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ON THE STRUCTURE AT INFINITY OF LINEAR DELAY SYSTEMS WITH APPLICATION TO THE DISTURBANCE DECOUPLING PROBLEM

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The disturbance decoupling problem is studied for linear delay systems. The structural approach is used to design a decoupling precompensator. The realization of the given precompensator by static state feedback is studied. Using various structural and geometric tools, a detailed description of the feedback is given, in particular, derivative of the delayed disturbance can be needed in the realization of the precompensator.

1. INTRODUCTION

The structure at infinity or the Smith–McMillan form at infinity are well known tools for the characterization of the solvability of some control problems such as model matching, disturbance decoupling, row-by-row decoupling. For linear finite dimensional systems see [12] for instance. The notion of zeros at infinity has been generalized to non-linear systems [4]. For linear infinite dimensional systems and in the particular case of bounded operators, the structure at infinity was introduced in [1], described in several equivalent ways and used to solve some control problems in [2]. The particular case of delay systems was studied in [3]. However the structure at infinity defined there is too weak to insure a good solution for control problems: indeed the potential precompensators may be anticipative (see also [11]). In [8] we introduced the concept of strong structure at infinity which is more convenient for infinite dimensional systems (and for delay systems as a particular case). This structure is only well defined for some classes of systems. The positive result is that if this structure at infinity is well available then all potential solutions of control problems are non-anticipative and may be realized by static state feedback. Here we use the weak structure at infinity of the system in order to design the precompensator, then this precompensator is decomposed in two parts: a strong proper precompensator which may be realized by static state feedback and a weak proper precompensator which can be realized by generalized static state feedback, feedback which contains the derivative of the measured and delayed disturbance. The results given here are in a general form at least for systems with commensurate delays. If the disturbance cannot be measured or is not smooth, then the disturbance decoupling problem cannot be solvable by generalized static state feedback.

A similar approach was used to solve the row-by-row decoupling problem for delay systems. The main result may be found in [10] presented while the present paper was in the reviewing process.

The paper is organized as follows. In Section 2 we describe the delay system considered in the paper and the problem of disturbance decoupling. In Section 3 we give basic notions and recall classical results concerning linear systems without delays, then we extend some notions and results for systems with delays in Section 4. In Section 5 we solve the disturbance decoupling problem for delay systems in a general framework.

2. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

2.1. System description

We consider the linear time-invariant systems with delays described by:

$$\begin{cases} \dot{x}(t) = A_0x(t) + A_1x(t-1) + B_0u(t) + D_0h(t) \\ y(t) = C_0x(t) \end{cases} \tag{1}$$

where $x(t) \in \mathcal{X} \approx \boldsymbol{R}^n$ is the state, $u(t) \in \mathcal{U} \approx \boldsymbol{R}^m$ is the control input, $h(t) \in \mathcal{H} \approx \boldsymbol{R}^q$ is the disturbance input, $y(t) \in \mathcal{Y} \approx \boldsymbol{R}^p$ is the output to be controlled. In order to simplify the notation and some computations, we limit ourselves to systems with single delay in the states. All results and considerations given here remain valid for systems with several commensurate delays in the state, the control and disturbance inputs and outputs.

The transfer function matrix of the control is

$$T(s, e^{-s}) = C_0(sI - A_0 - A_1e^{-s})^{-1}B_0$$

which may be expanded as follows

$$T(s, e^{-s}) = \sum_{j=0}^{\infty} T_j(s)e^{-js}, \tag{2}$$

where

$$T_j(s) = C_0(sI - A_0)^{-1}\left[A_1(sI - A_0)^{-1}\right]^j B_0.$$

Each matrix $T_j(s)$ may be decomposed using the following constant matrices introduced by Kirillova and Churakova (see for example [13]):

$$\begin{aligned} Q_i(j) &= A_0Q_{i-1}(j) + A_1Q_{i-1}(j-1), \\ Q_0(0) &= I, \ Q_i(j) = 0, \quad i < 0 \text{ or } j < 0. \end{aligned} \tag{3}$$

We have

$$T_j(s) = \sum_{i=0}^{\infty} C_0Q_i(j)\,B_0s^{-(i+1)}.$$

The expression may be obtained by a simple calculation using the relations (3), (see [11, 13]).

The same representation takes place for the transfer function matrix of the disturbance

$$T^D(s,e^{-s}) = C_0(sI - A_0 - A_1e^{-s})^{-1}D_0.$$

The corresponding matrix coefficients in the decomposition of $T^D(s,e^{-s})$ will be noted by $T_j^D(s)$.

Consider now a partial representation of the delay system given by the systems without delay (see [3]):

$$\left\{\begin{array}{rcl} \dot{z}_k(t) & = & \mathcal{A}_k z_k(t) + \mathcal{B}_k u(t) + \mathcal{D}_k q_k(t), \\ w_k(t) & = & \mathcal{C}_k z_k(t), \end{array}\right. \tag{4}$$

where $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{D}_k$ and $\mathcal{C}_k$ are given as by:

$$\mathcal{A}_k = \begin{bmatrix} A_0 & 0 & \cdots & 0 & 0 \\ A_1 & A_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_1 & A_0 \end{bmatrix}, \quad \mathcal{B}_k = \begin{bmatrix} B_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_0 \end{bmatrix},$$

$$\mathcal{C}_k = \begin{bmatrix} C_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_0 \end{bmatrix}, \quad \mathcal{D}_k = \begin{bmatrix} D_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_0 \end{bmatrix}.$$

If there are several delays, or if inputs and outputs also contain delays, then one has to add to the corresponding matrices blocks under the diagonal. Let $\Theta_k(s)$ and $\Theta_k^D(s)$ be the transfer function matrices of the control and disturbance of the systems (4). Then, it is easy to see that

$$\Theta_k(s) = \begin{bmatrix} T_0(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ T_k(s) & \cdots & T_0(s) \end{bmatrix}, \quad \Theta_k^D(s) = \begin{bmatrix} T_0^D(s) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ T_k^D(s) & \cdots & T_0^D(s) \end{bmatrix}.$$

In this sense one can say that the systems (4) give a partial representation of the system (1). In the time domain, the systems (4) describes the behavior of the delay system (1) for $t \in [0, k+1[$.

2.2. Problem formulation

Let be given the system (1) with a measurable disturbance $h(t)$. Find a precompensator $K(s,e^{-s})$ such that $T(s,e^{-s})\,K(s,e^{-s}) + T^D(s,e^{-s}) \equiv 0$ and which may be realized by generalized static state feedback of the form

$$u = F(e^{-s})\,x + G(s,e^{-s})\,h,$$

without anticipation. This means that $F(e^{-s})$ and $G(s, e^{-s})$ may be decomposed as

$$\begin{aligned} F(e^{-s}) &= F_0 + F_1 e^{-s} + F_2 e^{-2s} + \cdots, \\ G(s, e^{-s}) &= G_0 + G_1(s) e^{-s} + G_2(s) e^{-2s} + \cdots, \end{aligned}$$

with (possible) polynomial matrices $G_i(s), i \geq 1$. This assumption allows to give a more general solution for a very large class of delay systems. We shall discuss when this assumption may be replaced by the assumption of constant matrices $G_i(s) = G_i$, for $i \geq 1$ (see Corollary 8). If the problem is solvable we say that the *disturbance decoupling problem with measurement* is solvable. The corresponding precompensator $K(s, e^{-s})$ is called *realizable* or *causal*.

3. FINITE DIMENSIONAL SYSTEMS

The basic notion used is this paper is the notion of properness. Let us recall in this section the case of a classical linear system given by:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dh(t) \\ y(t) = Cx(t) \end{cases} \tag{5}$$

where $x(t) \in \mathcal{X} \approx \boldsymbol{R}^n$ is the state, $u(t) \in \mathcal{U} \approx \boldsymbol{R}^m$ is the control input, $h(t) \in \mathcal{H} \approx \boldsymbol{R}^q$ is the disturbance input, $y(t) \in \mathcal{Y} \approx \boldsymbol{R}^p$ is the output to be controlled. The transfer function matrix of the control and the disturbance are

$$T(s) = C(sI - A)^{-1}B, \quad T^D(s) = C(sI - A)^{-1}D.$$

The matrices $T(s)$ and $T^D(s)$ are rational and strictly proper. The properness being defined by the following.

Definition 1. A complex valued function $f(s)$ is called proper if $\lim f(s)$ is finite when $|s| \to \infty$. It is called strictly proper if this limit is 0. A matrix $B(s)$ is biproper if it is proper and its inverse is also proper.

As for linear systems in finite dimensional spaces one considers in fact only rational functions, properness means that the degree of the numerator is less than or equal to the degree of the denominator and strictly properness means that the equality cannot hold. A fundamental result is the existence of a canonical form at infinity (Smith–McMillan form at infinity) for strictly proper matrices (but also for more general matrices).

Theorem 2. There exists (non unique) biproper matrices $B_1(s)$ and $B_2(s)$ such that

$$B_1(s)T(s)\,B_2(s) = \begin{bmatrix} \Delta(s) & 0 \\ 0 & 0 \end{bmatrix},$$

where $\Delta(s) = \operatorname{diag}[s^{-n_1}, \ldots, s^{-n_r}]$. The integers n_i are called the orders of the zeros at infinity and the list of integers $\{n_1, \ldots, n_r\}$ is the *structure at infinity* and is noted by $\Sigma_\infty(C, A, B)$ or $\Sigma_\infty T(s)$.

The structure at infinity allows to describe the behavior of system at $t = 0$.

Another important tool which is useful to characterize several properties of linear systems is the maximal (A, B)-invariant subspace contained in $\operatorname{Ker} C$ (see [14]). It will be noted by $\mathcal{V}_*(C, A, B)$. We shall also use the alternative expression of this subspace given by Hautus:

$$\mathcal{V}_*(C, A, B) = \{x \in \operatorname{Ker} C : x = (sI - A)\xi(s) - B\omega(s)\},$$

with strictly proper ξ and ω such that $\xi(s) \in \operatorname{Ker} C$ for $|s| > s_0$. The following result is well known and established by several authors. Let $\mathcal{B}$ and $\mathcal{D}$ denote the images of B and D respectively.

Theorem 3. The following propositions are equivalent:

1. There exists a proper precompensator $K(s)$ such that $T(s)\,K(s) + T^D(s) \equiv 0$.
2. The disturbance is rejected by the feedback $u = Fx + Gh$, this means that
$$C(sI - A - BF)^{-1}(BG + D) = 0.$$
3. $\Sigma_\infty[T(s) \quad T^D(s)] = \Sigma_\infty[T(s) \quad 0]$.
4. $\mathcal{D} \subset \mathcal{V}_*(C, A, B) + \mathcal{B}$

The relation between the precompensator and the feedback is given by

$$K(s) = \left(I - F(sI - A)^{-1}B\right)^{-1}\left(F(sI - A)^{-1}D + G\right). \tag{6}$$

If the disturbance is not available, then the precompensator must be strictly proper, $G = 0$ in 1), condition 3) must be replaced by

$$\Sigma_\infty[s^{-1}T(s) \quad T^D(s)] = \Sigma_\infty[s^{-1}T(s) \quad 0]$$

and 4) by $\mathcal{D} \subset \mathcal{V}_*(C, A, B)$.

Proof. We need in this paper the proof of the equivalence 1) $\Leftrightarrow$ 4) and the relation (6). Suppose that $T(s)\,K(s) + T^D(s) \equiv 0$, this means that

$$C(sI - A)^{-1}\,(BK(s) + D) = 0$$

for all s, $|s| > s_0$. Then noting

$$\xi(s) = (sI - A)^{-1}\,(BK(s) + D)\;h,$$

where $h \in \mathcal{H}$ is an arbitrary fixed vector, one obtains

$$(sI - A)\,\xi(s) - BK(s)\,h = -Dh, \quad C\xi(s) = 0.$$

Let now v be the limit of $K(s)\,h$ when $s \to \infty$. Then $K(s)\,h = v + \omega(s)$, $\omega(s)$ is strictly proper, and

$$(sI - A)\,\xi(s) - B\omega(s) - Bv = -Dh,$$

for all $h \in \mathcal{H}$. This means that 4) holds: $\mathcal{D} \subset \mathcal{V}_*(C, A, B) + \mathcal{B}$.
Conversely, if $\mathcal{D} \subset \mathcal{V}_*(C, A, B) + \mathcal{B}$, then, $\{h_1, \ldots, h_q\}$ being a basis of $\mathbf{R}^q$, we have

$$Dh_i = (sI - A)\,\xi_i(s) - B\omega_i(s) - Bv_i, \quad i = 1, \ldots, q,$$

where $\xi_i(s)$ and $\omega_i(s)$ are strictly proper functions such that $\xi_i(s) \in \operatorname{Ker} C$. Let G be a matrix such that $v_i = Gh_i$ for $i = 1, \ldots, q$. Then

$$D = (sI - A)\Xi(s) - B\Omega(s) - BG,$$

with strictly proper matrices $\Xi(s)$ and $\Omega(s)$, aand with $C\Xi(s) = 0$. Finally we obtain

$$C(sI - A)^{-1}B(G + \Omega(s)) = -C(sI - A)^{-1}D.$$

Hence $T(s)\,K(s) + T^D(s) = 0$ with a proper $K(s)$.

Consider now the relation between the precompensator and the feedback. The closed loop control law $u = Fx + Gh$ gives:

$$\begin{aligned} & C(sI - A - BF)^{-1}(BG + D) \\ & \quad = \quad C(sI - A)^{-1}B\left(I - F(sI - A)^{-1}B\right)^{-1}G + C(sI - A - BF)^{-1}D, \end{aligned}$$

because $C(sI - A - BF)^{-1}B = C(sI - A)^{-1}B\left(I - F(sI - A)^{-1}B\right)^{-1}$. Using the relation:

$$(sI - A - BF)^{-1} = (sI - A)^{-1} + (sI - A - BF)^{-1}BF(sI - A)^{-1},$$

we get

$$C(sI - A - BF)^{-1}D = C(sI - A)^{-1}D + C(sI - A - BF)^{-1}BF(sI - A)^{-1}D.$$

Then

$$\begin{aligned} & C(sI - A - BF)^{-1}D \\ & \quad = \quad C(sI - A)^{-1}D + C(sI - A)^{-1}B\left(I - F(sI - A)^{-1}B\right)^{-1}F(sI - A)^{-1}D. \end{aligned}$$

Finally

$$\begin{aligned} & C(sI - A - BF)^{-1}(BG + D) \\ & \quad = C(sI - A)^{-1}B\left(I - F(sI - A)^{-1}B\right)^{-1}\left(F(sI - A)^{-1}D + G\right) + C(sI - A)^{-1}D. \end{aligned}$$

This gives

$$C(sI - A - BF)^{-1}(BG + D) = T(s)\,K(s) + T^D(s),$$

with $K(s) = \left(I - F(sI - A)^{-1}B\right)^{-1}\left(F(sI - A)^{-1}D + G\right)$ as in (6). □

4. STRUCTURAL NOTIONS FOR DELAY SYSTEMS

The transfer function matrix of a delay system is not rational. Moreover, it is not analytical at infinity. The notions of properness must be precised.

Definition 4. A complex valued function $f(s)$ is called weak proper if $\lim f(s)$ is finite when $s \in \mathbf{R}$ tends to ∞. It is called strictly weak proper if this limit is 0. A matrix $B(s)$ is weak biproper if it is weak proper and its inverse is also weak proper. Weak proper is replaced by strong proper if the same occurs when $\Re e(s) \to \infty$.

It is obvious that strong properness implies weak properness. If the function is analytical at infinity both notions coincide, because the limit at infinity is the same. The strong properness was used in [1] and [2] in the description of the structure at infinity for infinite dimensional systems. In [3] and [11] the weak notion was used in order to define the structure at infinity of delay systems and to solve some control problems.

However, in general, this structure at infinity cannot be used, in an efficient way, to solve control problems with non predictive control laws. For example, if the transfer function of the system is $T(s) = s^{-3}+s^{-2}e^{-s}$, the weak structure at infinity is s^{-3} since $T(s) = s^{-3}(1+se^{-s})$, and since $1+se^{-s}$ is weak biproper. Suppose one has to solve the disturbance decoupling problem for the disturbance given by $T^D(s) = s^{-3}$. As the structure at infinity of the control and the disturbance are the same, there exists a proper precompensator $K(s)$ such that $T(s)\,K(s) = T^D(s)$ (see [3], where the question is considered for the model matching problem). In fact the unique precompensator solving this problem is $K(s) = 1+se^{-s}$ which is not realizable by linear static state feedback without additional derivative even if one uses distributed delays.

If we consider the notion of strong properness to define the structure at infinity, some difficulties occur in the construction of the canonical form at infinity. For the given example the structure at infinity is not s^{-3} because $1+se^{-s}$ is not proper in the strong sense and it is not possible, in fact, to define a strong structure at infinity for the given example.

Let us recall the following results using the weak properness (see [3, 8]).

Theorem 5. There exist weak biproper matrices $B_1(s,e^{-s})$ and $B_2(s,e^{-s})$ such that

$$B_1(s,e^{-s})T(s,e^{-s})\,B_2(s,e^{-s}) = \begin{bmatrix} \Delta_0(s) & 0 & \cdots & 0 & 0 \\ 0 & \Delta_1(s)e^{-s} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \Delta_k(s)e^{-ks} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $\Delta_i(s) = \operatorname{diag}[s^{-n_{i,1}},\ldots,s^{-n_{i,j_i}}]$ and $n_{i,j} \le n_{i,j+1}$, $i = 1,\ldots,k$. The list of integers

$$\{n_{i,j},\ i = 1,\ldots,k; j = j_1,\ldots,j_i\}$$

is called the *weak structure at infinity* of the system $T(s,e^{-s})$ and is noted by $\Sigma^w_\infty T(s,e^{-s})$.

Theorem 6. There exist a weak proper precompensator $K(s,e^{-s})$ such that

$$T(s,e^{-s})\,K(s,e^{-s}) = T^D(s,e^{-s})$$

if and only if

$$\Sigma^w_\infty[T(s,e^{-s}) \quad T^D(s,e^{-s})] = \Sigma^w_\infty[T(s,e^{-s}) \quad 0].$$

Some additional assumptions may insure that the weak structure at infinity also gives a strong structure at infinity (the biproper matrices B_i are strongly biproper). In this case the precompensator is strongly proper and realizable by static state feedback. We shall see that the assumptions given in [8] may be weakened.

5. THE DISTURBANCE DECOUPLING PROBLEM FOR DELAY SYSTEMS

Our purpose is to give for a linear time delay system a more general solution for the disturbance decoupling problem.
The given problem was studied by several authors. Let us cite the paper of L. Pandolfi [6] which uses infinite dimensional approach. The difficulties appear with the definition of geometric tools like (A,B)-invariant subspaces. An algebraic approach was used in [5] to describe the causal precompensator. In [9] an abstract geometric approach is developed using Hautus' definition of (A,B)-invariant subspaces. The weak structure at infinity given in the previous section allows to give the following general formulation and solution for this control problem by generalized static state feedback.

Note that a similar approach was developped by the authors for the row-by-row decoupling problem (see [10], written while this paper was being reviewed).

Theorem 7. The following propositions are equivalent:

1. The disturbance decoupling problem for the delay system (1) is solvable by a weak proper precompensator:

$$T(s,e^{-s})\,K(s,e^{-s}) + T^D(s,e^{-s}) \equiv 0.$$

2. The weak structure at infinity verifies:

$$\Sigma^w_\infty[T(s,e^{-s}) \quad T^D(s,e^{-s})] = \Sigma^w_\infty[T(s,e^{-s}) \quad 0].$$

3. The disturbance decoupling problem is solvable by generalized static state feedback

$$u = F(e^{-s})\,x + G(s,e^{-s})\,h,$$

with

$$\begin{aligned} F(e^{-s}) &= F_0 + F_1e^{-s} + F_2e^{-2s} + \cdots, \\ G(s,e^{-s}) &= G_0 + G_1(s)e^{-s} + G_2(s)e^{-2s} + \cdots, \end{aligned}$$

with (possible) polynomial matrices $G_i(s)$, $i \geq 1$ and constant matrices F_i, $i \in \boldsymbol{N}$.

4. $\mathcal{D}_0 \subset \mathcal{V}_\Sigma(C,A,B) + \mathcal{B}_0$, where $\mathcal{D}_0$ and $\mathcal{B}_0$ are the images of D_0 and B_0 respectively, the subspace $\mathcal{V}_\Sigma(C,A,B)$ being given by

$$\mathcal{V}_\Sigma(C,A,B) = \left\{x \in \operatorname{Ker} C_0 : x = (sI - A_0 - A_1 e^{-s})\xi(s,e^{-s}) - B_0\omega(s,e^{-s})\right\},$$

with strictly weak proper ξ and ω such that $\xi(s,e^{-s}) \in \operatorname{Ker} C_0$ for $s > s_0$.

Proof. From Theorem 6 we have the equivalence of statements 1) and 2). The equivalence between 1) and 4) may be shown in the same formal way as for the finite dimensional systems (see the proof of this equivalence for Theorem 3).
Let us now show the equivalence between 1) and 3). In order to simplify the notations, let us put: $A = A_0 + A_1 e^{-s}$, $B = B_0$, $C = C_0, D = D_0$, $F = F(e^{-s})$ and $G = G(s,e^{-s})$.
Suppose that 3) holds. Then

$$C(sI - A - BF)^{-1}(BG + D) = 0.$$

A formal computation gives $T(s,e^{-s})\,K(s,e^{-s}) = -T^D(s,e^{-s})$ with

$$K(s,e^{-s}) = \left(I - F(sI-A)^{-1}B\right)^{-1}\left(F(sI-A)^{-1}D + G\right),$$

and $K(s,e^{-s})$ is weak proper. This means that 1) is verified.
Suppose now that there exists a weak proper precompensator $K(s,e^{-s})$ such that

$$T(s,e^{-s})\,K(s,e^{-s}) \equiv -T^D(s,e^{-s}).$$

The precompensator $K(s,e^{-s})$ may be decomposed as follows:

$$K(s,e^{-s}) = K_0(s) + K_1(s)e^{-s} + \cdots,$$

where $K_i(s)$ are rational matrices. The weak properness implies that $K_0(s)$ is proper in the classical sense, but $K_i(s)$, $i \geq 1$ may contain polynomial terms. Decompose $K_i(s)$, $i \geq 1$ in polynomial part and rational strictly proper part (this decomposition is unique). This yields to a decomposition of the precompensator as

$$K(s,e^{-s}) = K^1(s,e^{-s}) + K^2(s,e^{-s}),$$

where $K^1(s,e^{-s})$ is proper in the strong sense and $K^2(s,e^{-s})$ is proper in the weak sense. Clearly:

$$K^1(s,e^{-s}) = K_0^1(s) + K_1^1(s)e^{-s} + \cdots,$$

with proper rational matrices $K_i^1(s)$ and

$$K^2(s,e^{-s}) = K_1^2(s)e^{-s} + K_2^2(s)e^{-2s} + \cdots.$$

This gives $T(s,e^{-s})\,K^1(s,e^{-s}) = -T^D(s,e^{-s}) - T(s,e^{-s})\,K^2(s,e^{-s})$ or in detailed form:

$$\begin{aligned} C(sI-A)^{-1}BK^1(s,e^{-s}) &= -C(sI-A)^{-1}D - C(sI-A)^{-1}BK^2(s,e^{-s}) \\ &= -C(sI-A)^{-1}\left(D + BK^2(s,e^{-s})\right) \\ &= -C(sI-A)^{-1}E(s,e^{-s}) \\ &= -T^E(s,e^{-s}), \end{aligned}$$

where $E(s,e^{-s}) = D + BK^2(s,e^{-s})$ may be considered as a new disturbance which includes derivatives of the delayed disturbance. Now we let us show how to design F and G.
First design F^1 and G^1 such that the control law $u^1 = F^1x + G^1q$ realizes the precompensator $K_1(s,e^{-s})$ (Step 1) which satisfies $T(s,e^{-s})\,K_1(s,e^{-s}) \equiv -T^E(s,e^{-s})$, then we deduce $u = Fx + Gh$ (Step 2).

Step 1: We have $T(s,e^{-s})\,K^1(s,e^{-s}) \equiv -T^E(s,e^{-s})$. We can consider the partial representation of delay systems, as given in Section 2, by the systems (4). A simple computation gives that for all $k \in NN$, we have:

$$\Theta_k(s)\Gamma_k(s) + \Theta_k^D(s) = 0,$$

with proper precompensator $\Gamma_k(s)$ given by:

$$\Gamma_k(s) = \begin{bmatrix} K_0^1(s) & 0 & \cdots & 0 \\ K_1^1(s) & K_0^1(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K_k^1(s) & K_{k-1}^1(s) & \cdots & K_0^1(s) \end{bmatrix}.$$

This means that for the corresponding systems (4), the disturbance decoupling problem is solvable (Theorem 3) and the geometric conditions are verified for each k. Then the feedbacks may be chosen as in [3] and give

$$C(sI - A - BF^1)^{-1}(BG^1 + E) = T(s,e^{-s})\,K^1(s,e^{-s}),$$

where, for simplicity, the argument is omitted in F^1, G^1 and E:

$$\begin{aligned} F^1(e^{-s}) &= F_0 + F_1e^{-s} + F_2e^{-2s} + \cdots, \\ G^1(s,e^{-s}) &= G_0^1 + G_1^1e^{-s} + G_2^1e^{-2s} + \cdots, \end{aligned}$$

and $E(s,e^{-s}) = D + BK^2(s,e^{-s})$ is the new disturbance. The constant matrices F_i and G_i^1, $i = 0,1,\ldots$ are computed from the geometric conditions for the systems (4) for each k. The same considerations as in Theorem 3 gives the relation between the precompensator and the feedback:

$$K^1(s,e^{-s}) = \left(I - F^1(sI - A)^{-1}B\right)^{-1}\left(F^1(sI - A)^{-1}E + G^1\right),$$

hence K^1 is realized by static state feedback. This can be rewritten as:

$$K^1(s,e^{-s}) = F^1(sI - A)^{-1}E + F^1(sI - A)^{-1}BK^1(s,e^{-s}) + G^1.$$

Step 2: The relation obtained at the end of the *Step 1* with the equality

$$E(s,e^{-s}) = D + BK^2(s,e^{-s})$$

gives

$$\begin{aligned} &K^1(s,e^{-s}) \\ &\quad = F^1(sI - A)^{-1}(D + BK^2(s,e^{-s})) + F^1(sI - A)^{-1}BK^1(s,e^{-s}) + G^1. \end{aligned}$$

and then

$$K^1(s,e^{-s}) = F^1(sI-A)^{-1}D + F^1(sI-A)^{-1}BK(s,e^{-s}) + G^1.$$

Let us now put $G(s,e^{-s}) = G^1(e^{-s}) + K^2(s,e^{-s})$ and $F(e^{-s}) = F^1(e^{-s})$. Then from the previous expression of $K^1(s,e^{-s})$, we can get

$$\begin{aligned} K(s,e^{-s}) &= K^1(s,e^{-s}) + K^2(s,e^{-s}) \\ &= F(sI-A)^{-1}D + F(sI-A)^{-1}BK(s,e^{-s}) + G \end{aligned}$$

and then we obtain from this last relation that:

$$K(s,e^{-s}) = \left(I - F(sI-A)^{-1}B\right)^{-1}\left(F(sI-A)^{-1}D + G\right),$$

and then, as $T(s,e^{-s})\,K(s,e^{-s}) + T^D(s,e^{-s}) \equiv 0$, by hypothesis, we get

$$T_{FG}(s,e^{-s}) = C(sI-A-BF)^{-1}(BG+D) = 0.$$

The disturbance is decoupled by the closed loop control law. This ends the proof.□

Corollary 8. Suppose that

$$\Sigma_\infty^w[T(s,e^{-s}) \quad T^D(s,e^{-s})] = \Sigma_\infty^w[T(s,e^{-s}) \quad 0]$$

and that structures are obtained by strong biproper operations. Then the precompensator is strongly proper and is realizable by static state feedback, that is $G_i(s)$, $i \geq 1$ in the expression of $G(s,e^{-s})$ are constant.

P r o o f. The assumptions of the corollary imply that the weak structure at infinity is also the strong structure at infinity (see [8]), this gives $K(s,e^{-s}) = K^1(s,e^{-s})$, and then the *Step 2* in the previous proof is not needed. Then $G = G^1$ and $F = F^1$ and then

$$G(s,e^{-s}) = G_0 + G_1e^{-s} + G_2e^{-2s} + \cdots,$$

with constant matrices G_i for all integers i. The precompensator is realizable by static state feedback. No derivation of the delayed disturbance is needed. □

Let us precise in some example how the new disturbance is constructed using the initial one.
Suppose that the initial disturbance is one dimensional and $K^2(s,e^{-s}) = s^2e^{-s} + se^{-2s}$. Then the new disturbance is $q = [h \quad \dot{h} \quad \ddot{h}]'$, and

$$Eq(t) = E_0q(t) + E_1q(t-1) + E_2q(t-2),$$

with $E_0 = [D \quad 0 \quad 0]$, $E_1 = [0 \quad 0 \quad B_0]$ and $E_2 = [0 \quad B_0 \quad 0]$.

Note that in the classical finite dimensional case, one can consider the disturbance decoupling problem when the disturbance is not measurable. In this case, the solution is a strictly proper compensator and the feedback is of the form $u = Fx$. Here we can also consider the case when the precompensator is weakly strictly proper. The structural condition may be reformulated in this context. However, the weak proper part (even if it is *strictly* proper in the weak sense) needs, for it realization, the disturbance. Hence, this problem, except for some classes of systems, cannot be solved.

6. CONCLUSION

In order to solve in a general form and without prediction the disturbance decoupling problem for delay systems we use the weak structure at infinity which is well defined for linear time delay systems. The general solution is of feedback type. However we need some smoothness of the disturbance since (delayed) derivatives of the disturbance may be needed. This is the counterpart of the generality. For practical use this means that if the disturbance is not smooth enough, we need in fact very high gain in approximation. The results given here may be also considered, with some modification, for more general delay systems: systems with distributed delays or of neutral type.

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