

STATE OBSERVERS FOR NONLINEAR SYSTEMS WITH SMOOTH/BOUNDED INPUT¹

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It is known that for affine nonlinear systems the drift-observability property (i. e. observability for zero input) is not sufficient to guarantee the existence of an asymptotic observer for any input. Many authors studied structural conditions that ensure uniform observability of nonlinear systems (i. e. observability for any input). Conditions are available that define classes of systems that are uniformly observable.

This work considers the problem of state observation with exponential error rate for smooth nonlinear systems that meet or not conditions of uniform observability. In previous works the authors showed that drift-observability together with a smallness condition on the input is sufficient to ensure existence of an exponential observer. Here it is shown that drift-observability implies a kind of local uniform observability, that is observability for sufficiently small and smooth input. For locally uniformly observable systems two observers are presented: an exponential observer that uses derivatives of the input functions; an observer that does not use input derivatives and ensures exponential decay of the observation error below a prescribed level (high-gain observer). The construction of both observers is straightforward. Moreover the state observation is provided in the original coordinate system. Simulation results close the paper.

1. INTRODUCTION

Many authors pointed out the peculiarities of the problem of state observation for nonlinear dynamic systems in comparison with the much simpler linear case [1, 6, 8, 11–17]. A main property of nonlinear systems is that, differently from linear ones, state reconstructability in general depends on the input: drift-observability (i. e. observability for zero input) is not a sufficient condition for existence of an asymptotic observer for any input. This fact induced some authors to find conditions on nonlinear systems that ensure state reconstructability for any input. Classes of nonlinear systems are then defined for which observers can be constructed that work independently of the input applied (uniformly observable systems). However, such classes are characterized by limitative conditions, that are not met in many significant applications.

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Uniformly observable systems admit a map that transforms the input and output derivatives up to a given order into the system state. If such map can be explicitly computed then high-gain observers can be implemented: some estimates of input and output derivatives are used in the map to compute an estimate of the state [15]. Moreover, if the observer is inserted in a control loop in which the input is computed by a dynamic law, the input is known together with its derivatives up to the desired order. Then, estimates of the output derivatives together with the known input derivatives can be used to compute the state estimate [5, 10, 13]. The main problem with this approach is the availability of the above mentioned map.

In [6] it is shown that uniformly observable systems admit a sort of canonical coordinate system. In [7] an observer is presented that provides a state estimate in such coordinate system, so that the inverse coordinate transformation is needed to obtain the state estimate in the original coordinate system.

Following the researches presented in [2–4] this paper studies conditions not only on the system structure, but also on the input functions, that guarantee existence and practical computation of a state observer with exponential error decay. In [3–4] it is shown that an exponential observer can be constructed for systems that are not uniformly observable, provided that the input is sufficiently small. Both global and semi-global results are reported. In [4] the case of multi-input multi-output systems is also analysed.

In this paper a condition of uniform observability is defined for systems whose inputs are bounded together with their derivatives up to a given order (smooth/bounded inputs). Moreover, it is shown that smooth systems under drift-observability property own the uniform observability property for sufficiently small and smooth inputs. For systems that are uniformly observable for smooth/bounded inputs, two kind of observers are presented. The first is an exponential observer and uses the derivatives of the input function. Since in most open-loop applications input derivatives are not available, this observer is denoted *Theoretical Observer*. The second is an observer that does not use input derivatives, and therefore it is called *Practical Observer*, and allows exponential decay of the observation error below a prescribed level. For this reason it can be classified as a high-gain observer. However, differently from high-gain observers in [5, 10, 13] it does not require the explicit computation of the function that maps input and output derivative into the system state. Proofs of convergence are given for both observers.

Due to pages limitation, only global results are reported in this paper, that require stronger assumptions but are easier to prove, and only the case of single-input single-output nonlinear systems is analysed.

In this work nonlinear systems of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad (1.1)$$

$$y(t) = h(x(t)), \quad (1.2)$$

are considered, where $x(t) \in X \subseteq \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}$ and $y(t) \in \mathbb{R}$, $g(x)$ and $f(x)$ are $C^\infty(X)$ vector fields and $h(x)$ is a $C^\infty(X)$ function. It is assumed that the inputs applied to the system are such that the state is well defined and bounded for every time (no finite escape-time).

Throughout the paper the symbol $O_{a \times b}$ denotes a matrix of zeroes of dimension $a \times b$, the symbol I_k denotes the $k \times k$ identity matrix, and $\text{cn}(P)$ denotes the condition number of a matrix P . The following properties for symmetric positive definite matrices are used

$$\text{cn}(P) = \frac{\lambda_{\text{Max}}(P)}{\lambda_{\text{min}}(P)} = \text{cn}(P^{-1}), \quad \|P\| = \lambda_{\text{Max}}(P) = \frac{1}{\lambda_{\text{min}}(P^{-1})}, \quad (1.3)$$

where $\lambda_{\text{Max}}(P)$ and $\lambda_{\text{min}}(P)$ denote the maximum and minimum eigenvalues of matrix P , respectively.

2. PRELIMINARIES

Consider the vector Y_n of the first n output derivatives (from 0 to $n - 1$) for system (1.1)

$$Y_n = [y \ \dot{y} \ \dots \ y^{(n-1)}]^T. \quad (2.1)$$

It is well known that, for $u(t) \equiv 0$,

$$Y_n = \Phi(x) \quad (2.2)$$

where $\Phi(x)$ is the square map

$$\Phi(x) \triangleq \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} \quad (2.3)$$

(it is assumed that the reader is familiar with the concept of repeated Lie derivative of a function along a vector field). The knowledge of the vector Y_n at a given time t and of the inverse map $\Phi^{-1}(\cdot)$ would allow exact state reconstruction.

Definition 2.1. System (1.1), (1.2) is said *drift-observable* in an open set $\Omega \subseteq \mathbb{R}^n$ if the map $\Phi(x)$ is a diffeomorphism from Ω to $\Phi(\Omega)$. If $\Omega \equiv \mathbb{R}^n$ then the system (1.1), (1.2) is said *globally drift-observable*.

For systems that are drift-observable in Ω the Jacobian of the map $\Phi(\cdot)$

$$Q(x) \triangleq \frac{\partial \Phi(x)}{\partial x} \quad (2.4)$$

is nonsingular in Ω . Although in general the inverse map of $z = \Phi(x)$, that exists in $\Phi(\Omega)$ and is denoted $x = \Phi^{-1}(z)$, is difficult to compute, its Jacobian can be computed as

$$\left. \frac{\partial \Phi^{-1}(z)}{\partial z} \right|_{z=\Phi(x)} = Q^{-1}(x). \quad (2.5)$$

In the state observation problem it is important the following concept, that is a weaker version of the well-known concept of relative degree (see e. g. [9]).

Definition 2.2. The system (1.1), (1.2) is said to have *observation relative degree* r in a set $\Omega \in \mathbb{R}^n$ if

$$\begin{aligned} \forall x \in \Omega, \quad L_g L_f^s h(x) &= 0, \quad s = 0, 1, \dots, r - 2, \\ \exists x \in \Omega : L_g L_f^{r-1} h(x) &\neq 0. \end{aligned} \tag{2.6}$$

Note that nonlinear systems may not have relative degree, but they always have observation relative degree.

Consider now the expression of the output derivatives in the general case in which $u \neq 0$. From the definition of observation relative degree it follows that the output derivatives from 0 to $r - 1$ are functions of the state x only, while the r th derivative is also function of the input

$$\begin{aligned} y^{(k)} &= L_f^k h(x), \quad k = 0, 1, \dots, r - 1, \\ y^{(r)} &= L_f^r h(x) + L_g L_f^{r-1} h(x) u. \end{aligned} \tag{2.7}$$

It is readily proved that higher order derivatives are functions of the state x , of the input u and of its time derivatives until a suitable order. More precisely, if U_s denotes the vector of the first s time-derivatives of the input (from 0 to $s - 1$), that is

$$U_s \triangleq \left[u \quad \dot{u} \quad \dots \quad u^{(s-1)} \right]^T, \tag{2.8}$$

it can be readily proved that the k th output derivative can be written, for $k \geq r$, as

$$y^{(k)} = L_f^k h(x) + \psi_k(x, U_{k-r+1}) \tag{2.9}$$

where the function $\psi_k(x, U_{k-r+1})$ is recursively defined as

$$\begin{aligned} \psi_r(x, U_1) &\triangleq L_g L_f^{r-1} h(x) u, \\ \psi_k(x, U_{k-r+1}) &\triangleq L_g L_f^{k-1} h(x) u + L_f \psi_{k-1}(x, U_{k-r}) \\ &\quad + L_g \psi_{k-1}(x, U_{k-r}) u + \left[0 \quad \frac{\partial \psi_{k-1}}{\partial U_{k-r}} \right] U_{k-r+1}, \quad k > r. \end{aligned} \tag{2.10}$$

$\psi_k = 0$ for $k = 0, 1, \dots, r - 1$, is assumed also. Using the scalar functions $\psi_k(x, U_{k-r+1})$ for $k = 0, 1, \dots, n - 1$ a n -components vector function $\Psi(x, U_{n-r})$ can be defined such that

$$Y_n = \bar{\Phi}(x, U_{n-r}) \triangleq \Phi(x) + \Psi(x, U_{n-r}). \tag{2.11}$$

(The j th component of $\Psi(x, U_{n-r})$, with $r + 1 \leq j \leq n$, is $\psi_{j-1}(x, U_{j-r})$.) If $r = n$ the function Ψ vanishes, and formula (2.11) can be simply written as (2.2). It is also easy to check from definitions (2.10) that the function $\Psi(x, U_{n-r})$ satisfies the property

$$\Psi(x, 0) = 0, \quad \forall x \in \mathbb{R}^n. \tag{2.12}$$

In general, drift-observability of system (1.1), (1.2) does not imply invertibility of (2.11) for x . In general, invertibility of (2.11) for x strongly depends on the input, through the vector of derivatives U_{n-r} , that can be considered as parameters in the mapping $\bar{\Phi}(x, U_{n-r})$. Thus, the following definition can be given.

Definition 2.3. If for any U_{n-r} in a set $\bar{U} \subseteq \mathbb{R}^{n-r}$ the map $Y_n = \bar{\Phi}(x, U_{n-r})$ in (2.11) is a diffeomorphism from an open set $\Omega \subseteq \mathbb{R}^n$ in $\bar{\Phi}(\Omega, \bar{U})$, the system (1.1), (1.2) is said *uniformly observable* in $\Omega \times \bar{U}$. If $\Omega \equiv \mathbb{R}^n$ and $\bar{U} \equiv \mathbb{R}^{n-r}$ than the system (1.1), (1.2) is said *globally uniformly observable*. If $\Omega \equiv \mathbb{R}^n$ and $\bar{U} = \{U_{n-r} : \|U_{n-r}\| \leq u_M\}$, for a given $u_M > 0$, then system (1.1), (1.2) is said *uniformly observable* for smooth/bounded input.

For systems that are uniformly observable in $\Omega \times \bar{U}$ the Jacobian

$$\bar{Q}(x, U_{n-r}) \triangleq \frac{\partial \bar{\Phi}(x, U_{n-r})}{\partial x} \tag{2.13}$$

is nonsingular in $\Omega \times \bar{U}$. The inverse map of $\eta = \bar{\Phi}(x, U_{n-r})$, that exists in $\bar{\Phi}(\Omega) \times \bar{U}$ and is denoted $x = \bar{\Phi}^{-1}(\eta, U_{n-r})$, is difficult to compute in general. However, its Jacobian can be directly computed as

$$\left. \frac{\partial \bar{\Phi}^{-1}(\eta, U_{n-r})}{\partial \eta} \right|_{\eta = \bar{\Phi}(x, U_{n-r})} = \bar{Q}^{-1}(x, U_{n-r}). \tag{2.14}$$

If a system is uniformly observable, the knowledge of vectors Y_n and $U_{n-r} \in \bar{U}$ would allow exact state reconstruction. Note that in the case $r = n$ (maximal relative degree) the maps Φ and $\bar{\Phi}$ coincide, so that drift-observability guarantees state reconstructability for any input. Moreover the following theorem holds.

Theorem 2.4. If system (1.1), (1.2) is drift-observable in Ω , then there exists a sufficiently small spherical neighborhood \bar{U} of the origin such that the system is uniformly observable in $\Omega \times \bar{U}$ (uniformly observable for smooth/bounded input).

Proof. From (2.11) the map $\bar{\Phi}(x, U_{n-r})$ satisfies the property

$$\bar{\Phi}(x, 0) = \Phi(x). \tag{2.15}$$

As a consequence invertibility in Ω of $\Phi(x)$ ensures that $\bar{\Phi}(x, U_{n-r})$ can be solved for $x \in \Omega$ if $U_{n-r} = 0$. Since, by smoothness assumption for system (1.1), (1.2), the map $\bar{\Phi}(x, U_{n-r})$ is continuous w.r.t. U_{n-r} , then it can be solved for $x \in \Omega$ if U_{n-r} is sufficiently close to the origin. This means that there exists a spherical neighborhood \bar{U} of the origin with sufficiently small radius that ensures uniform observability in $\Omega \times \bar{U}$. \square

If the system (1.1), (1.2) is uniformly observable in $\Omega \times \bar{U}$, the map $\eta = \bar{\Phi}(x, U_{n-r})$ can be considered as a time-varying change of coordinates (U_{n-r} is a function of time), as long as $U_{n-r} \in \bar{U}$. Since $\eta_j = y^{(j-1)}$ for $j = 1, \dots, n$, and then $\dot{\eta}_j = \eta_{j+1}$ for $j = 1, \dots, n-1$, in η -coordinates the system is written

$$\begin{aligned} \dot{\eta} &= A_n \eta + B_n m(\eta, U_{n-r+1}), \\ y &= C_n \eta, \end{aligned} \tag{2.16}$$

where

$$m(\eta, U_{n-r+1}) \triangleq \left(L_f^n h(x) + \psi_n(x, U_{n-r+1}) \right) \Big|_{x=\bar{\Phi}^{-1}(\eta, U_{n-r})}, \tag{2.17}$$

and matrices $A_n \in \mathbb{R}^{n \times n}$, $B_n \in \mathbb{R}^n$ and $C_n \in \mathbb{R}^n$ are Brunovsky matrices

$$A_n = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad B_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2.18}$$

$$C_n = [1 \ 0 \ \dots \ 0 \ 0].$$

If the coordinate change $z = \Phi(x)$ is considered instead, the system (1.1), (1.2) can be written in the new coordinates as

$$\begin{aligned} \dot{z} &= A_n z + B_n L_f^n h(\Phi^{-1}(z)) + Q(x)g(x) \Big|_{x=\Phi^{-1}(z)} u, \\ y &= C_n z. \end{aligned} \tag{2.19}$$

The product of the Jacobian $Q(x)$ by the matrix $g(x)$ is

$$Q(x)g(x) = \begin{bmatrix} L_g h(x) \\ \vdots \\ L_g L_f^{n-1} h_j(x) \end{bmatrix}. \tag{2.20}$$

From the definition of observation relative degree in Ω , the first $r - 1$ rows of vector (2.20) are identically zero in Ω , so that (2.20) can be rewritten as

$$Q(x)g(x) = F H(x), \tag{2.21}$$

$$\text{where } F \triangleq \begin{bmatrix} O_{(r-1) \times (n-r+1)} \\ I_{n-r+1} \end{bmatrix}, \quad H(x) \triangleq \begin{bmatrix} L_g L_f^{r-1} h(x) \\ \vdots \\ L_g L_f^{n-1} h(x) \end{bmatrix}. \tag{2.22a}$$

It is also useful to define the function

$$L(x) \triangleq L_f^n h(x), \tag{2.22b}$$

so that system (2.19) can be rewritten

$$\begin{aligned} \dot{z} &= A_n z + B_n L(\Phi^{-1}(z)) + F H(\Phi^{-1}(z)) u, \\ y &= C_n z. \end{aligned} \tag{2.23}$$

The pair A_n, C_n defined in (2.18) is observable, and it is an easy matter to assign eigenvalues to the matrix $A_n - K C_n$, that has the companion structure

$$A_n - K C_n = \begin{bmatrix} -k_1 & 1 & \dots & 0 & 0 \\ -k_2 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -k_{n-1} & 0 & \dots & 0 & 1 \\ -k_n & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{2.24}$$

Let $K(\lambda)$ denote the vector that assigns eigenvalues $\lambda = (\lambda_1, \dots, \lambda_n)$. Matrix $A_n - K(\lambda)C_n$ is diagonalized by the Vandermonde matrix

$$V_n \triangleq V_n(\lambda) = \begin{bmatrix} \lambda_1^{n-1} & \dots & \lambda_1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \lambda_n^{n-1} & \dots & \lambda_n & 1 \end{bmatrix}, \tag{2.25}$$

so that

$$V_n(\lambda)(A_n - K(\lambda)C_n)V_n^{-1}(\lambda) = \text{diag}\{\lambda\} = \Lambda. \tag{2.26}$$

In [4] it is shown that if the n eigenvalues are chosen as a function of a positive parameter σ as $\lambda(\sigma) = (-\sigma, -\sigma^2, \dots, -\sigma^n)$, then

$$\lim_{\sigma \rightarrow \infty} \|V_n^{-1}(\lambda(\sigma))\| = 1. \tag{2.27}$$

3. THE OBSERVER FOR SYSTEMS WITH BOUNDED INPUT

In this section it is shown that the system

$$\dot{\hat{x}}(t) = f(\hat{x}(t)) + g(\hat{x}(t))u(t) + Q^{-1}(\hat{x}(t))K(y(t) - h(\hat{x}(t))), \tag{3.1}$$

with the constant gain matrix K properly chosen, is an exponential observer for the system (1.1),(1.2), provided that the input is suitably *small* and some technical conditions are satisfied. The results reported in this section are a modified version of those presented in [3] for the SISO case, and in [4] for the MIMO case. Although both local and global results are available, for shortness only global results are here reported.

Theorem 3.1. Let system (1.1),(1.2) satisfy the following hypotheses:

1. The system is drift-observable in \mathbb{R}^n , and the map $z = \Phi(x)$ is uniformly Lipschitz together with its inverse $x = \Phi^{-1}(z)$ in \mathbb{R}^n , with constants γ_Φ and $\gamma_{\Phi^{-1}}$ respectively;
2. the functions $H(\Phi^{-1}(z))$ and $L(\Phi^{-1}(z))$, defined in (2.22), are uniformly Lipschitz in \mathbb{R}^n , with Lipschitz constants $\gamma_{\bar{H}}$ and $\gamma_{\bar{L}}$ respectively;
3. a constant $u_M > 0$ exists such that $|u(t)| \leq u_M \forall t \geq 0$;
4. for a given $\alpha > 0$ a vector $K \in \mathbb{R}^n$ and a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ exist that satisfy the following H_∞ Riccati-like inequality

$$(A_n - KC_n)P + P(A_n - KC_n)^T + B_n B_n^T + u_M^2 F F^T + 2\alpha P + \gamma^2 P^2 \leq 0, \tag{3.2}$$

where $\gamma^2 = \gamma_{\bar{L}}^2 + \gamma_{\bar{H}}^2$.

Then the observer (3.1) is such that

$$\|x(t) - \hat{x}(t)\| \leq \mu e^{-\alpha t} \|x(0) - \hat{x}(0)\| \tag{3.3}$$

with $\mu = \sqrt{cn(P)}\gamma_\Phi\gamma_{\Phi^{-1}}$.

Proof. For system (1.1), (1.2) and for observer (3.1) consider the following coordinate transformations and the following definitions of observation errors

$$\begin{aligned} z &= \Phi(x), & e_x &\triangleq x - \hat{x}, \\ \hat{z} &= \Phi(\hat{x}), & e_z &\triangleq z - \hat{z}. \end{aligned} \tag{3.4}$$

From assumption (1) they are such that

$$\|e_z\| \leq \gamma_\Phi \|e_x\|, \quad \|e_x\| \leq \gamma_{\Phi^{-1}} \|e_z\|. \tag{3.5}$$

System (1.1), (1.2) can be written in z -coordinates as (2.19), while the observer is written

$$\dot{\hat{z}} = A_n \hat{z} + B_n L(\Phi^{-1}(\hat{z})) + FH(\Phi^{-1}(\hat{z}))u + K(y - C_n \hat{z}). \tag{3.6}$$

The dynamics of the observation error in z -coordinates is governed by the linear perturbed equation

$$\dot{e}_z = (A_n - KC_n)e_z + B_n v_1(z, \hat{z}) + Fv_2(z, \hat{z})u, \tag{3.7}$$

where

$$\begin{aligned} v_1(z, \hat{z}) &\triangleq L(\Phi^{-1}(z)) - L(\Phi^{-1}(\hat{z})), \\ v_2(z, \hat{z}) &\triangleq H(\Phi^{-1}(z)) - H(\Phi^{-1}(\hat{z})). \end{aligned} \tag{3.8}$$

From assumption (2) the perturbations satisfy the inequalities

$$\|v_1\| \leq \gamma_L \|e_z\|, \quad \|v_2\| \leq \gamma_H \|e_z\|. \tag{3.9}$$

In order to prove that $e_z(t)$ exponentially goes to zero, consider the positive definite function of e_z

$$\nu(e_z) = e_z^T P^{-1} e_z, \tag{3.10}$$

where the positive definite symmetric matrix P satisfies inequality (3.2). The derivative of ν along the error trajectory is

$$\begin{aligned} \dot{\nu} &= e_z^T (P^{-1}(A - KC) + (A - KC)^T P^{-1}) e_z \\ &\quad + v_1 B^T P^{-1} e_z + e_z^T P^{-1} B v_1 + u v_2^T F^T P^{-1} e_z + u e_z^T P^{-1} F v_2. \end{aligned} \tag{3.11}$$

The following inequalities can be easily checked

$$\begin{aligned} v_1 B_n^T P^{-1} e_z + e_z^T P^{-1} B_n v_1 &\leq e_z^T P^{-1} B_n B_n^T P^{-1} e_z + v_1^2, \\ u v_2^T F^T P^{-1} e_z + u e_z^T P^{-1} F v_2 &\leq u_M^2 e_z^T P^{-1} F F^T P^{-1} e_z + v_2^T v_2, \end{aligned} \tag{3.12}$$

and substituted in (3.11). Using (3.5) and inequality (3.2), after simple transformations one has

$$\dot{\nu} \leq -2\alpha\nu, \Rightarrow \nu(t) \leq e^{-2\alpha t}\nu(0), \tag{3.13}$$

(last implication is due to Gronwall's inequality). From this, recalling the definition (3.10) of ν , we have

$$\|e_z(t)\| \leq \sqrt{\text{cn}(P)}e^{-\alpha t}\|e_z(0)\|. \tag{3.14}$$

Given the properties (3.5), inequality (3.14) becomes

$$\|e_x(t)\| \leq \mu e^{-\alpha t}\|e_x(0)\|, \tag{3.15}$$

with $\mu = \sqrt{\text{cn}(P)}\gamma_\Phi\gamma_{\Phi^{-1}}$, and the theorem is proved. □

The uniform Lipschitz assumptions in \mathbb{R}^n in Theorem 3.1 are rather strong, and can be relaxed to prove local convergence of the observation error to zero (see [4]).

The key point for the existence of an exponential observer of the form (3.1) for a drift-observable system is the existence of a pair (K, P) that solves inequality (3.2).

An interesting point is that the H_∞ Riccati-like inequality admits solution (K, P) for any $\alpha > 0$ and $\gamma > 0$ if the term FF^T is not present in the expression.

Lemma 3.2. For any triple α, β, γ of positive real the H_∞ Riccati-like inequality

$$(A_n - KC_n)P + P(A_n - KC_n)^T + \beta^2 B_n B_n^T + 2\alpha P + \gamma^2 P^2 \leq 0, \tag{3.16}$$

admits solution (K, P) with P symmetric positive definite.

Proof. Choose vector K so to assign a set of real eigenvalues λ , and set $P = (V_n(\lambda)^T V_n(\lambda))^{-1}$. Left-multiplying (3.16) $V_n(\lambda)$ and right-multiplying it by $V_n^T(\lambda)$ the H_∞ Riccati-like inequality becomes

$$2\lambda + \beta^2 V_n B_n B_n^T V_n^T + 2\alpha I_n + \gamma^2 V_n^{-1} V_n^{-T} \leq 0. \tag{3.17}$$

Matrix inequality (3.17) is satisfied if the following scalar inequality holds

$$2 \max\{\lambda\} \leq -\beta^2 \|V_n B_n\|^2 - 2\alpha - \gamma^2 \|V_n^{-1}\|^2. \tag{3.18}$$

The product $V_n B_n$ and the norm $\|V_n B_n\|$ are easy to compute

$$V_n B_n = [1 \ \dots \ 1]^T \in \mathbb{R}^n, \Rightarrow \|V_n B_n B_n^T V_n^T\| = n. \tag{3.19}$$

With the choice of eigenvalues $\lambda(\sigma) = (-\sigma, -\sigma^2, \dots, -\sigma^n)$, with $\sigma > 1$ so that $\max\{\lambda\} = -\sigma$, inequality (3.18) becomes

$$-\sigma \leq -\alpha - \frac{1}{2}n\beta^2 - \frac{1}{2}\gamma^2 \|V_n^{-1}(\lambda(\sigma))\|, \tag{3.20}$$

Thanks to (2.27), inequality (3.20) can be satisfied for σ sufficiently large, and the lemma is proved. □

Theorem 3.1 and the properties of the H_∞ Riccati-like inequality (3.2) originate two important results:

1. Existence of exponential observers for systems driven by sufficiently small input;
2. existence of an observer with assigned exponential rate for systems that have observation relative degree equal to n and a bounded input.

The theorems that state these results are reported below.

Theorem 3.3. Let system (1.1), (1.2) satisfy the following hypotheses:

1. The system is drift-observable and the map $z = \Phi(x)$ and its inverse $x = \Phi^{-1}(z)$ are uniformly Lipschitz in \mathbb{R}^n with constants γ_Φ and $\gamma_{\Phi^{-1}}$, respectively;
2. the functions $H(\Phi^{-1}(z))$ and $L(\Phi^{-1}(z))$ are uniformly Lipschitz in \mathbb{R}^n , with Lipschitz constants $\gamma_{\bar{H}}$ and $\gamma_{\bar{L}}$ respectively;

Then for any $\alpha > 0$ there exist a gain vector $K \in \mathbb{R}^n$ and positive reals u_M, μ such that if $\|u(t)\| \leq u_M \forall t$ then the observer (3.1) gives

$$\|x(t) - \hat{x}(t)\| \leq \mu e^{-\alpha t} \|x(0) - \hat{x}(0)\|. \tag{3.21}$$

Proof. From Theorem 3.1, it is sufficient to show that for any positive α a sufficiently small u_M exists such that the H_∞ Riccati-like inequality (3.2) can be satisfied. This can be done by considering, for a given β , the inequality

$$(A_n - KC_n)P + P(A_n - KC_n)^T + B_n B_n^T + 2\alpha P + (\gamma^2 + \beta^2)P^2 \leq 0, \tag{3.22}$$

which admits solution K, P , as proved in Lemma 3.2.

Since $FF^T \leq I_n \leq \frac{1}{\lambda_{\min}^2(P)}P^2$, as it can be easily verified, one has

$$\beta^2 \lambda_{\min}^2(P) FF^T \leq \beta^2 P^2, \tag{3.23}$$

and thus the solution for (3.2) exists with $u_M \leq \beta^2 \lambda_{\min}^2(P)$.

This completes the proof. □

Remark 3.4. The sufficient conditions for the existence of an exponential observer given in Theorem 3.3 do not include the condition of observability for any input. It is the bound on the input that excludes *bad inputs*, i.e. inputs that make indistinguishable some system states.

Theorem 3.5. Let system (1.1), (1.2) satisfy the following hypotheses:

1. The system is drift-observable and the map $z = \Phi(x)$ and its inverse $x = \Phi^{-1}(z)$ are uniformly Lipschitz in \mathbb{R}^n with constants γ_Φ and $\gamma_{\Phi^{-1}}$, respectively;
2. the observability relative degree in \mathbb{R}^n is $r = n$;
3. the matrix functions $H(\Phi^{-1}(z))$ and $L(\Phi^{-1}(z))$ are uniformly Lipschitz in \mathbb{R}^n , with Lipschitz constants $\gamma_{\bar{H}}$ and $\gamma_{\bar{L}}$ respectively;
4. a constant $u_M > 0$ exists such that $\|u(t)\| \leq u_M \forall t \geq 0$;

Then for any $\alpha > 0$ there exists a gain vector $K \in \mathbb{R}^n$ and a positive real μ such that the observer (3.1) gives

$$\|x(t) - \hat{x}(t)\| \leq \mu e^{-\alpha t} \|x(0) - \hat{x}(0)\|. \tag{3.24}$$

Proof. From Theorem 3.1 it is sufficient to prove that with the given assumptions the H_∞ Riccati-like inequality (3.2) can always be satisfied. This happens because when $r = n$, then $F = B_n$ (see definition (2.22 a), and thus inequality (3.2) can be rewritten

$$(A_n - KC_n)P + P(A_n - KC_n)^T + (1 + u_M^2)B_n B_n^T + 2\alpha P + \gamma^2 P^2 \leq 0. \tag{3.25}$$

Lemma 3.2 ensures existence of solution (K, P) . □

The result of Theorem 3.5 has been proved in [2] using different arguments.

4. THE OBSERVER FOR SYSTEMS WITH SMOOTH/BOUNDED INPUT

Theorem 3.3 states that for systems with any relative degree an exponential observer can be designed if the input is sufficiently small. Moreover, Lemma 3.2 can be used to show that the smaller is the input, the faster can be chosen the exponential rate. If the system has observation relative degree $r = n$ and the input is bounded (not necessarily *small*) then an observer with arbitrary exponential rate can be designed (ineq. (3.16) admits solution for any triple α, β, γ).

In this section it is shown that in the case of relative degree $r < n$, an exponential observer with arbitrary exponential rate can be obtained if the derivatives of the input up to order $n - r$ are known and bounded. Obviously, in many application the derivatives of the input are not known, and this observer can not be constructed. For this reason this observed is called *theoretical*.

An observer that uses estimates of input derivatives is presented after. In this case the observation error is not driven to zero, but its norm can be reduced, with exponential rate, below a prescribed bound (high-gain observer). This kind of observer is called *practical*.

4.1. Theoretical observer

Here it is assumed that the input function is differentiable $n - r$ times, with derivatives uniformly bounded in $[0, +\infty)$. The observer considered has the form

$$\dot{\hat{x}}(t) = f(\hat{x}(t)) + g(\hat{x}(t))u(t) + \bar{Q}^{-1}(\hat{x}(t), U_{n-r})K(y(t) - h(\hat{x}(t))), \tag{4.1}$$

where $\bar{Q}(\hat{x}(t), U_{n-r})$ is defined in (2.13) and is the Jacobian of the map $\bar{\Phi}(x, U_{n-r})$. The observer (4.1) can be constructed as long as U_{n-r} allows invertibility of the Jacobian.

For brevity only global convergence of the observed state to the real one is proved in this section, under uniform observability assumption for smooth/bounded inputs

(remember that Theorem 2.4 shows that drift-observable systems are uniformly observable for smooth/bounded inputs), although local results can be derived under weaker assumptions.

Theorem 4.1. Let system (1.1),(1.2) satisfy the following hypotheses:

1. The vector of input derivatives is bounded by a positive constant \bar{u}_M , i.e. $\|U_{n-r+1}(t)\| \leq \bar{u}_M \quad \forall t \geq 0$ (smooth/bounded input);
2. for $\|U_{n-r}\| \leq \bar{u}_M$ (implied by hypothesis 1) the system is uniformly observable in \mathbb{R}^n and the map $\eta = \bar{\Phi}(x, U_{n-r})$ and its inverse $x = \bar{\Phi}^{-1}(\eta, U_{n-r})$ are globally uniformly Lipschitz w.r.t. x and η , respectively, with Lipschitz constants $\gamma_{\bar{\Phi}}$ and $\gamma_{\bar{\Phi}^{-1}}$ (computed for $\|U_{n-r}\| \leq \bar{u}_M$);
3. the function $m(\eta, U_{n-r+1})$ defined in (2.17) is uniformly Lipschitz w.r.t. η in \mathbb{R}^n , under assumption 1. Let γ_m be its Lipschitz constant for $\|U_{n-r+1}\| \leq \bar{u}_M$.

Then for any $\alpha > 0$ there exist a gain vector $K \in \mathbb{R}^n$ and a positive μ such that the observer (4.1) gives

$$\|x(t) - \hat{x}(t)\| \leq \mu e^{-\alpha t} \|x(0) - \hat{x}(0)\|. \tag{4.2}$$

Proof. From assumption 2 the map $\eta = \bar{\Phi}(x, U_{n-r})$ can be considered a time-varying change of coordinates. In η -coordinates system (1.1),(1.2) and observer (4.1) can be rewritten

$$\begin{aligned} \dot{\eta} &= A_n \eta + B_n m(\eta, U_{n-r+1}), \\ y &= C_n \eta, \\ \dot{\hat{\eta}} &= A_n \hat{\eta} + B_n m(\hat{\eta}, U_{n-r+1}) + K(y - C_n \hat{\eta}). \end{aligned} \tag{4.3}$$

Defining the function

$$v(\eta, \hat{\eta}, U_{n-r+1}) \triangleq m(\eta, U_{n-r+1}) - m(\hat{\eta}, U_{n-r+1}), \tag{4.4}$$

the observation error $e_\eta = \eta - \hat{\eta}$ in η coordinates is described by a linear perturbed system

$$\dot{e}_\eta = (A_n - KC_n)e_\eta + B_n v, \tag{4.5}$$

in which the perturbation, by Assumptions 1 and 3, satisfies the inequality

$$|v| \leq \gamma_m \|e_\eta\|. \tag{4.6}$$

In order to prove that a properly chosen gain matrix K drives $e_\eta(t)$ to zero with an assigned exponential rate α , consider a pair (K, P) that solves the H_∞ Riccati-like inequality

$$(A_n - KC_n)P + P(A_n - KC_n)^T + B_n B_n^T + 2\alpha P + \gamma_m^2 P^2 \leq 0. \tag{4.7}$$

Existence of solution for (4.7) for any α and γ_m is guaranteed by Lemma 3.2. Consider now the following positive definite function of the observation error

$$\nu(e_\eta) = e_\eta^T P^{-1} e_\eta. \tag{4.8}$$

Taking the derivative of ν along the error trajectory, after few passages yields

$$\dot{\nu} \leq e_\eta^T P^{-1} ((A_n - KC_n)P + P(A_n - KC_n)^T + B_n B_n^T + \gamma_m^2 P^2) P^{-1} e_\eta, \tag{4.9}$$

and thus, from (4.7),

$$\dot{\nu} \leq -2\alpha \nu, \Rightarrow \nu(t) \leq e^{-2\alpha t} \nu(0). \tag{4.10}$$

Recalling definition (4.8) of ν

$$\|e_\eta(t)\| \leq \sqrt{cn(P)} e^{-\alpha t} \|e_\eta(0)\|, \tag{4.11}$$

and using Lipschitz conditions in Assumption 2, inequality (4.2) is obtained with $\mu = \sqrt{cn(P)} \gamma_\Phi \gamma_{\Phi-1}$, and the thesis is proved. \square

Remark 4.2. As mentioned before, the observer (4.1) can be implemented only if input derivatives up to order $n - r - 1$ are known. It follows, obviously, that the observer can be always implemented if $r \leq n - 1$, since in this case no input derivative is needed (if $r = n$ the observer (4.1) coincides with observer (3.1)). Moreover, the observer can be implemented in all cases in which the generation model of input u is known (e.g. the input u is generated by a smooth controller or simply by a preprocessing filter).

4.2. Practical observer

With the assumption of existence and boundedness of the first $n - r$ derivatives for the input function in $[0, +\infty)$, the input can be thought as generated by the system

$$\begin{aligned} \dot{U}_{n-r} &= A_{n-r} U_{n-r} + B_{n-r} u^{(n-r)}, \\ u &= C_{n-r} U_{n-r}, \end{aligned} \tag{4.12}$$

where $A_{n-r}, B_{n-r}, C_{n-r}$ is a Brunovsky triple of order $n - r$. The asymptotic reconstruction of the input derivatives can be made using an observer for system (4.12). Let $x_a \in \mathbb{R}^{n-r}$ be an auxiliary state and $x_e = [x^T \ x_a^T]^T$ be an extended state $x_e \in \mathbb{R}^{2n-r}$. Consider now the observation problem applied to the augmented system

$$\begin{aligned} \dot{x}_e &= \bar{f}(x_e) + \bar{g}(x_e)w, \\ y &= \bar{h}(x_e), \\ u &= [0 \ C_{n-r}]x_e. \end{aligned} \tag{4.13}$$

where

$$\bar{f}(x_e) \triangleq \begin{bmatrix} f(x) + g(x)C_{n-r}x_a \\ A_{n-r}x_a \end{bmatrix}, \quad \bar{g}(x_e) \triangleq \begin{bmatrix} 0 \\ B_{n-r} \end{bmatrix}, \quad \bar{h}(x_e) \triangleq h(x). \tag{4.14}$$

The auxiliary variable x_a coincides with the vector of input derivatives U_{n-r} , while the new input w is the $(n - r)$ th input derivative, i. e. $w = u^{(n-r)}$, and is unknown. Thus, the problem into consideration is transformed into a state observation problem with an unknown input w and two known outputs y and u .

If system (1.1), (1.2) has observation relative degree r in a set $\Omega \subseteq \mathbb{R}^n$, from definitions (4.14) it follows that

$$\begin{aligned} L_{\bar{g}}L_{\bar{f}}^k\bar{h}(x_e) &= 0, \quad k = 0, 1, \dots, n - 2 \\ L_{\bar{g}}L_{\bar{f}}^{n-1}\bar{h}(x_e) &= L_gL_f^{r-1}h(x), \end{aligned} \tag{4.15}$$

and therefore $\exists x_e \in \Omega \times \mathbb{R}^{n-r} : L_{\bar{g}}L_{\bar{f}}^{n-1}\bar{h}(x_e) \neq 0$.

This means that system (4.13) has observation relative degree n .

With a little abuse of notation, the map $\bar{\Phi}(\cdot, \cdot)$ defined in (2.11) can be written as

$$\bar{\Phi}(x_e) = \bar{\Phi}(x, x_a) \triangleq \bar{\Phi}(x, U_{n-r})|_{U_{n-r}=x_a}, \tag{4.16}$$

that is also

$$\bar{\Phi}(x_e) = \begin{bmatrix} \bar{h}(x_e) \\ L_{\bar{f}}\bar{h}(x_e) \\ \vdots \\ L_{\bar{f}}^{n-1}\bar{h}(x_e) \end{bmatrix}. \tag{4.17}$$

Defining the square map

$$\begin{bmatrix} z \\ x_a \end{bmatrix} = \Phi_e(x_e) \triangleq \begin{bmatrix} \bar{\Phi}(x, x_a) \\ x_a \end{bmatrix}, \tag{4.18}$$

it can be easily recognized that $z_j = y^{(j-1)}$, $j = 1, \dots, n$ and, as a consequence, system (4.13) in the (z, x_a) -coordinates is written as

$$\begin{aligned} \dot{z} &= A_n z + B_n (\bar{m}(z, x_a) + \bar{n}(z, x_a)w), \\ \dot{x}_a &= A_{n-r} x_a + B_{n-r} w, \\ y &= C_n z, \\ u &= C_{n-r} x_a, \end{aligned} \tag{4.19}$$

where $\bar{m}(z, x_a) = L_{\bar{f}}^r \bar{h}(x_e)$ and $\bar{n}(z, x_a) = L_{\bar{g}}L_{\bar{f}}^{n-1}\bar{h}(x_e)$. Defining the matrices

$$\bar{Q}(x, x_a) \triangleq \frac{\partial \bar{\Phi}}{\partial x}, \quad \bar{Q}_a(x, x_a) \triangleq \frac{\partial \bar{\Phi}}{\partial x_a}, \tag{4.20}$$

the Jacobian of the map Φ_e and its inverse can be written

$$\frac{\partial \Phi_e}{\partial x_e} = \begin{bmatrix} \bar{Q} & \bar{Q}_a \\ 0 & I_{n-r} \end{bmatrix}, \quad \left(\frac{\partial \Phi_e}{\partial x_e} \right)^{-1} = \begin{bmatrix} \bar{Q}^{-1} & -\bar{Q}^{-1}\bar{Q}_a \\ 0 & I_{n-r} \end{bmatrix}. \tag{4.21}$$

The observer proposed for system (4.13), and therefore for (1.1), (1.2), is

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}) + g(\hat{x})C_{n-r}\hat{x}_a + \\ &\quad + \bar{Q}^{-1}(\hat{x}, \hat{x}_a) \left(K_1(y - h(\hat{x})) - \bar{Q}_a(\hat{x}, \hat{x}_a)K_2(u - C_{n-r}\hat{x}_a) \right), \quad (4.22) \\ \dot{\hat{x}}_a &= A_{n-r}\hat{x}_a + K_2(u - C_{n-r}\hat{x}_a). \end{aligned}$$

This is a block triangular system. The second block estimates the vector U_{n-r} of input derivatives. In (z, x_a) -coordinates the observer becomes

$$\begin{aligned} \dot{\hat{z}} &= A_n\hat{z} + B_n(\bar{m}(\hat{z}, \hat{x}_a) + \bar{n}(\hat{z}, \hat{x}_a)w) + K_1(y - C_n\hat{z}), \quad (4.23) \\ \dot{\hat{x}}_a &= A_{n-r}\hat{x}_a + K_2(u - C_{n-r}\hat{x}_a), \end{aligned}$$

Defining the functions

$$\begin{aligned} v_m(z, \hat{z}, x_a, \hat{x}_a) &= \bar{m}(z, x_a) - \bar{m}(\hat{z}, \hat{x}_a), \quad (4.24) \\ v_n(z, \hat{z}, x_a, \hat{x}_a) &= \bar{n}(z, x_a) - \bar{n}(\hat{z}, \hat{x}_a), \end{aligned}$$

the dynamics of the observation errors is described by equations

$$\dot{e}_z = (A_n - K_1C_n)e_z + B_n(v_m + v_nw) \quad (4.25)$$

$$\dot{e}_a = (A_{n-r} - K_2C_{n-r})e_a + B_{n-r}w. \quad (4.26)$$

For the linear system (4.26), the following lemma can be given (the proof is quite simple and is not reported for brevity).

Lemma 4.3. Assume that in (4.26) a bound $w_M > 0$ exists such that $|w(t)| \leq w_M$, $\forall t \geq 0$. For a given positive α_2 let (K_2, P_a) be a solution of the Lyapunov-like inequality

$$(A_{n-r} - K_2C_{n-r})P_a + P_a(A_{n-r} - K_2C_{n-r})^T + B_{n-r}B_{n-r}^T + 2\alpha_2P_a \leq 0, \quad (4.27)$$

(P_a symmetric and positive definite).

Then

$$\|e_a(t)\|^2 \leq \text{cn}(P_a)e^{-2\alpha_2t}\|e_a(0)\|^2 + \frac{\|P_a\|}{2\alpha_2}w_M^2. \quad (4.28)$$

Remark 4.4. Note that for any α_2 inequality (4.27) admits solutions (K_2, P_a) . Moreover, with the choice of eigenvalues $\lambda_i = -\sigma^i$, $i = 1, \dots, n - r$ for matrix $A_{n-r} - K_2(\lambda)C_{n-r}$, the norm $\|P_a\|$ can be made arbitrarily close to 1 (see (2.27) and proof of Lemma 3.2. Therefore, Lemma 4.3 asserts that a gain K_2 can be chosen so that the error e_a decays below a prescribed bound with a prescribed exponential rate α_2 . The bigger the constant α_2 the faster is the convergence and the smaller is the final error bound. Also the approach in [15] can be followed for high-gain estimation for the auxiliary system in (4.19).

In the next theorem the following function definition is needed

$$\varepsilon(t; \alpha_1, \alpha_2) \triangleq \begin{cases} \frac{e^{-2\alpha_1 t} - e^{-2\alpha_2 t}}{-2(\alpha_1 - \alpha_2)}, & \text{if } \alpha_1 \neq \alpha_2, \\ te^{-2\alpha_1 t}, & \text{if } \alpha_1 = \alpha_2. \end{cases} \quad (4.29)$$

Let $\alpha = \min(\alpha_1, \alpha_2)$. It can be proved that $\varepsilon(t; \alpha_1, \alpha_2) \leq 1/(2\alpha e) \quad \forall t \geq 0$.

Let $\bar{U}_{n-r}(\bar{u}_M, w_M)$ be the set of input functions u such that $\|U_{n-r}(t)\| \leq \bar{u}_M$ and $u^{(n-r)}(t) \leq w_M$ for $t \geq 0$.

Theorem 4.5. Let system (1.1), (1.2) satisfy the following hypotheses:

1. The map $z = \bar{\Phi}(x, x_a)$ admits inverse $x = \bar{\Phi}^{-1}(z, x_a)$ for all $x_a \in \mathbb{R}^{n-r}$ (global uniform observability). $\bar{\Phi}$ and $\bar{\Phi}^{-1}$ are Lipschitz w.r.t. both arguments, with Lipschitz constants $\gamma_{\bar{\Phi}}$ and $\gamma_{\bar{\Phi}^{-1}}$, respectively;
2. the functions $\bar{m}(z, x_a)$ and $\bar{n}(z, x_a)$ are Lipschitz w.r.t. both arguments; let $\gamma_{\bar{m}}$ and $\gamma_{\bar{n}}$ be the Lipschitz constants;
3. $u \in \bar{U}_{n-r}(\bar{u}_M, w_M)$;

Then there exist gain vectors K_1 and K_2 for the observer (4.22) such that for $t \geq 0$

$$\|e_x(t)\| \leq c_1 e^{-\alpha t} \|e_x(0)\| + (c_1 e^{-\alpha t} + c_2 \sqrt{\varepsilon(t; \alpha_1, \alpha_2)}) \|e_a(0)\| + c_3. \quad (4.30)$$

Moreover, K_1 and K_2 can be chosen so to make constants c_2 and c_3 arbitrarily small.

Proof. From Lemma 4.3 for any $\alpha_2 > 0$ a gain K_2 exists that ensures observation error decay for system (4.26) according to the law (4.28).

The state observation error dynamics in z -coordinates (4.25) consists of a linear system with nonlinear perturbations $v_{\bar{m}}$ and $v_{\bar{n}}$.

Assumption (2) states that

$$|v_{\bar{m}}| \leq \gamma_{\bar{m}} \left\| \begin{bmatrix} e_z \\ e_a \end{bmatrix} \right\|, \quad |v_{\bar{n}}| \leq \gamma_{\bar{n}} \left\| \begin{bmatrix} e_z \\ e_a \end{bmatrix} \right\|, \quad (4.31)$$

and therefore

$$\begin{aligned} v_{\bar{m}}^2 &\leq \gamma_{\bar{m}}^2 (e_z^T e_z + e_a^T e_a), \\ v_{\bar{n}}^2 &\leq \gamma_{\bar{n}}^2 (e_z^T e_z + e_a^T e_a). \end{aligned} \quad (4.32)$$

Now, given positive constants α_1 and β , consider a solution pair (K_1, P) (P symmetric and positive definite) of the H_∞ Riccati-like inequality

$$(A_n - K_1 C_n) P + P (A_n - K_1 C_n)^T + 2\beta^2 B_n B_n^T + 2\alpha_1 P + \frac{\gamma^2}{\beta^2} P^2 \leq 0, \quad (4.33)$$

where $\gamma^2 = \gamma_{\bar{m}}^2 + \gamma_{\bar{n}}^2 w_M^2$ (solution is ensured by Lemma 3.2).

Consider also the positive definite function of the error e_z

$$v = e_z^T P^{-1} e_z. \quad (4.34)$$

It is not difficult to derive the following inequality

$$\dot{\nu}(t) \leq -2\alpha_1\nu(t) + \frac{\gamma^2}{\beta^2}\|e_a(t)\|^2. \tag{4.35}$$

Substitution of (4.28) in (4.35), after few computations based on Gronwall inequality, gives

$$\begin{aligned} \nu(t) &\leq e^{-2\alpha_1 t}\nu(0) + \mu_1\|e_a(0)\|^2\varepsilon(t; \alpha_1, \alpha_2) + \frac{1-e^{-2\alpha_1 t}}{2\alpha_1}\mu_2 \\ \text{where } \mu_1 &= \text{cn}(P_a)\frac{\gamma^2}{\beta^2}, \quad \mu_2 = \frac{\|P_a\|}{2\alpha_2}\frac{\gamma^2}{\beta^2}w_M^2 \end{aligned} \tag{4.36}$$

and from definition (4.34)

$$\|e_z(t)\|^2 \leq \text{cn}(P)e^{-2\alpha_1 t}\|e_z(0)\|^2 + \|P\|\cdot\|e_a(0)\|^2\mu_1\varepsilon(t; \alpha_1, \alpha_2) + \|P\|\frac{1-e^{-2\alpha_1 t}}{2\alpha_1}\mu_2. \tag{4.37}$$

Using the following inequalities, implied by assumption (1),

$$\begin{aligned} \|e_z\|^2 &\leq \gamma_{\bar{\Phi}}^2(\|e_x\|^2 + \|e_a\|^2), \\ \|e_x\|^2 &\leq \gamma_{\bar{\Phi}^{-1}}^2(\|e_z\|^2 + \|e_a\|^2), \end{aligned} \tag{4.38}$$

easy computations show that the observation error in original coordinates satisfies inequality

$$\begin{aligned} \|e_x(t)\|^2 &\leq \gamma_{\bar{\Phi}}^2\gamma_{\bar{\Phi}^{-1}}^2\text{cn}(P)e^{-2\alpha_1 t}(\|e_x(0)\|^2 + \|e_a(0)\|^2) \\ &\quad + \gamma_{\bar{\Phi}^{-1}}^2\|P\|\left(\mu_1\|e_a(0)\|^2\varepsilon(t; \alpha_1, \alpha_2) + \frac{1-e^{-2\alpha_1 t}}{2\alpha_1}\mu_2\right). \end{aligned} \tag{4.39}$$

This inequality easily implies (4.30), with

$$\begin{aligned} c_1 &= \gamma_{\bar{\Phi}}\gamma_{\bar{\Phi}^{-1}}\sqrt{\text{cn}(P)} \\ c_2 &= \gamma_{\bar{\Phi}^{-1}}\sqrt{\|P\|\text{cn}(P_a)\frac{\gamma}{\beta}} \\ c_3 &= \gamma_{\bar{\Phi}^{-1}}\frac{\sqrt{\|P\|\|P_a\|w_M}}{2\beta\sqrt{\alpha_1\alpha_2}}. \end{aligned} \tag{4.40}$$

The proof is completed observing that α_2 can be chosen arbitrarily large while keeping $\|P_a\|$ arbitrarily close to 1, while α_1 and β can be made arbitrarily large while keeping $\|P\|$ arbitrarily close to 1. As a consequence constants c_2 and c_3 can be made arbitrarily small. This concludes the proof. \square

Remark 4.6. Inequality (4.30) can be expressed by stating that the observation error exponentially tends to be bounded by c_3 , with prescribed exponential rate. Therefore, the observer (4.22) is a high-gain observer. The main difference with other approaches, as in [15], is that the estimated state is computed in the original coordinate system, without explicit computation of the inverse map $x = \bar{\Phi}^{-1}(z, U_{n-r})$.

Remark 4.7. Assumption of global uniform observability made in Theorem 4.5 can be relaxed to uniform observability for smooth/bounded inputs in the case the error on the estimate of the input derivatives can be kept small (for instance consider the case in which $\|e_a(0)\|$ is zero or small in (4.28)).

5. NUMERICAL RESULTS

In this section the theoretical and practical observers are numerically tested on the third order nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + (x_3^2 - x_1) u, \\ \dot{x}_2 &= -x_3(1 + x_1^2), \\ \dot{x}_3 &= \sin x_3, \\ y &= x_1. \end{aligned} \tag{5.1}$$

The *observation relative degree* for this system is 1 in \mathbb{R}^3 (note that system (5.1) loses relative degree on the manifold $x_3^2 - x_1 = 0$). The maps $\Phi(x)$ and $\Psi(x, u, \dot{u})$ defined in the paper are

$$\Phi(x) = [x_1 \quad x_2 \quad -x_3(1 + x_1^2)]^T, \tag{5.2}$$

and

$$\Psi(x, u, \dot{u}) = \begin{bmatrix} 0 \\ (x_3^2 - x_1) u \\ (2x_3 \sin x_2 - x_2 - (x_3^2 - x_1) u) u + (x_3^2 - x_1) \dot{u} \end{bmatrix}. \tag{5.3}$$

It is easy to see that the map $z = \Phi(x)$ is globally invertible, and therefore the system is globally drift-observable. This means that if the input u and its derivative \dot{u} are sufficiently small, also the map $z = \Phi(x) + \Psi(x, u, \dot{u})$ is invertible (uniform observability for smooth/bounded inputs). The initial true and observed states considered in the simulations presented are

$$x(0) = [0.5 \ 0.5 \ 0.5]^T, \quad \hat{x}(0) = [0.5 \ 1 \ 1]^T. \tag{5.4}$$

and the input applied is

$$u(t) = u_0 + 2 \sin(2\pi t), \quad \text{with } u_0 = 5, \tag{5.5}$$

(with smaller u_0 all three observers worked well). With the chosen input ($u_0 = 5$) and initial state the observer (3.1) does not work at all. Therefore only simulation of the theoretical observer (4.1) and of the practical observer (4.22) are reported. For the observer (4.22) the auxiliary system is simply

$$\begin{aligned} \dot{x}_{a,1} &= x_{a,2}, \\ \dot{x}_{a,2} &= w. \end{aligned} \tag{5.6}$$

The initial observed state considered for the auxiliary system is $\hat{x}_a(0) = [5 \ 0]^T$. The gain K in (4.1) and K_1 in (4.22) are taken equal, and such to assign to matrix

$A_3 - KC_3$ eigenvalues $\lambda = -4 \cdot [1 \ 1.2 \ 1.2^2]$. In (4.22) the gain K_2 assigns to matrix $A_2 - K_2C_2$ eigenvalues $\lambda = -40 \cdot [1 \ 1.2]$. The plots of the numerical simulations are reported in Figures 1-3. As expected, the behavior of the theoretical observer is somewhat better than the behavior of the practical observer.

6. CONCLUSIONS

This work considers the problem of state observation with exponential error rate for smooth nonlinear systems that do not meet conditions of uniform observability. It is shown that drift-observability, together with a smoothness/boundedness condition on the input, is sufficient to ensure existence of an exponential observer. Three types of observers are presented, that can be constructed under drift-observability assumption only. The first observer presented is suitable for systems with maximal relative degree or for general nonlinear systems driven by sufficiently small input. The second type of observer requires the input derivatives up to a certain order, and gives exponential error decay in the case of input sufficiently smooth. The third observer presented, applicable in the case of smooth input, does not require input derivatives, and ensures exponential decay of the observation error below a prescribed level. Computer simulations show good behavior of the last two observers in a situation in which the first observer does not work.

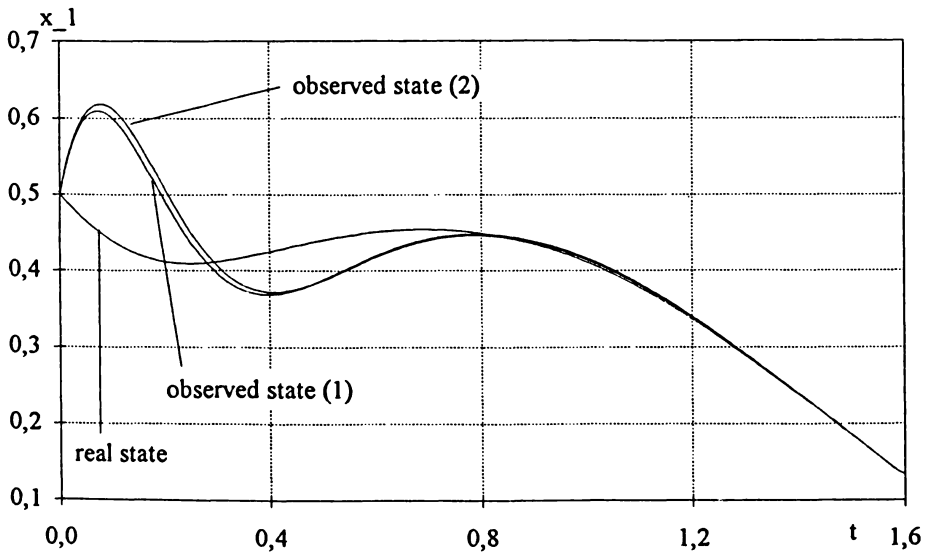


Fig. 1. Real and observed state variable x_1 .

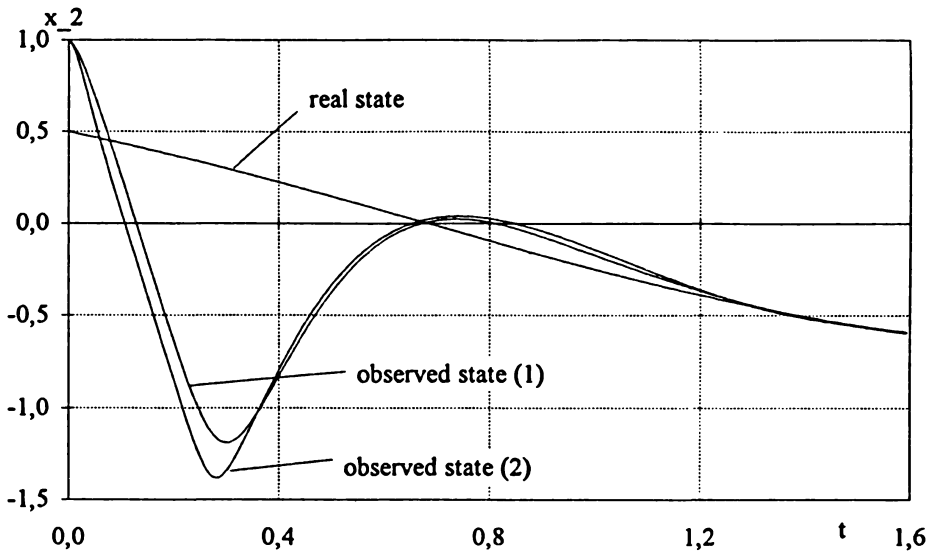


Fig. 2. Real and observed state variable x_2 .

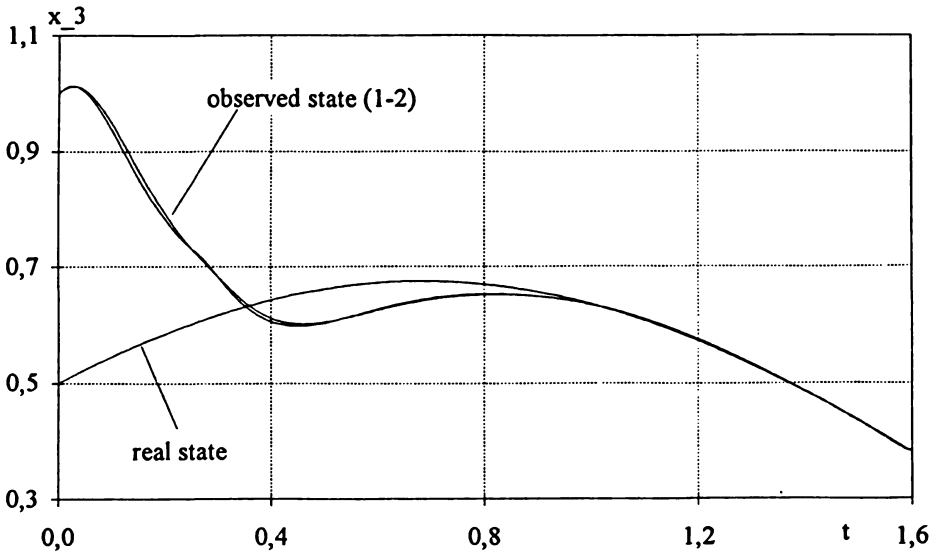


Fig. 3. Real and observed state variable x_3 .

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