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CONTROLLABILITY OF FUNCTIONAL DIFFERENTIAL SYSTEMS OF SOBOLEV TYPE IN BANACH SPACES

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Sufficient conditions for controllability of partial functional differential systems of Sobolev type in Banach spaces are established. The results are obtained using compact semigroups and the Schauder fixed point theorem. An example is provided to illustrate the results.

1. INTRODUCTION

The problem of controllability of linear and nonlinear systems represented by ordinary differential equations in finite dimensional spaces has been extensively studied. Several authors [5, 6, 12–14] have extended the concept to infinite dimensional systems in Banach spaces with bounded operators. Triggiani [17] established sufficient conditions for controllability of linear and nonlinear systems in Banach spaces. Exact controllability of abstract semilinear equations has been studied by Lasiecka and Triggiani [10]. Kwun et al [8] investigated the controllability and approximate controllability of delay Volterra systems by using a fixed point theorem. Recently Balachandran et al [1–3] studied the controllabilty and local null controllability of nonlinear integrodifferential systems and functional differential systems in Banach spaces and it was shown that the controllability problem in Banach spaces can be converted into one of a fixed-point problem for a single-valued mapping. The purpose of this paper is to study the controllability of Sobolev type partial functional differential systems in Banach spaces. The equation considered here serves as an abstract formulation of Sobolev type partial functional differential equations which arise in many physical phenomena [4, 7, 9, 11, 16].

Consider a nonlinear partial functional differential system of the form

$$\begin{aligned} \big(Ex(t)\big)' + Ax(t) &= Bu(t) + f\big(t, x_t\big), \quad t > 0 \\ x(t) &= \phi(t), \quad -r \le t \le 0 \end{aligned} \tag{1}$$

where the state $x(\cdot)$ takes values in a Banach space X and the control function $u(\cdot)$ is given in $L^2(J, U)$, the Banach space of admissible control functions with U a Banach space. B is a bounded linear operator from U into Y, a Banach space. The nonlinear operator $f : J \times C \to Y$ is continuous. Here $J = [0, a]$ and for a continuous function $x : J^* = [-r, a] \to X$, x_t is that element of $C = C([-r, 0] : X)$ defined by $x_t(s) = x(t+s)$, $-r \le s \le 0$. The domain $D(E)$ of E becomes a Banach space with norm $\|x\|_{D(E)} = \|Ex\|_Y$, $x \in D(E)$ and $C(E) = C([-r, 0] : D(E))$.

2. PRELIMINARIES

The operators $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$ satisfy the following hypotheses $[C_i]$ for $i = 1, 2, \ldots, 4$:

[C$_1$] A and E are closed linear operators,

[C$_2$] $D(E) \subset D(A)$ and E is bijective,

[C$_3$] $E^{-1} : Y \to D(E)$ is compact,

[C$_4$] For each $t \in [0, a]$ and for some $\lambda \in \rho(-AE^{-1})$, the resolvent set of $-AE^{-1}$, we have that the resolvent $R(\lambda : -AE^{-1})$ is a compact operator.

The hypotheses [C$_1$], [C$_2$] and the closed graph theorem imply the boundedness of the linear operator $AE^{-1} : Y \to Y$.

Lemma [15]. Let A be the infinitesimal generator of a uniformly continuous semigroup $S(t)$. If the resolvent $R(\lambda : A)$ of A is compact for every $\lambda \in \rho(A)$, then $S(t)$ is a compact semigroup.

From the above fact, $-AE^{-1}$ generates a compact semigroup $T(t)$, $t \ge 0$. Thus, $\max_{t \in J} \|T(t)\|$ is finite and so denote $M = \max_{t \in J} \|T(t)\|$.

Definition. The system (1) is said to be controllable on the interval J if for every continuous initial function ϕ defined on $[-r, 0]$ and every $x_1 \in X$ there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of (1) satisfies $x(a) = x_1$.

The following additional assumptions will be used:

[C$_5$] B is a bounded linear operator and the linear operator W from U into X defined by

$$Wu = \int_0^a E^{-1}T(a-s)\,Bu(s)\,\mathrm{d}s$$

has a bounded inverse operator W^{-1} defined on $L^2(J, U)/\ker W$.

[C$_6$] The function f satisfies the following two conditions:

(i) For each $t \in J$, the function $f(t, \cdot) : C \to Y$ is continuous, and for each $x \in C$ the function $f(\cdot, x) : J \to Y$ is strongly measurable.

(ii) For each natural number k, there is a function $g_k \in L^1(J)$ such that

$$\sup_{|x|\le k} |f(t,x_t)| \le g_k(t),$$

$$\underline{\lim}_{k\to\infty} \frac{1}{k}\int_0^a g_k(s)\,\mathrm{d}s = \alpha < \infty,$$

where α is a real number.

The solution of (1) is given by the integral equation

$$\begin{aligned} x(t) &= E^{-1}T(t)\,E\phi(0) + \int_0^t E^{-1}T(t-s)\,f(s,x_s)\,\mathrm{d}s \\ &\quad + \int_0^t E^{-1}T(t-s)\,Bu(s)\,\mathrm{d}s, \quad t>0, \\ x(t) &= \phi(t), \quad -r \le t \le 0. \end{aligned} \tag{2}$$

In the next section the Schauder fixed point theorem is used in order to establish the controllability theorem for equation (1) under above conditions.

3. MAIN RESULT

Theorem. If the assumptions [C_1] – [C_6] are satisfied, then the system (1) is controllable on J provided that

$$\alpha M\|E^{-1}\|\Big[1 + aM\|B\|\,\|W^{-1}\|\,\|E^{-1}\|\Big] < 1.$$

Proof. Using the assumption [C_5], for an arbitrary function $x(\cdot)$ define the control

$$u(t) = W^{-1}\left[x_1 - E^{-1}T(a)\,E\phi(0) - \int_0^a E^{-1}T(a-s)\,f(s,x_s)\,\mathrm{d}s\right](t).$$

It shall now be shown that when using this control, the operator S defined by

$$\begin{aligned} (Sx)(t) &= E^{-1}T(t)\,E\phi(0) + \int_0^t E^{-1}T(t-s)\,f(s,x_s)\,\mathrm{d}s \\ &\quad + \int_0^t E^{-1}T(t-s)\,Bu(s)\,\mathrm{d}s, \quad \text{for } t>0, \\ (Sx)(t) &= \phi(t), \quad \text{for } -r \le t \le 0, \end{aligned}$$

from $C(J^*,X)$ into itself, for each $x \in C(J^*,X)$, has a fixed point. This fixed point is then a solution of equation (1). Clearly,

$$(Sx)(a) = E^{-1}T(a)\,E\phi(0) + \int_0^a E^{-1}T(a-s)\,f(s,x_s)\,\mathrm{d}s$$

$$
\begin{aligned}
& + \int_0^a E^{-1}T(a-s)\,Bu(s)\,\mathrm{d}s \\
= \quad & E^{-1}T(a)\,E\phi(0) + \int_0^a E^{-1}T(a-\tau)\,f(s,x_s)\,\mathrm{d}s \\
& + \int_0^a E^{-1}T(a-s)\,BW^{-1}\Big[x_1 - E^{-1}T(a)\,E\phi(0) \\
& - \int_0^a E^{-1}T(a-\tau)\,f(\tau,x_\tau)\,\mathrm{d}\tau\Big]\,\mathrm{d}s. \\
= \quad & x_1.
\end{aligned}
$$

It can be easily verified that S maps $C(J^*,X)$ into itself continuously. For each natural number k let

$$B_k = \{x \in C(J^*,x) : x(0) = \phi(0),\ \|x(t)\| \le k,\ t \in J\}.$$

Then for each k, the set B_k is clearly a bounded, closed, convex subset in $C(J^*,X)$ and there exists a natural number K with $SB_K \subset B_K$. If this were not the case, then for each natural number k there is a function $x_k \in B_k$ with $Sx_k \notin B_k$; that is $\|Sx_k\| \ge k$. Then $1 < \frac{1}{k}\|Sx_k\|$, and so

$$1 \le \varliminf_{k\to\infty} k^{-1}\|Sx_k\|.$$

However,

$$
\begin{aligned}
& \varliminf_{k\to\infty} k^{-1}\|Sx_k\| \\
\le \quad & \varliminf_{k\to\infty} k^{-1}\Big\{M\|E^{-1}\|\,\|E\|\,\|\phi(0)\| + M\|E^{-1}\|\int_0^a g_k(s)\,\mathrm{d}s \\
& +M\|E^{-1}\|\,\|B\|\|W^{-1}\|\int_0^a \Big[\|x_1\| + \|E^{-1}\|\,M\|E\|\|\phi(0)\| \\
& +\|E^{-1}\|\,M\int_0^a g_k(\tau)\,\mathrm{d}\tau\Big]\,\mathrm{d}s\Big\} \\
= \quad & \alpha M\|E^{-1}\| + \alpha a M\|E^{-1}\|\,\|B\|\,\|W^{-1}\|\,\|E^{-1}\|\,M \\
= \quad & \alpha M\|E^{-1}\|[1 + aM\|B\|\,\|W^{-1}\|\,\|E^{-1}\|] \\
= \quad & \alpha M\|E^{-1}\|[1 + aM\|B\|\|W^{-1}\|\,\|E^{-1}\|] < 1,
\end{aligned}
$$

a contradiction. Hence, $SB_K \subset B_K$ for some positive integer K.

In fact, the operator S maps B_K into a compact subset of B_K. To prove this it is first shown that for every fixed $t \in J$ the set

$$V_K(t) = \{(Sx)(t) : x \in B_K\}$$

is a precompact in X. This is trivial for $t = 0$, since $V_K(0) = \{\phi(0)\}$. So let t, $0 < t \le a$, be fixed and let ϵ be a given real number satisfying $0 < \epsilon < t$. Define

$$\begin{aligned}(S_\epsilon x)(t) &= E^{-1}T(t)\,E\phi(0) + \int_0^{t-\epsilon} E^{-1}T(t-s)\,f(s,x_s)\,\mathrm{d}s \\ &\quad + \int_0^{t-\epsilon} E^{-1}T(t-s)\,Bu(s)\,\mathrm{d}s \\ &= E^{-1}T(t)\,E\phi(0) + \int_0^{t-\epsilon} E^{-1}T(t-s)\,f(s,x_s)\,\mathrm{d}s \\ &\quad + \int_0^{t-\epsilon} E^{-1}(t-s)\,BW^{-1}\Big[x_1 - E^{-1}T(a)\,E\phi(0) \\ &\quad - \int_0^a E^{-1}T(a-\tau)\,f(\tau,x_\tau)\,\mathrm{d}\tau\Big]\,\mathrm{d}s.\end{aligned}$$

Since $u(s)$ is bounded and $T(t)$ is compact, the set $V_\epsilon(t) = \{(S_\epsilon x)(t) : x \in B_K\}$ is a precompact set in X. Also, for $x \in B_K$, using the defined control $u(t)$ yields

$$\begin{aligned}&\|(Sx)(t) - (S_\epsilon x)(t)\| \\ =\ & \|E^{-1}T(t)\,E\phi(0) + \int_0^t E^{-1}T(t-s)\,f(s,x_s)\,\mathrm{d}s \\ &+ \int_0^t E^{-1}T(t-s)\,Bu(s)\,\mathrm{d}s \\ &- E^{-1}T(t)\,E\phi(0) - \int_0^{t-\epsilon} E^{-1}T(t-s)\,f(s,x_s)\,\mathrm{d}s \\ &- \int_0^{t-\epsilon} E^{-1}T(t-s)\,Bu(s)\,\mathrm{d}s \\ \le\ & \left\|\int_{t-\epsilon}^t E^{-1}T(t-s)\,f(s,x_s)\,\mathrm{d}s\right\| + \left\|\int_{t-\epsilon}^t E^{-1}T(t-s)\,Bu(s)\,\mathrm{d}s\right\| \\ \le\ & M\|E^{-1}\|\int_{t-\epsilon}^t |f(s,x_s)|\,\mathrm{d}s + M\|E^{-1}\|\,\|B\|\,\|W^{-1}\|\int_{t-\epsilon}^t \Big[\|x_1\| \\ &+ M\|E^{-1}\|\|E\|\,\|\phi(0)\| + M\|E^{-1}\|\int_0^a \|f(\tau,x_\tau)\|\,\mathrm{d}\tau\Big]\,\mathrm{d}s \\ \le\ & M\|E^{-1}\|\int_{t-\epsilon}^t g_K(s)\,\mathrm{d}s + M\|E^{-1}\|\,\|B\|\|W^{-1}\|\int_{t-\epsilon}^t \Big[\|x_1\| \\ &+ M\|E^{-1}\|\,\|E\|\,\|\phi(0)\| + M\|E^{-1}\|\int_0^a g_K(\tau)\,\mathrm{d}\tau\Big]\,\mathrm{d}s.\end{aligned}$$

Since g_K, $h_K \in L^1(J)$, it follows that $\|(Sx)(t) - (S_\epsilon x)(t)\|$ is finite by the uniform boundedness principle. Thus, there are precompact sets arbitrarily close to the set $V_K(t)$, and so $V_K(t)$ is precompact in X.

Next it is shown that $SB_K = \{Sx : x \in B_K\}$ is an equicontinuous family of functions. Let $x \in B_K$ and $t, \tau \in J$ such that $0 < t < \tau$, then

$$\begin{aligned}
&\|(Sx)(t) - (Sx)(\tau)\| \\
\leq\ & \|T(t) - T(\tau)\|\,\|E^{-1}\|\,\|E\|\,\|\phi(0)\| \\
&+ \int_0^t \|T(t-s) - T(\tau - s)\|\,\|E^{-1}\|\|f(s, x_s)\|\,\mathrm{d}s \\
&+ \int_t^\tau \|T(\tau - s)\|\,\|E^{-1}\|\,\|f(s, x_s)\|\,\mathrm{d}s \\
&+ \int_0^t \|T(t-s) - T(\tau - s)\|\,\|E^{-1}\|\|B\|\,\|W^{-1}\|\Big[\|x_1\| \\
&+\|E^{-1}\|\,\|T(a)\|\,\|E\|\,\|\phi(0)\| \\
&+ \int_0^a \|E^{-1}\|\,\|T(a-\tau)\|\|f(\tau, x_\tau)\|\,\mathrm{d}\tau\Big]\,\mathrm{d}s \\
&+ \int_t^\tau \|T(\tau - s)\|\,\|E^{-1}\|\,\|B\|\|W^{-1}\|\Big[\|x_1\| + \|E^{-1}\|\,\|T(a)\|\,\|E\|\|\phi(0)\| \\
&+ \int_0^a \|E^{-1}\|\,\|T(a-\tau)\|\,\|f(\tau, x_\tau)\|\,\mathrm{d}\tau\Big]\,\mathrm{d}s \\
\leq\ & \|T(t) - T(\tau)\|\,\|E^{-1}\|\,\|E\|\|\phi(0)\| \\
&+ \int_0^t \|T(t-s) - T(\tau - s)\|\,\|E^{-1}\|g_K(s)\,\mathrm{d}s \\
&+ \int_t^\tau \|T(t-s)\|\,\|E^{-1}\|g_K(s)\,\mathrm{d}s \\
&+ \int_0^t \|T(t-s) - T(\tau - s)\|\,\|E^{-1}\|\|B\|\,\|W^{-1}\|\Big[\|x_1\| \\
&+\|E^{-1}\|\,\|T(a)\|\|E\|\,\|\phi(0)\| + \int_0^a \|E^{-1}\|\,\|T(a-\tau)\|g_K(\tau)\,\mathrm{d}\tau\Big]\,\mathrm{d}s \\
&+ \int_t^\tau \|T(\tau - s)\|\,\|E^{-1}\|\,\|B\|\,\|W^{-1}\|\Big[\|x_1\| \\
&+\|E^{-1}\|\,\|T(a)\|\,\|E\|\,\|\phi(0)\| + \int_0^a \|E^{-1}\|\|T(a-\tau)\|g_K(\tau)\,\mathrm{d}\tau\Big]\,\mathrm{d}s.
\end{aligned}$$

Now $T(t)$ is continuous in the uniform operator topology for $t > 0$. Since $T(t)$ is compact and g_K, $h_K \in L^1(J)$, the right hand side of above inequality tends to zero as $t \to \tau$. Thus, SB_K is equicontinuous and also bounded. By the Arzela-Ascoli theorem SB_K is precompact in $C(J^*, X)$. Hence S is a completely continuous operator on $C(J^*, X)$. From the Schauder fixed point theorem, S has a fixed point in B_K. Any fixed point of S is a mild solution of (1) on J satisfying $(Sx)(t) = x(t) \in X$. Thus, the system (1) is controllable on J. □

4. EXAMPLE

The above result is illustrated by showing its applicability to a delay partial differential equation with a nonlinear function satisfying the Carathéodory condition.

Consider the following differential equation with a control term

$$\frac{\partial}{\partial t}\Big(z(t,y) - z_{yy}(t,y)\Big) - z_{yy}(t,y) = Bu(t) + f\big(t, z(t-r,y)\big) \tag{3}$$

$$\text{where } y \in [0,\pi],\ t \geq 0$$

$$z(t,0) = z(t,\pi) = 0, \quad t \geq 0$$

$$z(t,y) = \phi(t,y), \qquad 0 \leq y \leq \pi,\ -r \leq t \leq 0$$

where $f : J \times C \to Y$ is continuous. It is assumed that the following conditions hold with $X = Y = L^2[0,\pi]$:

[A$_1$] The operator $B : U \subset J \to Y$ is an identity operator.

[A$_2$] The linear operator $W : U \to X$ is defined by

$$Wu = \int_0^a E^{-1}T(a-s)\,Bu(s)\,ds$$

and has a bounded invertible operator W^{-1} defined on $L^2(J,U)/\ker W$.

[A$_3$] The nonlinear operator $f : J \times C \to Y$ satisfies the following three conditions:

(i) For each $t \in J$, $f(t, z_t)$ is continuous,

(ii) For each $z_t \in C$, $f(t, z_t)$ is measurable,

(iii) There is a constant $\nu (0 < \nu < 1)$ and a function $h \in L^1(J)$ such that for all $(t, z_t) \in J \times C$

$$\|f(t,z_t)\| \leq h(t)|z|^\nu.$$

Define the operators $A : D(A) \subset X \to Y$, $E : D(E) \subset X \to Y$ by

$$\begin{aligned} Az &= -z_{yy}, \\ Ez &= z - z_{yy}, \end{aligned}$$

respectively, where each domain $D(A)$, $D(E)$ is given by

$$\{z \in X : z,\ z_y \text{ are absolutely continuous},\ z_{yy} \in X,\ z(t,0) = z(t,\pi) = 0\}.$$

Define an operator $F : J \times C \to Y$ by

$$F(t,w)(y) = f\big(t, w(-r)(y)\big).$$

Then the above problem (3) can be formulated abstractly as

$$\begin{aligned} &\big(Ez(t)\big)' + Az(t) = Bu(t) + F(t, z_t), \\ &z(t) = \phi(t), \quad -r \leq t \leq 0. \end{aligned}$$

Also, A and E can be written, respectively, as (see [11])

$$Az = \sum_{n=1}^{\infty} n^2 \langle z, z_n \rangle z_n, \qquad z \in D(A),$$
$$Ez = \sum_{n=1}^{\infty} (1+n^2) \langle z, z_n \rangle z_n, \quad z \in D(E),$$

where $z_n(y) = \sqrt{2/\pi}\, \sin ny$, $n = 1, 2, \dots$ is the orthonormal set of eigenfunctions of A. Furthermore, for $z \in X$ we have

$$E^{-1}z = \sum_{n=1}^{\infty} \frac{1}{(1+n^2)} \langle z, z_n \rangle z_n,$$
$$-AE^{-1}z = \sum_{n=1}^{\infty} \frac{-n^2}{(1+n^2)} \langle z, z_n \rangle z_n,$$
$$T(t)z = \sum_{n=1}^{\infty} e^{\frac{-n^2}{(1+n^2)}t} \langle z, z_n \rangle z_n.$$

It is easy to see that $-AE^{-1}$ generates a strongly continuous semigroup $T(t)$ on Y and that $T(t)$ is compact with $\|T(t)\| \le e^{-t}$ for each $t > 0$. Also, the operator f satisfies condition [C_6], [18]. So all the conditions stated in the above theorem are satisfied. Hence the system (3) is controllable on J.

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