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# PROBABILISTIC PROPOSITIONAL CALCULUS WITH DOUBLED NONSTANDARD SEMANTICS[1]

IVAN KRAMOSIL

The classical propositional language is evaluated in such a way that truthvalues are subsets of the set of all positive integers. Such an evaluation is projected in two different ways into the unit interval of real numbers so that two real-valued evaluations are obtained. The set of tautologies is proved to be identical, in all the three cases, with the set of classical propositional tautologies, but the induced evaluations meet some natural properties of probability measures with respect to nonstandard supremum and infimum operations induced in the unit interval of real numbers.

## 1. INTRODUCTION

Even if there are the probability theory, and the mathematical statistics based on this theory, which have been playing, since the 18th century, the role of the dominant mathematical tool for uncertainty quantification and processing, the attempts to build alternative mathematical apparata for these sakes, paradigmatically more close to formalized logical deductive calculi, are also numerous, important and interesting. The resulting mathematical models are usually subsumed under the common general notion "non-classical logics" and can be divided, roughly speaking, into three groups.

(i) *Modal logics* follow the pattern which emphasize rather the qualitative than the quantitative aspects of the notions like possibility or necessity. This goal is reached by enriching the language of an appropriate logical calculus by new symbols for the functors like "it is possible that..." or "it is necessary that...", and by choosing a collection of axioms for the original as well as for the new, modal functors. Such a choice leads, as a rule, to a compromise between the intuitions and the common language feelings and conotations behind the modal functors, and the methodological (meta-logical) demands which must be obeyed when creating a deductive formalized system. The qualitative character of modal logics is also demonstrated by the fact that when defining semantical models of these logics based on the space of possible worlds (Kripke semantics), what matters is the fact whether the subsets of possible

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worlds, corresponding to some formulas, are empty or finite, or whether their complements possess these qualitative properties. However, if these subsets are beyond the scope of these extremal cases, their relative or absolute sizes (extends), measured by some quantitative numerical measure, do not play any important role.

(ii) *Fuzzy logics* are oriented toward quantification and processing of the notions like vagueness, impreciseness or fuzziness. They are based on the idea that formulas of the formalized language in question may be interpreted as taking not only the two qualitative values "true" and "false", but also some values "between these two ones". From the formal point of view this goal is reached in such a way that the classical qualitative truthvalues are identified with the extremal points 1 (true) and 0 (false) of the unit interval of real numbers, and the formulas are supposed to be able to take also truthvalues identified with (some or all) real numbers from the inside of the unit interval, i. e., from $(0,1)$. There are numerous variants of such formalized calculi based on different systems of functors and quantifies and on different ways of interpretations of these functors and quantifiers as functions from the truthvalues of the composing more elementary formula(s) to the truthvalue of the resulting composed formula. In every case, two aspects are emphasized by fuzzy logics: (1) the extensional character of all functors and quantifiers, i. e., as just mentioned, truthvalues of composed formulas are functions of truthvalues of their components, and (2) the notions of vagueness, impreciseness or fuzziness, to the description and processing of which fuzzy logics are applied, are supposed to be qualitatively different from the notions of uncertainty and randomness described and processed by probabilistic and statistical tools, and they are also supposed to be of extensional character or at least to be allowed to be processed by formal tools preserving the extensional character of functors.

(iii) *Probabilistic logics* copy fuzzy logics as far as the truthvalues ranging over the unit interval of real numbers are concerned. However, probabilistic logics insist on the possibility to understand these values as probabilities, even if this demand implies the non-extensionality of the used functions (contrary to fuzzy logics when the possibility to interprete truthvalues as probabilities is abandoned in every case when it conflicts the demand of extensionality of all functions and quantifiers). Hence, probabilistic logics can be seen as alternative apparata, if related to probability theory and mathematical statistics, for uncertainty and randomness quantification and processing based rather on the paradigmatical and methodological grounds of deductive formalized systems than on the grounds of measure theory, real functions and integral calculus, as it is the case of probability theory and mathematical statistics.

In what follows, we shall try to propose and develop probabilistic propositional calculus with a boolean-valued semantic. This semantic will induce two real-valued semantics, one of them being extensional, the other one being intensional (i. e. non-extensional), but conserving and copying the flexibility of probabilistic measures when various kinds and degrees of stochastical (statistical) dependence among propositions taken as random events are considered.

## 2. CLASSICAL SYNTAX AND BOOLEAN-VALUED SEMANTICS FOR PROPOSITIONAL CALCULUS

Let us briefly recall the syntax of the classical propositional calculus. There are numerous alternative and equivalent presentations of this syntax, just for the sake of unambiguity let us choose the well-known formalization introduced by Church in [1].

There is an infinite sequence $p_1, p_2, \ldots$ of *propositional variables*, by convention we shall write also $q$, $r$, $s$, $q_1$, $r_1$, $s_1$, $q_2, \ldots$ instead of $p_2, p_3, \ldots$ The set of all propositional variables will be denoted by *Var*. There is one unary functor $\neg$ called *negation* and one binary functor $\rightarrow$ called *implication*. The only auxiliary symbols are brackets ( and ), by convenience also other types of brackets will be used.

Each propositional variable is a *well-formed formula* (w. f. f.). If $A$ and $B$ are w. f. f.'s, then $(\neg A)$ and $(A \rightarrow B)$ are also w. f. f.'s. By convention, the outermost pair of brackets can be omitted. In order to abbreviate our notation we shall write $A \vee B$ instead of $(\neg A) \rightarrow B$, $A \wedge B$ instead of $\neg((\neg A) \vee (\neg B))$ and $A \equiv B$ instead of $(A \rightarrow B) \wedge (B \rightarrow A)$. Let $\mathcal{L}$ denote the set of all w. f. f.'s.

The three following w. f. f.'s, namely

$$\begin{aligned} &p \rightarrow (q \rightarrow p), \\ &(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)), \\ &(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow q) \end{aligned} \tag{1}$$

are called *axioms*. All axioms are also *theorems*. Moreover, if w. f. f.'s $A \rightarrow B$ and $A$ are theorems, then $B$ is also theorem (the so called *modus ponens* deduction rule). If a w. f. f. $A$ is theorem and $B$ is a w. f. f., then the w. f. f. $S^p_B\, A$, resulting when all occurrences of the propositional variable $p$ in $A$ are replaced by the w. f. f. $B$, is also a theorem. The set of all theorems will be denoted by *Ded*.

A sequence $A_1, A_2, \ldots, A_n$ of w. f. f.'s is called *proof*, if for each $i \leq n$ either $A_i$ is an axiom, or there is $j < i$, $p \in Var$ and w. f. f. $B$ such that $A_i$ is $S^p_B\, A_j$, or there are indices $j$, $k < i$ such that $A_j$ is $A_k \rightarrow A_i$. Evidently, a w. f. f. $A$ is a theorem (or: deducible formula, or: provable formula), iff there is a proof such that $A$ is the last formula in this proof.

The classical semantics for the propositional calculus (for the language $\mathcal{L}$) is defined as follows. A (*classical*) *evaluation* of $\mathcal{L}$ is a mapping $e^* : Var \rightarrow \{0, 1\}$ (with the interpretation $0$ = false, $1$ = true) uniquely extended to $e : \mathcal{L} \rightarrow \{0, 1\}$, setting $e(p) = e^*(p)$ for $p \in Var$, $e(\neg A) = 1 - e(A)$, and $e(A \rightarrow B) = \max\{1 - e(A), e(B)\}$, i.e. $e(\neg A) = 1$ iff $e(A) = 0$, and $e(A \rightarrow B) = 0$ iff $e(A) = 1$ and $e(B) = 0$. A w. f. f. $A \in \mathcal{L}$ is called a (*classical*) *tautology*, if $e(A) = 1$ for each $e^* : Var \rightarrow \{0, 1\}$, obviously, $e^*$ is the restriction of $e$ to *Var*, in symbols $e^* = e \restriction Var$. The set of all (classical) tautologies will be denoted by $Taut_{\mathrm{cl}}$. The well-known elementary completeness theorem for propositional calculus reads that $Taut_{\mathrm{cl}} = Ded$, cf. [1] or any elementary textbook on mathematical logic.

A Boolean-valued semantics for $\mathcal{L}$ can be introduced in this way. Let $\mathcal{N} = \{0, 1, 2, \ldots\}$ be the set of all non-negative integers, let $\mathcal{N}^+ = \mathcal{N} - \{0\} = \{1, 2, \ldots\}$, let $\mathcal{P}_0 = \mathcal{P}(\mathcal{N}^+)$ be the power-set of all sets of positive integers. A *Boolean-valued*, or a $\mathcal{P}_0$-*valued*, to be more correct, *evaluation* of $\mathcal{L}$ is a mapping $e^* = Var \to \mathcal{P}_0$ uniquely extended to $e : \mathcal{L} \to \mathcal{P}_0$, setting $e(p) = e^*(p)$ for $p \in Var$, $e(\neg A) = \mathcal{N}^+ - e(A)$, and $e(A \to B) = (\mathcal{N}^+ - e(A)) \cup e(B)$. A w.f.f. $A \in \mathcal{L}$ is called *Boolean tautology* or $\mathcal{P}_0$-*tautology*, if $e(A) = \mathcal{N}^+$ for each $e^* : Var \to \mathcal{P}_0$. The set of all $\mathcal{P}_0$-tautologies will be denoted by $Taut_b$. As in the classical case, obviously $e^* = e \restriction Var$.

## 3. COMPLETENESS THEOREM FOR PROPOSITIONAL CALCULUS WITH BOOLEAN-VALUED SEMANTICS

**Lemma 1.** For each propositional w.f.f. $A$, $A \in Ded$ iff $e(A) = \mathcal{N}^+$ for all $\mathcal{P}_0$-valued evaluations $e : \mathcal{L} \to \mathcal{P}_0$ such that $e^*(p) \in \{\emptyset, \mathcal{N}^+\}$ holds for each $p \in Var$, here $\emptyset$ denotes the empty subset of $\mathcal{N}^+$.

Proof. Let $e^* : Var \to \{0,1\}$ be a classical evaluation, let $F(e^*) : Var \to \{\emptyset, \mathcal{N}^+\}$ be defined in such a way that $F(e^*)(p) = \mathcal{N}^+$, if $e^*(p) = 1$, and $F(e^*)(p) = \emptyset$, if $e^*(p) = 0$. An easy verification of the evaluation rules for $\neg$ and $\to$ yields that for each $A \in \mathcal{L}$, $e(A) = 1$ iff $F(e)(A) = \mathcal{N}^+$, here $F(e)$ extends $F(e^*)$ from $Var$ to $\mathcal{L}$ and $F$ is obviously a one-to-one mapping between $\{0,1\}^{Var}$ and $\{\emptyset, \mathcal{N}^+\}^{Var}$. Hence, $e(A) = \mathcal{N}^+$ for all $e^* : Var \to \{\emptyset, \mathcal{N}^+\}$ iff $e(A) = 1$ for all $e^* : Var \to \{0,1\}$, so that $e(A) = \mathcal{N}^+$ for all $e^* : Var \to \{\emptyset, \mathcal{N}^+\}$ iff $A \in Taut_{cl}$. But $Taut_{cl} = Ded$, so that the proof is completed. □

**Lemma 2.** If $A$ is an axiom of the classical propositional calculus, i.e., one of the formulas in (1), then for each $\mathcal{P}_0$-valued evaluation $e^* : Var \to \mathcal{P}_0$, $e(A) = \mathcal{N}^+$. In other notation, if $A$ is an axiom of the classical propositional calculus, then $A \in Taut_b$.

Proof. Given $e^* : Var \to \mathcal{P}_0$, let $P = e^*(p)$, $Q = e^*(q)$, $Q = e^*(r)$, $P^c = \mathcal{N}^+ - P = e(\neg p)$, $P \to Q = P^c \cup Q = (\mathcal{N}^+ - e^*(p)) \cup e^*(q) = e(p \to q)$, etc., be abbreviated notations for the corresponding subsets of $\mathcal{N}^+$. Then

$$e(p \to (q \to p)) = P \to (Q \to P) = P^c \cup Q^c \cup P = \mathcal{N}^+, \tag{2}$$

$$\begin{aligned} & e((p \to (q \to r))) \to ((p \to q)) \to (p \to r))) \qquad (3)\\ = \; & (P \to (Q \to R)) \to ((P \to Q) \to (P \to R)) \\ = \; & (P^c \cup Q^c \cup R) \to ((P^c \cup Q) \to (P^c \cup R)) \\ = \; & (P^c \cup Q^c \cup R)^c \cup ((P^c \cup Q)^c \cup (P^c \cup R)) \\ = \; & (P \cap Q \cap R^c) \cup ((P \cap Q^c) \cup P^c \cup R) \\ = \; & ((P \cap Q \cap R^c)) \cup (P \cap Q^c) \cup P^c \cup R \\ \supset \; & (P \cap R^c) \cup P^c \cup R = \mathcal{N}^+, \end{aligned}$$

$$\begin{aligned} & e((\neg q \to \neg p) \to (p \to q)) = (Q^c \to P^c) \to (P \to Q) \qquad (4)\\ = \; & (Q \cup P^c) \to (P^c \cup Q) = (Q \cup P^c)^c \cup P^c \cup Q = (Q^c \cap P) \cup P^c \cup Q = \mathcal{N}^+. \end{aligned}$$

The assertion is proved. □

**Lemma 3.** Let $A \in Taut_b$, let $A \rightarrow B \in Taut_b$. Then $B \in Taut_b$. Let $A \in Taut_b$, let $p \in Var$, let $B \in \mathcal{L}$. Then $S_b^p A \in Taut_b$. On other words, the set $Taut_b \subset \mathcal{L}$ is closed with respect to both the deduction rules of the classical propositional calculus (modus ponens and substitution).

Proof. Let $A \in Taut_b$, let $A \rightarrow B \in Taut_b$, let $e^* : Var \rightarrow \mathcal{P}_0$. Then $e(A) = \mathcal{N}^+$, $e(A \rightarrow B) = (\mathcal{N}^+ - e(A)) \cup e(B) = \mathcal{N}^+$, hence, $e(B) = \mathcal{N}^+$. Let $A, B \in \mathcal{L}$, let $p \in Var$, let $e : Var \rightarrow \mathcal{P}_0$. Let $e_1^* : Var \rightarrow \mathcal{P}_0$ be defined in such a way that $e_1^*(p) = e(B)$, $e_1^*(q) = e^*(q)$ for all $q \in Var$, $q \neq p$. Due to the extensionality of the mapping $e$, $e(S_B^p A)$ depends on $B$ only through $e(B)$, namely, $e(S_B^p A) = e_1(A)$. If $A \in Taut_b$, then $e_1(A) = \mathcal{N}^+$, hence, $e(S_B^p A) = \mathcal{N}^+$ as well, so that $S_B^p A \in Taut_b$. The assertion is proved. □

**Theorem 1.** (Completeness theorem for propositional calculus with $\mathcal{P}_0$-valued semantics)

$$Ded = Taut_b. \tag{5}$$

Verbally, each w.f.f. $A$ of the classical propositional calculus is deducible (is a theorem) iff $e(A) = \mathcal{N}^+$ for all $e^* : Var \rightarrow \mathcal{P}_0 = \mathcal{P}(\mathcal{N}^+)$.

Proof. Lemmas 2 and 3 immediately yield that $Ded \subset Taut_b$. If $A \in Taut_b$, then $e(A) = \mathcal{N}^+$ for each $e^* : Var \rightarrow \mathcal{P}_0$, in particular, $e(A) = \mathcal{N}^+$ for each $e^* : Var \rightarrow \{\emptyset, \mathcal{N}^+\}$. So, $A \in Taut_{\mathrm{cl}}$ by Lemma 1, but $Taut_{\mathrm{cl}} = Ded$ by the completeness theorem for the propositional calculus with the classical semantics. Hence, $Taut_b = Ded$. □

We must admit that all the statements presented and proved till now are rather trivial consequences of the fact that set-theoretic operations of complement, union and intersection are defined in such a way that the Boolean algebra of all subsets of a basic space copies (or: translates) the Boolean algebra of propositions defined by the classical propositional functors. In the next chapter we shall take profit of the fact that in the case of $\mathcal{P}_0$ (the Boolean algebra of all subsets of the set $\mathcal{N}^+$ of positive integers), the elements of $\mathcal{P}_0$ can be uniquely encoded by real numbers from the (Cantor subset of the) unit interval $\langle 0, 1\rangle$.

## 4. TWO NONSTANDARD NUMERICAL SEMANTICS FOR PROPOSITIONAL CALCULUS INDUCED BY BOOLEAN-VALUED SEMANTICS

Let $\mathcal{B} = \{0,1\}^\infty$ be the set of all infinite binary sequences, let $\mathcal{C}$ be the well-known Cantor subset of the unit interval of real numbers; let us recall that $\mathcal{C}$ is the set of all numbers $x \in \langle 0, 1\rangle$ for which there exists a ternary decomposition (decomposition to the base 3), not containing any occurrence of the numeral 1. Let $\chi : \mathcal{P}_0 = \mathcal{P}(\mathcal{N}^+) \rightarrow \mathcal{B}$ be defined in such a way that $\chi(A) = \langle \chi(A)(1), \chi(A)(2), \ldots \rangle$ and $\chi(A)(i) = 1$ iff $i \in A$. Hence, $\chi(A)$ is the characteristic function or identifier of the set $A$ of non-negative integers. Let $\varphi : \mathcal{P}_0 \to \mathcal{C}$ be defined in such a way that

$$\varphi(A) = \sum_{i=1}^{\infty} 2\chi(A)(i)\, 3^{-i}, \tag{6}$$

we shall also take $\varphi$ as $\varphi : \mathcal{B} \to \mathcal{C}$ and write $\varphi(\chi(A))$ instead of $\varphi(A)$. Both the mappings $\chi$ and $\varphi$ are obviously one-to-one mappings between $\mathcal{P}_0$ and $\mathcal{B}$ ($\mathcal{P}_0$ and $\mathcal{C}$, resp.). Set also

$$w(A) = \lim_{n\to\infty} (1/n) \sum_{i=1}^{n} \chi(A)(i), \tag{7}$$

if this limit value is defined, $w(A)$ being undefined otherwise.

As $\varphi : \mathcal{P}_0 \to \mathcal{C}$ is one-to-one, the inverse mapping $\varphi^{-1} : \mathcal{C} \to \mathcal{P}_0$ is uniquely defined, so that the following three operations over the Cantor set $\mathcal{C}$ are defined for each $x$, $y \in \mathcal{C}$:

$$\begin{aligned} 1\dot{-}x &= \varphi(\mathcal{N}^+ - \varphi^{-1}(x)), \\ x \vee y &= \varphi(\varphi^{-1}(x) \cup \varphi^{-1}(y)) \\ x \wedge y &= \varphi(\varphi^{-1}(x) \cap \varphi^{-1}(y)). \end{aligned} \tag{8}$$

An easy calculation yields that $1\dot{-}x$ agrees with $1 - x$ for the usual subtraction but, in general, $x \vee y \neq \max\{x, y\}$ and $x \wedge y \neq \min\{x, y\}$ for the usual operations max and min in $\langle 0, 1\rangle$. In more detail, if $x$, $y \in \mathcal{C}$ are such that their corresponding (and obviously uniquely defined) ternary decompositions not containing 1 are $\langle x_1, x_2, \dots\rangle \in \{0,2\}^\infty$ and $\langle y_1, y_2, \dots\rangle \in \{0,2\}^\infty$, then $x \vee y$ is defined by the ternary decomposition $\langle z_1, z_2, \dots\rangle$ such that $z_i = \max\{x_i, y_i\}$ for each $i \in \mathcal{N}^+$, consequently, $x \vee y = \sum_{i=1}^{\infty}(\max\{x_i, y_i\})\, 3^{-i}$, similarly $x \wedge y = \sum_{i=1}^{\infty}(\min\{x_i, y_i\})\, 3^{-i}$. E.g., if $x = 1/3 = 0,0222\dots$ and $y = 2/3 = 0,2000\dots$, then $x \vee y = 1$ and $x \wedge y = 0$ (their alternative decompositions $x = 0,1000\dots$ and $y = 0,1222\dots$ do not meet the condition not to contain any occurrence of 1).

*Nonstandard c-valued evaluations* of w. f. f.'s of the propositional calculus can be defined in two ways which are evidently equivalent. Let $e^* : Var \to \mathcal{P}$ be given. Then the mapping $e_c : \mathcal{L} \to \mathcal{C}$ defined by $e_c(A) = \varphi(e(A))$ for each w.f.f. $A \in \mathcal{L}$ is called (nonstandard) $c$-valued evaluation of $\mathcal{L}$. Equivalently, let $e_c^* : Var \to \mathcal{C}$ be a mapping ascribing to each variable $p$ a real number $e_c^*(p)$ from the Cantor set, let $e_c : \mathcal{L} \to \mathcal{C}$ be uniquely defined in such a way that $e_c(p) = e_c^*(p)$ for each $p \in Var$, $e_c(\neg A) = 1\dot{-}e_c(A)$, and $e_c(A \to B) = (1\dot{-}e_c(A)) \vee e_c(B)$ for each $A$, $B \in \mathcal{L}$. Then $e_c$ is a (nonstandard) $c$-valued evaluation of $\mathcal{L}$. Obviously, $e_c^* = e_c \restriction Var$.

**Theorem 2.** (Completeness theorem for propositional calculus with nonstandard $c$-valued semantics)

Let $Taut_c$ be the set of all *w. f. f.*'s $A$ from $\mathcal{L}$ such that $e_c(A) = 1$ for each (nonstandard) $c$-valued evaluation $e_c^* : Var \to \mathcal{C}$. Then $Taut_c = Ded$.

P r o o f. Due to Theorem 1, the only we have to prove is that $Taut_c = Taut_b$. Clearly, $A \in Taut_b$ iff $e(A) = \mathcal{N}^+$ for each $e^* : Var \to \mathcal{P}_0$, hence, iff $\varphi(e(A)) = 1$

for each such $e^*$, as $\varphi^{-1}(1) = \mathcal{N}^+$. As there is an obvious $1-1$ mapping between $P_0^{\mathit{Var}}$ and $\mathcal{C}^{\mathit{Var}}$, we can conclude that $A \in \mathit{Taut}_b$ iff $\varphi(e(A)) = e_c(A) = 1$ for each $e_c^* : \mathit{Var} \to \mathcal{C}$, i.e., iff $A \subset \mathit{Taut}_c$. The assertion is proved. □

Theorem 2 is nothing else than a rather trivial consequence of the fact that the projection $\varphi$ of the power-set $\mathcal{P}_0$ of all subsets of $\mathcal{N}^+$ into the unit interval of real numbers has been defined rather with the aim to encode unambiguously subsets of $\mathcal{N}^+$ by reals than to quantify somehow their respective sizes. We have paid for such an encoding projection by the fact that the binary relation $\leq_*$ on $\mathcal{C}$, defined for each $x, y \in C$ by $x \leq_* y$ iff $x \vee y = y$ (or, what can be proved to be the same, iff $x \wedge y = x$). i.e., the relation with respect to which $\vee$ and $\wedge$ fulfill the properties of supremum and infimum operations, is just a *partial* ordering on $\mathcal{C}$, copying the partial ordering of $\mathcal{P}_0$ by the relation of set-theoretic inclusion. As can be easily proved, $x \leq_* y$ holds iff $x_i \leq y_i$ holds for each $i \in \mathcal{N}^+$, where $\langle x_1, x_2, \ldots \rangle$ and $\langle y_1, y_2, \ldots \rangle$ are the corresponding ternary decompositions from $\{0,1\}^\infty$. It follows immediately, that for each $x, y \in C$, $x \leq_* y$ implies that $x \leq y$ for the usual (linear) ordering $\leq$ in $\langle 0,1 \rangle$ but not, in general, vice versa, e.g., neither $1/3 \leq_* 2/3$ nor $2/3 \leq_* 1/3$ hold. Nevertheless, the following statement can be proved.

**Lemma 4.** (i) For each w.f.f.'s $A, B \in \mathcal{L}$, and for each $\mathcal{P}_0$-valued evaluation $e^* : \mathit{Var} \to \mathcal{P}_0$, $e(A \to B) = \mathcal{N}^+$ iff $e(A) \subset e(B)$,

(ii) For each w.f.f.'s $A, B \in \mathcal{L}$, and for each $c$-valued evaluation $e_c^* : \mathit{Var} \to \mathcal{C}$, $e_c(A \to B) = 1$ iff $e_c(A) \leq_* e_c(B)$. Hence, if $e_c(A \to B) = 1$, then $e_c(A) \leq e_c(B)$.

Proof. Let $A, B \in \mathcal{L}$, let $e^* : \mathit{Var} \to \mathcal{P}_0$. then $e(A \to B) = (\mathcal{N}^+ - e(A)) \cup e(B) = \mathcal{N}^+$ yields that $e(A) \subset e(B)$, at the same time, $e(A) \subset e(B)$ yields that $e(A \to B) = \mathcal{N}^+$. Let $e_c^* : \mathit{Var} \to \mathcal{C}$, let $e : \mathit{Var} \to \mathcal{P}_0$ be the $\mathcal{P}_0$-valued evaluation such that $e_c(D) = \varphi(e(D))$ for all $D \in \mathcal{L}$. Hence, $e_c(A \to B) = 1$ iff $e(A \to B) = \mathcal{N}^+$, and this holds by (i) iff $e(A) \subset e(B)$. However, $e(A) \subset e(B)$ holds iff $\varphi(e(A)) = e_c(A) \leq_* \varphi(e(B)) = e_c(B)$ holds. The assertion is proved. □

Given a $\mathcal{P}_0$-valued evaluation $e^* : \mathit{Var} \to \mathcal{P}_0$, there exists still another way how to project the subsets of $\mathcal{N}^+$ into $\langle 0,1 \rangle$ than the mapping $\varphi$. Namely, (*nonstandard*) *w-evaluation* of w.f.f.'s of the propositional language $\mathcal{L}$ is a mapping $e_w : \mathcal{L} \to \langle 0,1 \rangle$ such that there exists a $\mathcal{P}_0$-valued evaluation $e^* : \mathit{Var} \to \mathcal{P}_0$ with this property: for each $A \in \mathcal{L}$ the value $w(\chi(e(A))) = \lim_{n\to\infty}(1/n)\sum_{i=1}^{n} \chi(e(A))(i)$ exists and $e_w(A) = w(\chi(e(A)))$. In order to simplify our notation we shall write $w(e(A))$ instead of $w(\chi(e(A)))$. Nonstandard $w$-evaluations differ substantially from the $\mathcal{P}_0$-valued and $c$-valued ones, as they cannot be defined in the recurrent way beginning from evaluations of variables and extended recurrently (inductively) to all w.f.f.'s using rules for all functors. In other words, $\mathcal{P}_0$-valued and $c$-valued evaluations are *extensional*, but $w$-evaluations are not, hence, they are *intensional*. E.g., if $e(A) = \{1, 3, 5, 7, 9, \ldots\}$ for a w.f.f. $A \in \mathcal{L}$, then $e(\neg A) = \{2, 4, 6, 8, \ldots\}$, hence, $w(e(A)) = w(e(\neg A)) = 1/2$, but $w(e(A \vee (\neg A))) = w(\mathcal{N}^+) = 1$, and $w(e(A \vee A)) = w(e(A)) = 1/2$, so that $w(e(A \vee B))$ is not defined, in general, by the values $w(e(A))$ and $w(e(B))$.

Setting

$$Taut_w = \{A \in \mathcal{L} : w(e(A)) \text{ is defined and } w(e(A)) = 1 \text{ for each } e^* : Var \to \mathcal{P}_0\} \tag{9}$$

we obtain easily that $Taut_b \subset Taut_w$ holds, as if $e(A) = \mathcal{N}^+$, then trivially $w(e(A)) = 1$. The inverse inclusion $Taut_w \subset Taut_b$ can be also easily proved. By contradiction, let $A \in Taut_w - Taut_b$. Then there exists $e^* : Var \to \mathcal{P}_0$ such that $w(e(A)) = 1$, but $e(A) \neq \mathcal{N}^+$ and $A \notin Taut_{\text{cl}}$ $(= Taut_b)$. So, there exists a classical evaluation $e^*_{\text{cl}} : Var \to \{0, 1\}$ such that $e_{\text{cl}}(A) = 0$. Consequently, if $e^* : Var \to \mathcal{P}_0$ is such that $e^*(p) = \mathcal{N}^+$, if $e^*_{\text{cl}}(p) = 1$, and $e^*(p) = \emptyset$ (the empty subset of $\mathcal{N}^+$), if $e^*_{\text{cl}}(p) = 0$ for all $p \in Var$, then $e(A) = \emptyset$, hence, $w(e(A))) = 0$. So, $A \notin Taut_w$, as there exists $e^* : Var \to \mathcal{P}_0$ with $w(e(A)) \neq 1$. We have arrived at a contradiction, so that $Taut_w \subset Taut_b$.

In fact, we have proved more than the equality $Taut_w = Taut_b$. Our notion of $w$-tautology can be (seemingly) weakened by setting

$$Taut_{ww} = \{A \in \mathcal{L} : w(e(A)) \text{ is defined and } w(e(A)) > 0 \text{ for each } e^* : Var \to \mathcal{P}_0\}. \tag{10}$$

What we have proved can be explicitely stated as follows.

**Theorem 3.** (Completeness theorem for propositional calculus with respect to classical, boolean and nonstandard semantics)

Under the notations introduced above,

$$Taut_{ww} = Taut_w = Taut_b = Taut_c = Taut_{\text{cl}} = Ded. \tag{11}$$

## 5. NONSTANDARD SEMANTICS AND THEIR RELATION TO PROBABILISTIC AND POSSIBILISTIC MEASURES

As it is well-known, probability measure is not extensional, hence, there is no function $G : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$ such that, for a probability space $\langle \Omega, \mathcal{A}, P\rangle$ and for all $A$, $B \in \mathcal{A}$ the equalities

$$P(A \cup B) = G(P(A), P(B)) \tag{12}$$

$$P(A \cap B) = P(\Omega - ((\Omega - A) \cup (\Omega - B))) = 1 - G(1 - P(A), 1 - P(B)) \tag{13}$$

would hold. Obviously, when $P(A) = 1/2$, then $P(\Omega - A) = 1/2$ as well, but $P(A \cup A) = 1/2 \neq 1 = P(A \cup (\Omega - A))$. So, looking for an appropriate mathematical tool for uncertainty quantification and processing, we are at the very beginning of our considerations faced to the necessity to choose between the intensionality of probabilistic measures and the extensionality of some other models, comparing the relative advantages and disadvantages of both the approaches. However, the ideas and results explained above bring us to the conclusion that the ultimate character of this choice is closely related to the classical linear ordering of the unit interval of real numbers and to the resulting operations of supremum and infimum, and that using the nonstandard ordering and operations presented in the foregoing chapter we could combine the extensional and the probabilistic properties of the numerical uncertainty degrees in a much large extent than in the case of the classical structures over $\langle 0, 1 \rangle$.

**Theorem 4.** Let $A, B \in \mathcal{L}$ be w.f.f.'s, let $e^* : Var \to \mathcal{P}_0$ be a $\mathcal{P}_0$-valued evaluation of $\mathcal{L}$, let $e_w(A)$, $e_w(B)$ and $e_w(A \wedge B)$ be defined (let us recall that $e_w(A) = w(\chi(e(A))) = \lim_{n\to\infty}(1/n)\sum_{i=1}^{n} \chi(e(A))(i))$, let $e_c(A) = \varphi(e(A)) = \sum_{i=1}^{\infty} 2\chi(e(A))(i)\, 3^{-i}$. Then

$$e_c(A \vee B) = e_c(A) + e_c(B) - e_c(A \wedge B), \tag{14}$$
$$e_w(A \vee B) = e_w(A) + e_w(B) - e_w(A \wedge B). \tag{15}$$

In particular, if $\neg(A \wedge B) \in Taut_{\mathrm{cl}}$, then

$$e_c(A \vee B) = e_c(A) + e_c(B), \tag{16}$$
$$e_w(A \vee B) = e_w(A) + e_w(B), \tag{17}$$

and

$$e_c(T) = e_w(T) = 1 \tag{18}$$

for each $T \in Taut_{\mathrm{cl}}$. Hence, $e_c$ and $e_w$ both possess the properties of finite additive probability measures.

Proof. Let $A, B \in \mathcal{L}$, let $e^* : Var \to \mathcal{P}_0$. then

$$e_c(A) = \varphi(e(A)) = \sum_{i=1}^{\infty} 2(\chi(e(A))(i))\, 3^{-i} = \sum_{i \in e(A)} 2 \cdot 3^{-i}, \tag{19}$$

as $\chi(e(A))(i) = 1$, if $i \in e(A)$, $\chi(e(A))(i) = 0$ otherwise. If $A, B$ are such that $\neg(A \wedge B) \in Taut_{\mathrm{cl}}$, then $\neg(A \wedge B) \in Taut_b$, so that $\mathcal{N}^+ - e(A \wedge B) = \mathcal{N}^+$ and $e(A \wedge B) = \emptyset$. But $e(A \wedge B) = e(A) \cap e(B)$, so that, for $A \vee B$

$$\begin{aligned} e_c(A \vee B) &= \sum_{i \in e(A \vee B)} 2 \cdot 3^{-i} = \sum_{i \in e(A) \cup e(B)} 2 \cdot 3^{-i} \\ &= \sum_{i \in e(A)} 2 \cdot 3^{-i} + \sum_{i \in e(B)} 2 \cdot 3^{-i} = e_c(A) + e_c(B) \end{aligned} \tag{20}$$

and (16) is proved. Moreover, if $A \leftrightarrow B \in Taut_{\mathrm{cl}}$, then $A \to B \in Taut_{\mathrm{cl}}$ and $B \to A \in Taut_{\mathrm{cl}}$, so that, by Lemma 4, $e(A) \subset e(B)$ and $e(B) \subset e(A)$, hence, $e(A) = e(B)$. Considering a general case of $A, B \in \mathcal{L}$ and applying the results just obtained to the formulas $(A \wedge \neg B) \vee (A \wedge B)$ and $(B \wedge \neg A) \vee (A \vee B)$, we obtain immediately that

$$\begin{aligned}
&\neg((A \wedge \neg B) \wedge (A \wedge B)) \in Taut_{\mathrm{cl}}, \\
&\neg((B \wedge \neg A) \wedge (A \wedge B)) \in Taut_{\mathrm{cl}}, \\
&((A \wedge \neg B) \vee (A \wedge B)) \leftrightarrow A \in Taut_{\mathrm{cl}}, \\
&((B \wedge \neg A) \vee (A \wedge B)) \leftrightarrow B \in Taut_{\mathrm{cl}}, \\
&((A \wedge \neg B) \vee (B \wedge \neg A) \wedge (A \wedge B)) \leftrightarrow A \vee B \in Taut_{\mathrm{cl}},
\end{aligned} \tag{21}$$

so that

$$\begin{aligned}
e_c(A \vee B) &= \sum_{i \in e(A \vee B)} 2 \cdot 3^{-i} = \sum_{i \in e(A \wedge \neg B)} 2 \cdot 3^{-i} \\
&\quad + \sum_{i \in e(A \wedge B)} 2 \cdot 3^{-i} + \sum_{i \in e(B \wedge \neg A)} 2 \cdot 3^{-i} \\
&= \left( \sum_{i \in e(A \wedge \neg B)} 2 \cdot 3^{-i} + \sum_{i \in e(A \wedge B)} 2 \cdot 3^{-i} \right) \\
&\quad + \left( \sum_{i \in e(B \wedge \neg A)} 2 \cdot 3^{-i} + \sum_{i \in e(A \wedge B)} 2 \cdot 3^{-i} \right) - \sum_{i \in e(A \wedge B)} 2 \cdot 3^{-i} \\
&= \sum_{i \in e(A)} 2 \cdot 3^{-i} + \sum_{i \in e(B)} 2 \cdot 3^{-i} - \sum_{i \in e(A \wedge B)} 2 \cdot 3^{-i} \\
&= e_c(A) + e_c(B) - e_c(A \wedge B)
\end{aligned} \tag{22}$$

and (14) is proved.

The proof for $e_w$ is similar. If $\neg(A \wedge B) \in Taut_{\mathrm{cl}}$, then $(e(A) \cap [n]) \cap (e(B) \cap [n]) = \emptyset$ for each $n \in \mathcal{N}^+$, where $[n] = \{1, 2, \ldots, n\}$ is the initial segment of $\mathcal{N}^+$ of the length $n$. So, $\neg(A \wedge B) \in Taut_{\mathrm{cl}}$ implies that

$$\begin{aligned}
e_w(A \wedge B) &= \lim_{n \to \infty} (1/n) \sum_{i=1}^{n} \chi(e(A \vee B))(i) \\
&= \lim_{n \to \infty} (1/n) \operatorname{card}\{i \leq n : i \in e(A \vee B)\} \\
&= \lim_{n \to \infty} (1/n) \operatorname{card}(e(A \vee B) \cap [n]) \\
&= \lim_{n \to \infty} (1/n) [\operatorname{card}(e(A) \cap [n]) + \operatorname{card}(e(B) \cap [n])] \\
&= \lim_{n \to \infty} (1/n) \operatorname{card}(e(A) \cap [n]) + \lim_{n \to \infty} (1/n) \operatorname{card}(e(B) \cap [n]) \\
&= e_w(A) + e_w(B),
\end{aligned} \tag{23}$$

supposing that $e_w(A)$ and $e_w(B)$ are defined, so that (17) holds. Considering the general case of formulas $A$, $B \in \mathcal{L}$, supposing that the values $e_w(A)$, $e_w(B)$ and $e_w(A \wedge B)$ are defined, and applying (23) to the formulas $A \wedge \neg B$, $B \wedge \neg A$, and

$A \wedge B$, we obtain in the same way as above, that

$$\begin{aligned} &\mathrm{card}(e(A \vee B) \cap [n]) = \mathrm{card}(e(A \wedge \neg B) \cap [n]) \\ &\quad + \mathrm{card}(e(B \wedge \neg A) \cap [n]) + \mathrm{card}(e(A \wedge B) \cap [n]) \\ = \ &\mathrm{card}(e(A) \cap [n]) + \mathrm{card}(e(B) \cap [n]) - \mathrm{card}(e(A \wedge B) \cap [n]), \end{aligned} \tag{24}$$

so that (15) immediately follows when all the limit values are defined. As $e(T) = \mathcal{N}^+$ for each $T \in Taut_{\mathrm{cl}}$, (18) immediately follows, so that the theorem is proved. □

As can be almost obviously seen, but as it is perhaps worth being stated explicitly, Theorem 4 can be generalized to the case of finite disjunctions; let us limit to the case of logically disjoint components. Let $A_1, A_2, \dots, A_n$ be w. f. f.'s of $\mathcal{L}$ such that $\neg(A_i \wedge A_j) \in Taut_{\mathrm{cl}}$ holds for each $i, j \leq n,\ i \neq j$, let $\bigvee_{i=1}^{n} A_i$ abbreviate $A_1 \vee A_2 \vee \cdots \vee A_n$. Then

$$e_c\left(\bigvee_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} e_c(A_i), \tag{25}$$

and supposing that $e_w(A_i)$ for each $i \leq n$ is defined, also $e_w\left(\bigvee_{i=1}^{n} A_i\right)$ is defined and

$$e_w\left(\bigvee_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} e_w(A_i). \tag{26}$$

Generalized forms of (14) and (15) can be also deduced. However, the situation differs principially when considering the *σ-additivity* (the countable additivity) of the evaluations $e_c$ and $e_w$. As the language $\mathcal{L}$ does not allow to define disjunctions of infinitely many formulas, we have to formalize the next statement in a slightly modified way.

**Theorem 5.** Let $A_1, A_2, \dots$ be an infinite sequence of formulas of $\mathcal{L}$, such that $\neg(A_i \wedge A_j) \in Taut_{\mathrm{cl}}$ holds for each $i, j \geq 1,\ i \neq j$, let $e^* : Var \to \mathcal{P}_0$ be a $\mathcal{P}_0$-evaluation of $\mathcal{L}$. Then

$$\varphi\left(\bigcup_{i=1}^{\infty} e(A_i)\right) = \sum_{i=1}^{\infty} \varphi(e(A_i)) = \sum_{i=1}^{\infty} e_c(A_i). \tag{27}$$

Hence, the difference is that the value $\varphi\left(\bigcup_{i=1}^{\infty} e(A_i)\right)$ cannot be written as $e_c\left(\bigvee_{i=1}^{\infty} A_i\right)$ because of the fact that $\bigvee_{i=1}^{\infty} A_i$ is not a w. f. f. of $\mathcal{L}$.

Proof. Like as in the proof of Theorem 4 we obtain that $e(A_i) \cap e(A_j) = \emptyset$ for each $i, j \geq 1,\ i \neq j$, so that

$$\varphi\left(\bigcup_{i=1}^{\infty} e(A_i)\right) = \sum_{j=1}^{\infty} 2\varphi\left(\bigcup_{i=1}^{\infty} e(A_i)\right)(i)\, 3^{-i} \tag{28}$$

$$= \sum_{j \in \bigcup_{i=1}^{\infty} e(A_i)} 2 \cdot 3^{-j} = \sum_{i=1}^{\infty} \sum_{j \in e(A_i)} 2 \cdot 3^{-j}$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} 2\chi(e(A_i))(j) \, 3^{-j} = \sum_{i=1}^{\infty} e_c(A_i)$$

and the assertion is proved. □

For the evaluation $e_w$, however, an analogy of (27) does not hold. Or, let $A_1, A_2, \ldots \in \mathcal{L}$, and $e^* : Var \to \mathcal{P}_0$ be such that $e(A_i) = \{i\}$ for each $i \in \mathcal{N}^+$. Then $e_w(A_i) = \lim_{n\to\infty}(1/n) = 0$ for each $i \in \mathcal{N}^+$, but $\bigcup_{i=1}^{\infty} e(A_i) = \mathcal{N}^+$, so that $\varphi(\bigcup_{i=1}^{\infty} e(A_i)) = 1 \neq \sum_{i=1}^{\infty} e_w(A_i)$. Such $A_1, A_2, \ldots \in \mathcal{L}$ and $e^*$ always exist, take simply $A_i = p_i \in Var$, $e(A_i) = e^*(p_i) = \{i\} \in \mathcal{P}_0$. Cf. [2] for more details concerning the conditions when finitely additive measures can be extended uniquely to $\sigma$-additive ones.

As a matter of fact, it was just our aim to arrive at the $\sigma$-additivity of the mapping $e_c$, taken as mapping from $\mathcal{P}_0$ into $\langle 0, 1\rangle$, what forced us to take the mapping $\varphi$ as a one-to-one mapping between $\{0,1\}^\infty$ and the set of its values,namely between $\{0,1\}^\infty$ and the Cantor subset of $\langle 0,1\rangle$. What we have to pay is the consequence that not every real number from the unit interval can be ascribed by $e_c$ to a w. f. f. from $\mathcal{L}$, e. g., for no w. f. f. $A$ the equality $e_c(A) = 1/2$ can hold, as $1/2 \notin \mathcal{C}$. All the unit interval as the space of truthvalues can be ranged, in a sense, by the evaluation $e_w$ which is, on the other side, intensional. Let us describe this solution very briefly, referring to [3] or [4] for more detail.

**Lemma 5.** Let $0 \le q_i \le 1$, $i \in \mathcal{N}^+$, be a probability distribution on $\mathcal{N}^+$, so that $\sum_{i=1}^{\infty} q_i = 1$. Then there exist real numbers $\alpha_{ij}$, $i$, $j \in \mathcal{N}^+$, $\alpha_{ij}$ $\{0,1\}$, such that

(i) if $q_i = 1$, then $\alpha_{ij} = 1$ for each $j \in \mathcal{N}^+$,

(ii) if $q_i = 0$, then $\alpha_{ij} = 0$ for each $j \in \mathcal{N}^+$,

(iii) if $0 < q_i < 1$, then $\lim_{n\to\infty}(1/n)\sum_{j=1}^{n} \alpha_{ij} = q_i$,

(iv) for each $j \in \mathcal{N}^+$ there is just one $i \in \mathcal{N}^+$ such that $\alpha_{ij} = 1$.

Proof. Cf. Theorem 4 in [4]. □

Let us recall that the condition that $\alpha_{ij} = 1$ ($= 0$, resp.) for all $j \in \mathcal{N}^+$ if $q_i = 1$ ($= 0$, resp.) is substantial and cannot be replaced by a weaker condition that $\lim_{n\to\infty}(1/n)\sum_{j=1}^{n} \alpha_{ij} = 1$ ($= 0$, resp.). In other terms, the extremal points of the unit interval of real numbers must be represented by the "standard" 1 and "standard" 0, not by some of their nonstandard variants.

**Theorem 6.** Let $\langle q_1, q_2, \ldots\rangle$ be a probability distribution on $\mathcal{N}^+$. Then there exists a $\mathcal{P}_0$-valued evaluation $e^* : Var \to \mathcal{P}_0$ such that $e_c^*(p_i) = q_i$ for each $i \in \mathcal{N}^+$, here $Var = \{p_1, p_2, \ldots\}$.

Proof. Obvious, the only we have to do is to set $e^*(p_i) = \{j \in \mathcal{N}^+ : \alpha_{ij} = 1\}$ where $\{\alpha_{ij}\}_{i=1}^{\infty}$ is a sequence satisfying Lemma 5 with respect to $\langle q_1, q_2, \ldots \rangle$. □

The approach applied here in the most elementary case of propositional calculus can be extended also to the case of first-order predicate calculus, cf. [5]

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*RNDr. Ivan Kramosil, DrSc., Institute of Computer Science – Academy of Sciences of the Czech Republic, Pod vodárenskou věží 2, 182 07 Praha 8. Czech Republic.*
*e-mail: kramosil@uivt.cas.cz*