the case when the interpolation nodes and the identification moments are invariant, the non-zero entries of the vectors Φ^A and Φ^B do not depend on the identification moment. By other words, the non-zero parts of vectors Φ^A and Φ^B remain the same, but their positions are shifted with time.

Respecting the weights definition (see Section 2.4.), the corresponding data vector is formed: $d_k = [y_{t_k}, \ldots, y_{t_{k-m_A}}, u_{t_k}, \ldots, u_{t_{k-m_B}}]'$. Introducing the vector of unknown coefficients $\Theta = [\Phi^A, \Phi^B]'$ converts (12) to the form of standard ARX model:

$$\Theta'\, d_k + e_k = 0, \quad k = 0, 1, 2, \ldots \tag{13}$$

The model (13) is linear in unknown constant parameters Θ. To identify them, the Recursive Least Squares (RLS) algorithm [10] can be applied directly. The technique of restricted forgetting [9] is available in the case of a system with varying parameters.

3.2. Control synthesis

To perform the control synthesis we shall optimise the expected value $\mathcal{E}[\cdot]$ of the loss function

$$J = \frac{1}{T} \int_0^T [q_y^2\, y^2(t) + q_u^2\, u^2(t)]\, \mathrm{d}t, \tag{14}$$

specified by the control horizon T (potentially infinite) and by the output q_y and input q_u penalties. Assuming the approximation error is negligible in (9), the loss function in terms of approximated values of the given signals is

$$J \approx \tilde{J} = \frac{1}{T} \int_0^T [q_y^2\, \tilde{y}^2(t) + q_u^2\, \tilde{u}^2(t)]\, \mathrm{d}t. \tag{15}$$

Substituting the values of the approximated signals from (9), the integral quadratic performance index can be written

$$\tilde{J} = \boldsymbol{y}\, Q^y\, \boldsymbol{y}' + \boldsymbol{u}\, Q^u\, \boldsymbol{u}', \tag{16}$$

where $\boldsymbol{y}$ and $\boldsymbol{u}$ are the vectors of the output and the input samples; Q^y and Q^u are penalising matrices:

$$Q^y = \frac{q_y^2}{T} \int_0^T \boldsymbol{\phi}^{y\prime}(t)\, \boldsymbol{\phi}^y(t)\, \mathrm{d}t, \quad Q^u = \frac{q_u^2}{T} \int_0^T \boldsymbol{\phi}^{u\prime}(t)\, \boldsymbol{\phi}^u(t)\, \mathrm{d}t, \tag{17}$$

with $\boldsymbol{\phi}^u$ and $\boldsymbol{\phi}^y$ stand for pre-defined basis functions, Q^y and Q^u denote sparse, a few diagonal, symmetric matrices (in the case when the interpolating nodes for the input and the output signals are equidistant). For example, in the case of the first order spline approximation, the matrices are tridiagonal with elements ($x = u, y$ and $f = \phi^u, \phi^y$, respectively):

$$Q_{ij}^x = \begin{cases} \displaystyle\int_{t_{i-1}^x}^{t_{i+1}^x} f_i^2(t)\, \mathrm{d}t, & \text{if } i = j \\ \displaystyle\int_{t_i^x}^{t_{i+1}^x} f_i(t) f_{i+1}(t)\, \mathrm{d}t, & \text{if } |j - i| = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

Assuming the complete knowledge of the model parameters Θ, we can build a over-parametrised regression model that directly relates the sampled values of the output signal to the input. Then, the standard LQ design can be performed. It becomes an adaptive controller when the certainty equivalence technique is applied. Minimising of integral quadratic loss is solved by well justified algorithm [4], developed for regression models and discrete time quadratic loss.

The control synthesis is performed on square root factors of involved matrices in order to achieve numerically stable algorithm.

4. CONCLUSIONS

The paper considers a spline-based approach to the design of the hybrid adaptive controller. The use of splines for the modelling of a controlled process results in a very compact description of the controlled system (even for the system with complex models), that leads to the simple design of the controller. Besides, the proposed approach gives freedom to choose non-equidistant spline nodes, which possesses better modelling, especially for short sampling periods. In the case of smoothing splines (which have more general meaning than the interpolation ones), the approximate model include also filtration of measured data.

With the adopted model (6), the control design is straightforward. It consists of the following key stages:

- The involved signals are approximated by splines with nodes that include sampling time instants. This converts the model into a standard autoregressive-regressive model (ARX) which is linear in unknown constant parameters. To estimate the unknown system parameters, RLS method can be applied.

- Control design for LQ criterion is performed. Despite of the non-standard weighting, the standard technique is applicable.

- The predictive controller model working on the sampled signals is built by using certainty equivalence strategy (using parameter estimates instead of parameters). Its regressor consists of sampled data, possibly filtered.

ACKNOWLEDGEMENT

The reported research has been supported by the Grant Agency of the Czech Republic under grants 102/00/P045, 102/99/1564 and 102/99/1292.

(Received May 19, 1997.)

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