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ON NONCOOPERATIVE NONLINEAR DIFFERENTIAL GAMES[1]

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Noncooperative games with systems governed by nonlinear differential equations remain, in general, nonconvex even if continuously extended (i.e. relaxed) in terms of Young measures. However, if the individual payoff functionals are "enough" uniformly convex and the controlled system is only "slightly" nonlinear, then the relaxed game enjoys a globally convex structure, which guarantees existence of its Nash equilibria as well as existence of approximate Nash equilibria (in a suitable sense) for the original game.

1. INTRODUCTION AND PROBLEM FORMULATION

The concept of Nash equilibria [10] for noncooperative games requires typically a convex structure both of sets of admissible strategies and of the individual payoff functionals. This represents a severe restriction on the class of problems investigated. Considering games involving a controlled system governed by differential equations, called differential games, a relaxation (i.e. a natural extension by continuity) in terms of the Young measures [22] (also called relaxed controls or mixed strategies) as e.g. in Balder [1 – 3], Gamkrelidze [6], Krasovskiĭ and Subbotin [8], Nikol'skiĭ [12], Nowak [13], or Warga [21] can help to some extent: it can convexify the originally nonconvex sets of admissible strategies as well as original payoffs with respect to the strategies no matter how nonlinear they are. However, the required convexity structure of the relaxed problem still represents a considerable restriction: in general cases, only controlled systems linear with respect to the state can be treated. Sometimes, a special interplay of the data enables us to admit nonlinear systems, too; it seems that the only reference to such phenomena is Lenhart et al [9] for the case of a certain special elliptic game.

In this paper we want to pursue this idea in a more general manner by combination of the relaxation with the technique used in other occasions to prove sufficiency of the maximum principle; see Gabasov and Kirillova [5; Section VII.2] or, for the case of general integral processes, also Schmidt [18]. (Yet, we will use it more carefully to avoid the discrepancy of requirements of uniform convexity and boundedness

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of the first derivative simultaneously, which appeared incidentally in [5].) By this technique one can show the convex structure for the relaxed problem even if the controlled system is "slightly" nonlinear with respect to the state on the assumption that the individual payoff functionals are "enough" uniformly convex with respect to the state. To reduce technicalities, we confine ourselves to the cases where the state and the strategies are additively coupled, the nonlinearities in the strategies being arbitrary.

We want to illustrate here this idea on the simplest case where the controlled system is governed by ordinary differential equations, the number of players is only two, and the sets of admissible strategies are bounded in L^∞-norm. Let us only remark that the generalization to many-player games with partial differential equations or integral equations and with admissible strategies bounded only in an L^p-norm is possible; cf. in particular [16] for a game with systems of elliptic equations.

Hence, we will consider the following two-person non-cooperative game for a system of nonlinear ordinary differential equations, having additively coupled state and strategy terms, with individual payoffs also additively coupled (except the term $\pm\varphi(t,u_1,u_2)$ which couples the strategies in a general manner):

$$
(\mathcal{G})\quad \left\{\begin{array}{lll}
\text{Find Nash equilibrium} \left\{\begin{array}{l} \displaystyle\int_0^T g_1(t,y)+h_1(t,u_1)+\varphi(t,u_1,u_2)\,\mathrm{d}t \\ \displaystyle\int_0^T g_2(t,y)+h_2(t,u_2)-\varphi(t,u_1,u_2)\,\mathrm{d}t \end{array}\right. & & \begin{array}{r}(1^{\text{st}}\text{ player payoff})\\ (2^{\text{nd}}\text{ player payoff})\end{array}\\
\text{subject to} & \dfrac{\mathrm{d}y}{\mathrm{d}t}=G(t,y)+H_1(t,u_1)+H_2(t,u_2) & \text{(state equation)}\\
& y(0)=y_0, & \text{(initial condition)}\\
& u_1(t)\in S_1(t), & \text{(strategy}\\
& u_2(t)\in S_2(t)\ \text{for a.a. } t\in(0,T), & \text{constraints)}\\
& y\in W^{1,p}(0,T;\mathbb{R}^n) & \\
& u_1\in L^\infty(0,T;\mathbb{R}^{m_1}),\quad u_2\in L^\infty(0,T;\mathbb{R}^{m_2}). &
\end{array}\right.
$$

where $g_l:(0,T)\times\mathbb{R}^n\to\mathbb{R}$, $G:(0,T)\times\mathbb{R}^n\to\mathbb{R}^n$, $h_l:(0,T)\times\mathbb{R}^{m_l}\to\mathbb{R}$, $H_l:(0,T)\times\mathbb{R}^{m_l}\to\mathbb{R}^n$ and $\varphi:(0,T)\times\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}\to\mathbb{R}$, $y_0\in\mathbb{R}^n$, and moreover $S_l:(0,T)\rightrightarrows\mathbb{R}^{m_l}$ are multivalued mappings, $l=1,2$. Moreover, the notation for the Sobolev space $W^{1,p}(0,T;\mathbb{R}^n)=\{y\in L^\infty(0,T;\mathbb{R}^n);\ \mathrm{d}y/\mathrm{d}t\in L^p(0,T;\mathbb{R}^n)\}$ and for the Lebesgue spaces $L^p(0,T;\mathbb{R}^n)$ and $L^\infty(0,T;\mathbb{R}^{m_l})$ is standard. Supposing $p\in(1,+\infty]$, the basic data qualification we will need are the following:

$$g_l,\ G,\ h_l,\ H_l,\ \varphi \text{ are Carathéodory functions,} \tag{1.1a}$$

i.e. measurable in $t\in(0,T)$ and continuous in the resting variables,

$$\exists a\in L^p(0,T)\ \exists b\in\mathbb{R}:\quad |G(t,r)|\le a(t)+b|r|,\qquad |H_l(t,s)|\le a(t), \tag{1.1b}$$

$$\exists\,\ell \in L^1(0,T):\quad |G(t,r_1) - G(t,r_2)| \le \ell(t)|r_1 - r_2|, \tag{1.1c}$$

$$\exists\, a \in L^1(0,T):\quad |g_l(t,r)| \le a(t),\quad |h_l(t,s)| \le a(t),\quad |\varphi(t,s_1,s_2)| \le a(t), \tag{1.1d}$$

S_l is bounded, closed-valued and has a measurable graph, i. e. (1.1e)

$$\{(t,s) \in (0,T)\times \mathbb{R}^{m_l};\ s \in S_l(t)\} \ \in\ \Sigma_{\text{Lebesgue}}(0,T) \otimes \Sigma_{\text{Borel}}(\mathbb{R}^{m_l}),$$

where $l = 1,2$ and $\Sigma_{\text{Lebesgue}}(0,T)$ and $\Sigma_{\text{Borel}}(\mathbb{R}^{m_l})$ denote respectively the σ-algebra of Lebesgue measurable subsets of $(0,T)$ and the Borel σ-algebra on $\mathbb{R}^{m_l}$.

The strategies u_1 and u_2 represent the strategies of the particular players while y is the state response. The game $(\mathcal{G})$ has got the structure of finding Nash equilibria of the payoffs $\Phi_1, \Phi_2 : U_1 \times U_2 \to \mathbb{R}$, defined by

$$\Phi_l(u_1,u_2) := \int_0^T g_l(t,y(u_1,u_2)) + h_l(t,u_1) - (-1)^l \varphi(t,u_1,u_2)\,\mathrm{d}t \tag{1.2}$$

with $y = y(u_1,u_2) \in W^{1,p}(0,T;\mathbb{R}^n)$ being the unique solution to the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = G(t,y) + H_1(t,u_1) + H_2(t,u_2)\ ,\qquad y(0) = y_0\ , \tag{1.3}$$

while the sets of admissible strategies U_1 and U_2 are defined by

$$U_l := \{u \in L^\infty(0,T;\mathbb{R}^{m_l});\ u_l(t) \in S_l(t) \text{ for a. a. } t \in (0,T)\}, \tag{1.4}$$

where $l = 1,2$ distinguishes the particular players. Let us recall that the pair of strategies $(u_1,u_2) \in U_1 \times U_2$ is called a Nash equilibrium for the game $(\mathcal{G})$ if

$$\Phi_1(u_1,u_2) = \min_{\tilde u_1 \in U_1} \Phi_1(\tilde u_1,u_2) \quad \& \quad \Phi_2(u_1,u_2) = \min_{\tilde u_2 \in U_2} \Phi_2(u_1,\tilde u_2). \tag{1.5}$$

Such equilibria, however, do not exist unless quite strong data qualification are imposed. Instead of seeking a precise equilibrium, it is practically satisfactory to find at least an approximate equilibrium. In analogy with minimizing sequences used standardly in cooperative situations, here it is natural to speak about the so-called equilibrium sequences introduced in [15]: a sequence of strategies $\{(u_1^k,u_2^k)\}_{k\in\mathbb{N}}$ will be called an equilibrium sequence if

$$\exists\,\Psi_1 : U_1 \to \mathbb{R}:\quad \lim_{k\to\infty} \Phi_1(\cdot,u_2^k) = \Psi_1 \ \text{ point-wise}, \tag{1.6a}$$

$$\exists\,\Psi_2 : U_2 \to \mathbb{R}:\quad \lim_{k\to\infty} \Phi_2(u_1^k,\cdot) = \Psi_2 \ \text{ point-wise}, \tag{1.6b}$$

$$\lim_{k\to\infty} \Psi_1(u_1^k) = \inf_{u\in U_1} \Psi_1(u) \quad \& \quad \lim_{k\to\infty} \Psi_2(u_2^k) = \inf\nolimits_{u\in U_2} \Psi_2(u). \tag{1.6c}$$

This definition just means that the sequences $\{u_1^k\}_{k\in\mathbb{N}}$ and $\{u_2^k\}_{k\in\mathbb{N}}$ are minimizing with respect to limit payoff functions determined by the sequence of opponent's strategies. A bit other concept has been invented by Patrone [14]: $\{(u_1^k,u_2^k)\}_{k\in\mathbb{N}}$ is called an asymptotically Nash equilibrium if

$$\exists\,\{(\varepsilon_1^k,\varepsilon_2^k)\}_{k\in\mathbb{N}}\subset(0,+\infty)\times(0,+\infty):\ \lim_{k\to\infty}\varepsilon_1^k=0\ \ \&\ \ \lim_{k\to\infty}\varepsilon_2^k=0, \tag{1.7a}$$

$$\Phi_1(u_1^k,u_2^k)\le\inf_{\tilde u\in U_1}\Phi_1(\tilde u,u_2^k)+\varepsilon_1^k\ \ \&\ \ \Phi_2(u_1^k,u_2^k)\le\inf_{\tilde u\in U_2}\Phi_2(u_1^k,\tilde u)+\varepsilon_2^k. \tag{1.7b}$$

The pair (u_1^k,u_2^k) satisfying (1.7b) is also called $(\varepsilon_1^k,\varepsilon_2^k)$-equilibrium; see also Kindler [7], Tan, Yu and Yuan [19] or Tijs [20]. It should be emphasized that, contrary to minimizing sequences whose existence in cooperative situations is always guaranteed, the equilibrium sequences or asymptotically Nash equilibria in the sense of Patrone [14] need not exist in general, cf. also [14; Example 3]. However, we will prove their existence under a suitable data qualification; cf. Corollaries 3.1 and 3.2. Let us also note that, in general, one cannot suppose any relation between (1.6) and (1.7). However, it holds:

Proposition 1.1. Let $\{(u_1^k,u_2^k)\}_{k\in\mathbb{N}}$ be an equilibrium sequence such that (1.6a,b) hold not only point-wise but even uniformly, i.e. let (1.6) hold together with

$$\lim_{k\to\infty}\sup_{u\in U_1}|\Phi_1(u,u_2^k)-\Psi_1(u)|=0\ \ \&\ \ \lim_{k\to\infty}\sup_{u\in U_2}|\Phi_2(u_1^k,u)-\Psi_2(u)|=0. \tag{1.8}$$

Then $\{(u_1^k,u_2^k)\}_{k\in\mathbb{N}}$ is also an asymptotically Nash equilibrium in the sense of Patrone [14].

Proof. Put $a_l^k:=\Psi_l(u_l^k)-\inf_{u\in U_l}\Psi_l(u)$ for $l=1,2$ and $b_1^k:=\sup_{u\in U_1}|\Phi_1(u,u_2^k)-\Psi_1(u)|$ and $b_2^k:=\sup_{u\in U_2}|\Phi_2(u_1^k,u)-\Psi_2(u)|$. Since, for any $u\in U_1$, it holds $\Psi_1(u)\ge\Psi_1(u_1^k)-a_1^k$ and $|\Phi_1(u,u_2^k)-\Psi_1(u)|\le b_1^k$, we can estimate

$$\Phi_1(u_1^k,u_2^k)\ \le\ \Psi_1(u_1^k)+b_1^k\ \le\ \Psi_1(u)+a_1^k+b_1^k\ \le\ \Phi_1(u,u_2^k)+a_1^k+2b_1^k.$$

Analogously, we get also $\Phi_2(u_1^k,u_2^k)\le\Phi_2(u_1^k,u)+a_2^k+2b_2^k$ for any $u\in U_2$. Therefore, the pair (u_1^k,u_2^k) forms an $(a_1^k+2b_1^k,a_2^k+2b_2^k)$-equilibrium. By (1.6c), we know that $\lim_{k\to\infty}a_l^k=0$ and, by (1.8), we also know that $\lim_{k\to\infty}b_l^k=0$. This proves $\{(u_1^k,u_2^k)\}_{k\in\mathbb{N}}$ to be an asymptotically Nash equilibrium. □

2. A RELAXED GAME AND ITS STRUCTURE

Following ideas by Young [22], for $l=1,2$, we extend the sets of admissible strategies U_l from (1.4) to the set of admissible relaxed strategies

$$\bar U_l:=\{\nu\in\mathcal{Y}(0,T;\mathbb{R}^{m_l});\ \mathrm{supp}(\nu_t)\subset S_l(t)\ \text{for a.a.}\ t\in(0,T)\}, \tag{2.1}$$

where $\mathrm{supp}(\nu_t)$ stands for the support of the measure ν_t and the set of the so-called Young measures (cf. [22] where however $C_0(\mathbb{R}^{m_l})^*$ instead of $\mathrm{rca}(\mathbb{R}^{m_l})$ is used) is defined by

$$\mathcal{Y}(0,T;\mathbb{R}^{m_l}):=\{\nu\in L^\infty_{\mathrm{w}}(0,T;\mathrm{rca}(\mathbb{R}^{m_l})):\ \nu_t:=\nu(t)\ \text{is a probability measure for a.a.}\ t\in(0,T)\} \tag{2.2}$$

where $L^\infty_{\rm w}(0,T;{\rm rca}(\mathbb{R}^{m_l})) \cong L^1(0,T;C_0(\mathbb{R}^{m_l}))^*$ denotes the linear space of weakly measurable mappings $t \mapsto \nu_t : (0,T) \to {\rm rca}(\mathbb{R}^{m_l})$ and ${\rm rca}(\mathbb{R}^{m_l}) \cong C_0(\mathbb{R}^{m_l})^*$ stands for Radon measures on $\mathbb{R}^{m_l}$, and $C_0(\mathbb{R}^{m_l})$ denotes the Banach space of continuous functions $\mathbb{R}^{m_l} \to \mathbb{R}$ vanishing at infinity. The natural (norm,weak*)-continuous imbedding $i_l : L^\infty(0,T;\mathbb{R}^{m_l}) \to \mathcal{Y}(0,T;\mathbb{R}^{m_l})$ is defined by $i_l : u \mapsto \nu$ with $\nu_t = \delta_{u(t)}$ where $\delta_s \in {\rm rca}(\mathbb{R}^{m_l})$ denotes the Dirac measure supported at $s \in \mathbb{R}^{m_l}$. Let us note that $i_l(U_l) \subset \bar{U}_l$.

The relaxed game is then created by the continuous extension of the original game $(\mathcal{G})$ from $U_1 \times U_2$ to $\bar{U}_1 \times \bar{U}_2$, which gives:

$$
(\mathcal{RG})\left\{
\begin{array}{ll}
\text{Find Nash equilibrium} & \left\{
\begin{array}{l}
J_1(\nu_1,\nu_2,y) := \displaystyle\int_0^T \Big(g_1(t,y) + \int_{\mathbb{R}^{m_1}} h_1(t,s_1)\,\nu_{1,t}({\rm d}s_1) \\
\qquad\qquad + \displaystyle\int_{\mathbb{R}^{m_1}} \int_{\mathbb{R}^{m_2}} \varphi(t,s_1,s_2)\nu_{2,t}({\rm d}s_2)\,\nu_{1,t}({\rm d}s_1) \Big)\,{\rm d}t \\
J_2(\nu_1,\nu_2,y) := \displaystyle\int_0^T \Big(g_2(t,y) + \int_{\mathbb{R}^{m_2}} h_2(t,s_2)\,\nu_{2,t}({\rm d}s_2) \\
\qquad\qquad - \displaystyle\int_{\mathbb{R}^{m_1}} \int_{\mathbb{R}^{m_2}} \varphi(t,s_1,s_2)\,\nu_{2,t}({\rm d}s_2)\nu_{1,t}({\rm d}s_1) \Big)\,{\rm d}t
\end{array}\right. \\
\text{subject to} & \dfrac{{\rm d}y}{{\rm d}t} = G(t,y) + \displaystyle\int_{\mathbb{R}^{m_1}} H_1(t,s)\,\nu_{1,t}({\rm d}s) + \int_{\mathbb{R}^{m_2}} H_2(t,s)\,\nu_{2,t}({\rm d}s) \\
& y(0) = y_0, \\
& {\rm supp}(\nu_{1,t}) \subset S_1(t)\ ,\ \ {\rm supp}(\nu_{2,t}) \subset S_2(t) \quad \text{for a. a. } t \in (0,T), \\
& y \in W^{1,p}(0,T;\mathbb{R}^n),\ \ \nu_1 \in \mathcal{Y}(0,T;\mathbb{R}^{m_1}),\ \ \nu_2 \in \mathcal{Y}(0,T;\mathbb{R}^{m_2}).
\end{array}\right.
$$

To investigate the structure of the relaxed game $(\mathcal{RG})$ more in detail, let us define the extended payoffs $\bar{\Phi}_1, \bar{\Phi}_2 : \bar{U}_1 \times \bar{U}_2 \to \mathbb{R}$ by

$$
\bar{\Phi}_l(\nu_1,\nu_2) := J_l(\nu_1,\nu_2,y(\nu_1,\nu_2))\ , \qquad l = 1,2, \tag{2.3}
$$

where $y = y(\nu_1,\nu_2) \in W^{1,p}(0,T;\mathbb{R}^n)$ denotes the unique solution to the initial-value problem

$$
\frac{{\rm d}y}{{\rm d}t} = G(t,y) + \int_{\mathbb{R}^{m_1}} H_1(t,s)\,\nu_{1,t}({\rm d}s) + \int_{\mathbb{R}^{m_2}} H_2(t,s)\,\nu_{2,t}({\rm d}s), \quad y(0) = y_0. \tag{2.4}
$$

Obviously, $(\mathcal{RG})$ just represents a Nash equilibrium search over $\bar{U}_1 \times \bar{U}_2$ of the extended payoffs $\bar{\Phi}_1$ and $\bar{\Phi}_2$; this means we are to find $(\nu_1,\nu_2) \in \bar{U}_1 \times \bar{U}_2$ such that

$$
\bar{\Phi}_1(\nu_1,\nu_2) = \min_{\tilde{\nu}_1 \in \bar{U}_1} \bar{\Phi}_1(\tilde{\nu}_1,\nu_2) \quad \text{and} \quad \bar{\Phi}_2(\nu_1,\nu_2) = \min_{\tilde{\nu}_2 \in \bar{U}_2} \bar{\Phi}_2(\nu_1,\tilde{\nu}_2). \tag{2.5}
$$

The main results about $(\mathcal{RG})$ and relations between $(\mathcal{RG})$ and $(\mathcal{G})$ are supported by the properties stated in the following four lemmas.

Lemma 2.1. Let (1.1e) be valid. Then, for $l = 1,2$, the set of admissible relaxed strategies $\bar{U}_l$ defined by (2.1) is convex and weakly* compact, and contains densely the set of original strategies U_l imbedded into $L^\infty_{\mathrm{w}}(0,T;\mathrm{rca}(\mathbb{R}^{m_l}))$ via the imbedding i_l.

Proof. See Sainte–Beuve [17; Corollary 4] for a general case. For a special S_l such result can be also found e. g. in Gamkrelidze [6; Theorem II.2] or in Warga [21; Theorem IV.2.6], or also in [15; Theorem 3.1.6 and Remark 3.1.10]. □

Lemma 2.2. Let (1.1a–d) be valid. Then, for $l = 1,2$, the extended payoff $\bar\Phi_l$ from (2.3)–(2.4) is a separately (weak*×weak*)-continuous extension of the original payoff $\bar\Phi_l$ from (1.2)–(1.3). Moreover, $\bar\Phi_1 + \bar\Phi_2 : \bar U_1 \times \bar U_2 \to \mathbb{R}$ is jointly (weak*×weak*)-continuous.

Proof. The (weak*,weak)-continuity of the mapping $\mathcal{Y}(0,T;\mathbb{R}^{m_l}) \to L^p(0,T;\mathbb{R}^{m_l}) : \nu_l \mapsto \big(t \mapsto \int_{\mathbb{R}^{m_l}} H_l(t,s)\,\nu_{l,t}(\mathrm{d}s)\big)$ for $l = 1,2$ is obvious. From this one gets by standard arguments, including also the compactness of the imbedding $W^{1,p}(0,T;\mathbb{R}^n) \subset L^\infty(0,T;\mathbb{R}^n)$, the (weak*×weak*,weak)-continuity of the mapping $\mathcal{Y}(0,T;\mathbb{R}^{m_1})\times \mathcal{Y}(0,T;\mathbb{R}^{m_2}) \to W^{1,p}(0,T;\mathbb{R}^n) : (\nu_1,\nu_2) \mapsto y = y(\nu_1,\nu_2)$ from (2.4). As this mapping $(\nu_1,\nu_2) \mapsto y(\nu_1,\nu_2)$ is continuous to the norm topology of $L^\infty(0,T;\mathbb{R}^n)$ and also the Nemytskiĭ mapping $y \mapsto [t \mapsto g_l(t,y(t))] : L^\infty(0,T;\mathbb{R}^n) \to L^1(0,T)$ is continuous, the functional

$$(\nu_1,\nu_2) \mapsto \int_0^T g_l(t,y(\nu_1,\nu_2))\,\mathrm{d}t \tag{2.6}$$

is (weak*×weak*)-continuous, too. Also

$$(\nu_1,\nu_2) \mapsto \int_0^T \int_{\mathbb{R}^{m_l}} h_l(t,s)\,\nu_{l,t}(\mathrm{d}s)\,\mathrm{d}t \tag{2.7}$$

is obviously continuous. The remaining term in the payoff functional, i. e.

$$(\nu_1,\nu_2) \mapsto \int_0^T \int_{\mathbb{R}^{m_1}} \int_{\mathbb{R}^{m_2}} \varphi(t,s_1,s_2)\,\nu_{2,t}(\mathrm{d}s_2)\,\nu_{1,t}(\mathrm{d}s_1)\,\mathrm{d}t \tag{2.8}$$

is, however, not jointly (weak*×weak*)-continuous in a general case, cf. e. g. Warga [21; Sections IX.2 and X.0.1] or also [15; Example 3.6.18]. On the other hand, by the Fubini theorem, one can show that the functional (2.8) is separately (weak*×weak*)-continuous; cf. [15; Lemma 3.6.14]. Altogether, the continuity of (2.6)–(2.8) imply the separate (weak*×weak*)-continuity of each payoff $\bar\Phi_1$ and $\bar\Phi_2$. Moreover, by the continuity of (2.6)–(2.7), $\bar\Phi_1 + \bar\Phi_2$ is jointly (weak*×weak*)-continuous because the critical term (2.8), which is possibly not jointly continuous, disappears in the sum of payoffs. □

To investigate the geometrical properties of $\bar\Phi_l$ with $l = 1,2$, we will have to calculate its Gâteaux differential with respect to the geometry coming from the linear space $L^\infty_{\mathrm{w}}(0,T;\mathrm{rca}(\mathbb{R}^{m_l}))$ containing $\bar U_l$. This is, in fact, a standard task undertaken within derivation of the maximum principle for the relaxed strategies.

The needed Fréchet differentiability with respect to y can be guaranteed by the following assumptions on the partial derivative of g_l and G with respect to the variable r, denoted respectively by $g_l'(t,r)$ and $G'(t,r)$:

$$\exists\, a \in L^1(0,T)\ \ \exists\, b : \mathbb{R} \to \mathbb{R} \text{ continuous}: \ \ |g_l'(t,r)| \le a(t) + b(|r|), \tag{2.9a}$$
$$|g_l'(t,r_1) - g_l'(t,r_2)| \le (a(t) + b(|r_1|) + b(|r_2|))|r_1 - r_2|,$$

$$\exists\, a \in L^p(0,T)\ \ \exists\, b : \mathbb{R} \to \mathbb{R} \text{ continuous}: \ \ |G'(t,r)| \le a(t) + b(|r|), \tag{2.9b}$$
$$|G'(t,r_1) - G'(t,r_2)| \le (a(t) + b(|r_1|) + b(|r_2|))|r_1 - r_2|,$$

where $l = 1, 2$. The maximum principle involves the so-called adjoint terminal-value problem

$$\frac{\mathrm{d}\lambda_l}{\mathrm{d}t} = -\lambda_l(t) G'(t, y(t)) - g_l'(t, y(t))\ , \quad \lambda_l(T) = 0. \tag{2.10}$$

Under the assumption (2.9), the problem (2.10) possesses precisely one solution $\lambda_l \in W^{1,1}(0,T;\mathbb{R}^n)$. Following a procedure by Gabasov and Kirillova [5; Section VII.2] or (for the general integral processes) by Schmidt [18] developed to prove sufficiency of the maximum principle for optimal control problems, we can establish the following increment formula. Let us formulate it for $\bar{\Phi}_1(\cdot, \nu_2)$, the other needed case $\bar{\Phi}_2(\nu_1, \cdot)$ being entirely analogous.

Lemma 2.3. Let (1.1a–d) and (2.9) be valid, let $\nu_1, \tilde{\nu}_1 \in \bar{U}_1$ and $\nu_2 \in \bar{U}_2$, let $y = y(\nu_1, \nu_2) \in W^{1,p}(0,T;\mathbb{R}^n)$ be defined by (2.4), and let $\lambda_1 \in W^{1,1}(0,T;\mathbb{R}^n)$ solve (2.10) with $l = 1$. Then

$$\begin{aligned} \bar{\Phi}_1(\tilde{\nu}_1, \nu_2) - \bar{\Phi}_1(\nu_1, \nu_2) \ = \ & \int_0^T \int_{\mathbb{R}^{m_1}} \mathcal{H}_1^{\nu_2,\lambda_1}(t,s)[\tilde{\nu}_{1,t} - \nu_{1,t}](\mathrm{d}s)\,\mathrm{d}t \\ & + \int_0^T [\Delta_{g_1}(t) + \lambda_1(t)\Delta_G(t)]\mathrm{d}t \end{aligned} \tag{2.11}$$

where the "Hamiltonian" $\mathcal{H}_1^{\nu_2,\lambda}$ is defined by

$$\mathcal{H}_1^{\nu_2,\lambda}(t,s_1) := \lambda(t)\, H_1(t,s_1) + h_1(t,s_1) + \int_{\mathbb{R}^{m_2}} \varphi(t,s_1,s_2)\, \nu_{2,t}(\mathrm{d}s_2)\ , \tag{2.12}$$

and the second-order correcting terms Δ_{g_1} and Δ_G are defined by

$$\Delta_{g_1}(t) := g_1(t,\tilde{y}(t)) - g_1(t,y(t)) - g_1'(t,y(t))\,(\tilde{y}(t) - y(t))\ , \tag{2.13}$$

$$\Delta_G(t) := G(t,\tilde{y}(t)) - G(t,y(t)) - G'(t,y(t))\,(\tilde{y}(t) - y(t))\ , \tag{2.14}$$

with $\tilde{y} = y(\tilde{\nu}_1, \nu_2) \in W^{1,p}(0,T;\mathbb{R}^n)$ being the solution to the initial-value problem (2.4) with $\tilde{\nu}_1$ in place of ν_1.

Proof. Using successively the formula (2.12), the equation (2.4) both for ν_1 and for $\tilde{\nu}_1$, the by-parts integration and the adjoint equation (2.10), we can calculate:

$$\bar{\Phi}_1(\tilde{\nu}_1,\nu_2)-\bar{\Phi}_1(\nu_1,\nu_2)-\int_0^T\int_{\mathbb{R}^{m_1}}\mathcal{H}_1^{\nu_2,\lambda_1}(t,s)[\tilde{\nu}_{1,t}-\nu_{1,t}](\mathrm{d}s)\,\mathrm{d}t$$
$$=\int_0^T g_1(t,\tilde{y}(t))-g_1(t,y(t))-\int_{\mathbb{R}^{m_l}}\lambda_1(t)\,H_1(t,s_1)[\tilde{\nu}_t-\nu_t](\mathrm{d}s_1)\,\mathrm{d}t$$
$$=\int_0^T\Big[g_1(t,\tilde{y}(t))-g_1(t,y(t))+\lambda_1(t)\Big(G(t,\tilde{y}(t))-G(t,y(t))-\frac{\mathrm{d}(\tilde{y}(t)-y(t))}{\mathrm{d}t}\Big)\Big]\,\mathrm{d}t$$
$$=\int_0^T\Big[g_1(t,\tilde{y}(t))-g_1(t,y(t))+\lambda_1(t)\,(G(t,\tilde{y}(t))-G(t,y(t)))+\frac{\mathrm{d}\lambda_1}{\mathrm{d}t}(\tilde{y}(t)-y(t))\Big]\,\mathrm{d}t$$
$$=\int_0^T\Big[\,g_1(t,\tilde{y}(t))-g_1(t,y(t))-g_1'(t,y(t))\,(\tilde{y}(t)-y(t))$$
$$+\lambda_1(t)\,(G(t,\tilde{y}(t))-G(t,y(t))-G'(t,y(t))\,(\tilde{y}(t)-y(t)))\,\Big]\,\mathrm{d}t$$
$$=:\int_0^T\Delta_{g_1}(t)+\lambda_1(t)\Delta_G(t)\,\mathrm{d}t.$$ □

The formula (2.11) enables us to investigate convexity of the extended cost functional $\bar{\Phi}_1(\cdot,\nu_2)$. Of course, analogous considerations apply also to $\bar{\Phi}_2(\nu_1,\cdot)$. Let us take $B_R:=\{r\in\mathbb{R}^n;\ |r|\le R\}$ a sufficiently large ball so that $[y(u_1,u_2)](t)\in B_R$ for any $u_1\in U_1$, $u_2\in U_2$ and any $t\in(0,T)$, where $y(u_1,u_2)\in W^{1,p}(0,T;\mathbb{R}^n)$ denotes the unique solution to (1.3); this means we can put $R:=\sup_{u_1\in U_1}\sup_{u_2\in U_2}\|y(u_1,u_2)\|$ $C(0,T;\mathbb{R}^n)$. Furthermore, let

$$a_1(t):=\sup_{|r|\le R}|g_1'(t,r)|\,,\quad a_2(t):=\sup_{|r|\le R}|g_2'(t,r)|\,,\quad A(t):=\sup_{|r|\le R}|G'(t,r)|.\tag{2.15}$$

Note that (2.9) ensures certainly $a_1,a_2,A\in L^1(0,T)$. Thus we may put

$$b_1(t):=\int_t^T a_1(\tau)\,\mathrm{d}\tau\,,\quad b_2(t):=\int_t^T a_2(\tau)\,\mathrm{d}\tau\,,\quad B(t):=\int_t^T A(\tau)\,\mathrm{d}\tau.\tag{2.16}$$

Lemma 2.4. Let (1.1) and (2.9) be valid, and let $G(t,\cdot)$ be twice continuously differentiable, and let $g_l(t,\cdot)$ be uniformly convex on B_R in the sense

$$\forall r,\tilde{r}\in B_R:\quad g_l(t,\tilde{r})-g_l(t,r)-g_l'(t,r)\,(\tilde{r}-r)\ge c_l(t)|\tilde{r}-r|^2\tag{2.17}$$

with a modulus c_l satisfying

$$c_l(t)\ge\frac{1}{2}b_l(t)\mathrm{e}^{B(t)}\sup_{|r|\le R}|G''(t,r)|\tag{2.18}$$

for $l=1,2$. Then, for any $\nu_1\in\bar{U}_1$ and $\nu_2\in\bar{U}_2$, the extended payoffs $\bar{\Phi}_1(\cdot,\nu_2):\bar{U}_1\to\mathbb{R}$ and $\bar{\Phi}_2(\nu_1,\cdot):\bar{U}_2\to\mathbb{R}$ defined in (2.3)–(2.4) are convex.

Proof. Let us show the case $\bar{\Phi}_1(\cdot,\nu_2)$; the opponent's case $\bar{\Phi}_2(\nu_1,\cdot)$ being analogous.

From the adjoint equation (2.10) with $l=1$, we can estimate $\mathrm{d}|\lambda_1|/\mathrm{d}t \le |\mathrm{d}\lambda_1/\mathrm{d}t| \le A(t)|\lambda_1(t)| + a_1(t)$ so that by the Gronwall inequality one gets

$$|\lambda_1(t)| \ \le \ \left(\int_t^T a_1(\tau)\mathrm{e}^{-\int_t^T A(\theta)\,\mathrm{d}\theta}\mathrm{d}\tau\right)\mathrm{e}^{\int_t^T A(\tau)\,\mathrm{d}\tau}. \tag{2.19}$$

To simplify the notation, using (2.16) we can also (a bit more pessimistically) estimate

$$|\lambda_1(t)| \ \le \ b_1(t)\mathrm{e}^{B(t)}. \tag{2.20}$$

By the Taylor expansion, we can estimate

$$|G(t,\tilde{y}(t)) - G(t,y(t)) - G'(t,y(t))\,(\tilde{y}(t)-y(t))| \le \frac{1}{2}\sup_{|r|\le R}|G''(t,r)|\;|\tilde{y}(t)-y(t)|^2.$$

Then (2.17) with (2.18) and (2.20) ensure

$$\begin{aligned}\Delta_{g_1}(t) + \lambda_1(t)\Delta_G(t) \ &\ge \ c_1(t)|\tilde{y}(t)-y(t)|^2 - \frac{1}{2}|\lambda_1(t)|\sup_{|r|\le R}|G''(t,r)|\;|\tilde{y}(t)-y(t)|^2\\ &\ge \ \left(c_1(t) - \frac{1}{2}b_1(t)\mathrm{e}^{B(t)}\sup_{|r|\le R}|G''(t,r)|\right)|\tilde{y}(t)-y(t)|^2 \ \ge \ 0\end{aligned}$$

so that the second right-hand term in (2.11) is non-negative; note that by Lemma 2.1 and by the continuity of the mapping $(\nu_1,\nu_2)\mapsto y(\nu_1,\nu_2)$ we have $|[y(\nu_1,\nu_2)](t)|\le R$ for any $(\nu_1,\nu_2)\in\bar{U}_1\times\bar{U}_2$, which makes (2.17)–(2.18) effective. By [15; Section 4.3], the first right-hand term in (2.11) represents just the Gâteaux differential of $\bar{\Phi}_1(\cdot,\nu_2):\bar{U}_1\to\mathbb{R}$, i.e.

$$\begin{aligned}[\nabla_{\nu_1}\bar{\Phi}_1(\nu_1,\nu_2)](\tilde{\nu}_1-\nu_1) \ &:= \ \lim_{\varepsilon\searrow 0}\frac{\bar{\Phi}_1(\nu_1+\varepsilon(\tilde{\nu}_1-\nu_1),\nu_2) - \bar{\Phi}_1(\nu_1,\nu_2)}{\varepsilon}\\ &= \ \int_0^T\int_{\mathbb{R}^{m_1}}\mathcal{H}_1^{\nu_2,\lambda_1}(t,s)[\tilde{\nu}_{1,t}-\nu_{1,t}](\mathrm{d}s)\,\mathrm{d}t\end{aligned}$$

with $\nu_1,\tilde{\nu}_1\in\bar{U}_1$ arbitrary and with the adjoint state λ_1 and the Hamiltonian $\mathcal{H}_1^{\nu_2,\lambda}$ defined respectively by (2.10) and (2.12). Therefore we obtained

$$\bar{\Phi}_1(\tilde{\nu}_1,\nu_2) - \bar{\Phi}_1(\nu_1,\nu_2) - [\nabla_{\nu_1}\bar{\Phi}_1(\nu_1,\nu_2)](\tilde{\nu}_1-\nu_1) \ge 0\ , \tag{2.21}$$

for all $\nu_1,\tilde{\nu}_1\in\bar{U}_1$, which just says that $\bar{\Phi}_1(\cdot,\nu_2)$ is convex on $\bar{U}_1$. □

3. MAIN RESULTS

We are now ready to formulate the main achievements: existence of the Nash equilibria for the relaxed game $(\mathcal{RG})$, relations between $(\mathcal{RG})$ and the original game $(\mathcal{G})$, as well as existence of equilibrium sequences for $(\mathcal{G})$. We will use the Nikaidô and Isoda generalization [11] of the classical Nash theorem [10]; this generalization admits only separately continuous payoffs and is equivalent with the Brouwer fixed-point theorem, as pointed out by Kindler [7; Remark 1.2].

Theorem 3.1. Let (1.1), (2.9) and (2.17)–(2.18) for $l = 1, 2$ be valid. Then:

(i) The relaxed game $(\mathcal{RG})$ possesses at least one Nash equilibrium.

(ii) Every Nash equilibrium of the relaxed game $(\mathcal{RG})$ can be attained by an equilibrium sequence for the original game $(\mathcal{G})$ imbedded via $i_1 \times i_2$ into $L^\infty_{\rm w}(0,T;\mathrm{rca}(\mathbb{R}^{m_1})) \times L^\infty_{\rm w}(0,T;\mathrm{rca}(\mathbb{R}^{m_2}))$.

(iii) Conversely, every equilibrium sequence for the original game $(\mathcal{G})$ (imbedded via $i_1 \times i_2$) has a weakly* convergent subsequence and the limit of every such a subsequence is a Nash equilibrium for the relaxed game $(\mathcal{RG})$.

Proof. By Lemmas 2.1, 2.2 and 2.4, we can see that the relaxed problem $(\mathcal{RG})$ represents a game over the convex compact sets $\bar{U}_1$ and $\bar{U}_2$ for the separately continuous payoffs $\bar{\Phi}_1$ and $\bar{\Phi}_2$ whose sum is jointly continuous and such that $\bar{\Phi}_1(\cdot,\nu_2)$ and $\bar{\Phi}_2(\nu_1,\cdot)$ are convex. This just guarantees, by Nikaidô and Isoda [11], that $(\mathcal{RG})$ has got at least one Nash equilibrium, as claimed in (i).

The points (ii) and (iii) follow by Lemmas 2.1 and 2.2. In fact, $\Psi_1 : U_1 \to \mathbb{R}$ and $\Psi_2 : U_2 \to \mathbb{R}$ in (1.6) are defined by

$$\Psi_1 = \bar{\Phi}_1(\cdot,\nu_2) \circ i_1 \quad \& \quad \Psi_2 = \bar{\Phi}_2(\nu_1,\cdot) \circ i_2 , \tag{3.1}$$

for details see [15; Proposition 7.1.1] and realize the metrizability of the weak* topology on $\bar{U}_1$ and $\bar{U}_2$, which allows us to work in terms of sequences instead of nets. □

Corollary 3.1. Under the assumptions of Theorem 3.1, the original game $(\mathcal{G})$ possesses an equilibrium sequence, i.e. a sequence $\{(u_1^k, u_2^k)\}_{k\in\mathbb{N}}$ satisfying (1.6).

Proof. It follows straightforwardly by the points (i) and (ii) of Theorem 3.1. □

Corollary 3.2. Let the assumptions of Theorem 3.1 be satisfied and let $\varphi \equiv 0$. Then the original game $(\mathcal{G})$ possesses an asymptotically Nash equilibrium, i.e. a sequence $\{(u_1^k, u_2^k)\}_{k\in\mathbb{N}}$ satisfying (1.7).

Proof. Take $\nu_1 \in \bar{U}_1$ and $\nu_2 \in \bar{U}_2$ arbitrary. Furthermore, take a sequence $\{u_2^k\}_{k\in\mathbb{N}} \subset U_2$ generating $\nu_2 \in \bar{U}_2$ in the sense $i_2(u_2^k) \to \nu_2$ weakly* in $L^\infty_{\rm w}(0,T;$ $\mathrm{rca}(\mathbb{R}^{m_2}))$ and put $y = y(\nu_1,\nu_2)$ and $y^k = y(\nu_1, i_2(u_2^k))$. In view of (2.4), we can estimate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|y^k - y| \;&\le\; \left|\frac{\mathrm{d}}{\mathrm{d}t}(y^k - y)\right| \;\le\; |G(t,y^k(t)) - G(t,y(t))| \\ &+\; \left|H_2(t,u_2^k(t)) - \int_{\mathbb{R}^{m_2}} H_2(t,s)\,\nu_{2,t}(\mathrm{d}s)\right| \;\le\; a(t)|y^k(t) - y(t)| + c^k(t) ,\end{aligned}$$

where $a \in L^1(0,T)$ comes from (1.1c) and $c^k \in L^p(0,T)$ abbreviates $c^k(t) := |H_2(t,u_2^k(t)) - \int_{\mathbb{R}^{m_2}} H_2(t,s)\,\nu_{2,t}(\mathrm{d}s)|$. Likewise (2.19)–(2.20), we can estimate by means of the Gronwall inequality

$$|y^k(t)-y(t)| \le \left(\int_0^t c^k(\tau)e^{-\int_0^t a(\theta)\,d\theta}d\tau\right)e^{\int_0^t a(\tau)\,d\tau} \le \left(\int_0^t c^k(\tau)\,d\tau\right)e^{\int_0^t a(\tau)\,d\tau}. \tag{3.2}$$

Since $i_2(u_2^k) \to \nu_2$ weakly*, we have $c^k \to 0$ weakly in $L^p(0,T)$ and therefore, using also the compactness of the imbedding $W^{1,p}(0,T) \subset L^\infty(0,T)$, we have got $\sup_{t\in(0,T)} \int_0^t c^k(\tau)\,d\tau \to 0$. In view of (3.2), this shows that $y^k \to y$ in $L^\infty(0,T;\mathbb{R}^n)$ and this convergence is uniform with respect to $\nu_1 \in \bar{U}_1$. This also imply the convergence $\int_0^T g_1(t,y^k(t))\,dt \to \int_0^T g_1(t,y(t))\,dt$ uniformly with respect to $\nu_1 \in \bar{U}_1$.

Assuming $\varphi = 0$, this shows (1.6a) uniformly, i.e.

$$\lim_{k\in N} \sup_{u_1\in U_1} |\Phi_1(u_1,u_2^k) - \Psi_1(u_1)| = 0 \tag{3.3}$$

with Ψ_1 defined in (3.1).

Just analogously, we can also show that (1.6b) holds uniformly provided $\{u_1^k\}_{k\in\mathbb{N}}$ is such a sequence that $\{i_1(u_1^k)\}_{k\in\mathbb{N}}$ weakly* converges.

In view of Theorem 3.1(i–ii), there is an equilibrium sequence $\{(u_1^k,u_2^k)\}_{k\in\mathbb{N}}$ for $(\mathcal{G})$ such that both $\{i_1(u_1^k)\}_{k\in\mathbb{N}}$ and $\{i_2(u_2^k)\}_{k\in\mathbb{N}}$ weakly* converges. Then, by the above arguments, (1.8) holds so that by Proposition 1.1 the sequence $\{(u_1^k,u_2^k)\}_{k\in\mathbb{N}}$ represents an asymptotically Nash equilibrium. □

Remark 3.1. If the controlled system is linear with respect to the state, i.e. $G(t,\cdot)$ is affine, then obviously $G'' \equiv 0$ and one can take $c_l \equiv 0$ in (2.17) which then just requires $g_1(t,\cdot)$ and $g_2(t,\cdot)$ to be merely convex; cf. e.g. Balder [1], Bensoussan [4] or Nowak [13], or also [15; Section 7.3].

Remark 3.2. The assumption $\varphi = 0$ in Corollary 3.2 is inevitable because otherwise the uniform convergence (3.3) cannot be expected. Indeed, one can use the example $\varphi(t,s_1,s_2) = s_1 s_2$ which induces by the formula (2.8) a separately (weak*×weak*)-continuous functional $\bar{U}_1 \times \bar{U}_2 \to \mathbb{R}$ which is, however, not jointly (weak*×weak*)-continuous; cf. [15; Example 3.6.18].

Remark 3.3. If the multivalued mapping $S_l : (0,T) \rightrightarrows \mathbb{R}^{m_l}$ acting as strategy constraints in $(\mathcal{G})$ is not bounded, several sophisticated approaches based on Chacon's biting lemma and the Dunford–Pettis theorem must be still incorporated. For details we refer to [15; Section 7.3] where only systems linear with respect to the state are considered, however.

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