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OPTIMALITY CONDITIONS FOR NONCONVEX VARIATIONAL PROBLEMS RELAXED IN TERMS OF YOUNG MEASURES

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The scalar nonconvex variational problems of the minimum-energy type on Sobolev spaces are studied. As the Euler–Lagrange equation dramatically looses selectivity when extended in terms of the Young measures, the correct optimality conditions are sought by means of the convex compactification theory. It turns out that these conditions basically combine one part from the Euler–Lagrange equation with one part from the Weierstrass condition.

1. INTRODUCTION

We will deal with the following variational problem, related with various minimum energy principles in continuum mechanics (and not only there):

$$\text{(VP)} \qquad \text{minimize} \quad \int_\Omega \varphi(x, y(x), \nabla y(x))\,\mathrm{d}x, \qquad y \in W_0^{1,p}(\Omega),$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $W_0^{1,p}(\Omega)$ the Sobolev space of functions $y : \Omega \to \mathbb{R}$ with $\nabla y \in L^p(\Omega; \mathbb{R}^n)$ and with zero traces on the boundary $\partial\Omega$ of Ω, $\varphi : \Omega \times (\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R} : (x, r, s) \mapsto \varphi(x, r, s)$ a Carathéodory function with an appropriate growth specified latter, and $1 < p < +\infty$, $n \geq 1$. We are especially interested in the case when $\varphi(x, r, \cdot)$ is not convex.

Let us remind the Euler–Lagrange necessary optimality conditions for (VP) in the weak formulation: if y solves (VP), then

$$\operatorname{div} \lambda = \varphi'_r(y, \nabla y) \qquad \text{in} \quad W^{-1,p/(p-1)}(\Omega) \tag{1.1}$$

with $\lambda \in L^{p/(p-1)}(\Omega; \mathbb{R}^{nm})$ called "momentum" (cf. [9]) and defined, under appropriate growth conditions, by

$$\lambda = \varphi'_s(y, \nabla y) \qquad \text{in} \quad L^{p/(p-1)}(\Omega; \mathbb{R}^n) \tag{1.2}$$

where $\varphi'_r(y, \nabla y)$ abbreviates the function $x \mapsto \varphi'_r(x, y(x), \nabla y(x))$ with $\varphi'_r(x, r, s)$ the derivative of $\varphi(x, \cdot, s)$, and analogous meaning has also $\varphi'_s(y, \nabla y)$. Then $W^{-1,p/(p-1)}(\Omega)$

is the dual space to $W_0^{1,p}(\Omega)$ under the duality pairing induced from $L^2(\Omega)$. For $n=1$ let us still remind the Weierstrass condition ($x\in\Omega$, $s\in\mathbb{R}^n$):

$$\varphi(x,y(x),s)-\varphi(x,y(x),\nabla y(x))\geq\lambda(x)\cdot(s-\nabla y(x)) \tag{1.3}$$

with λ again from (1.2); cf. [23, Sect. 48.8] or [9].

It is well known that, without any convexity assumption about $\varphi(x,r,\cdot)$, the solution of (VP) may fail to exist. Of course, (1.1) – (1.2) need not have any solution, either. This is due to the lack of monotonicity of $\varphi_s'(y,\cdot)$ in (1.2). The minimizing sequences of (VP) will then typically exhibit more and more rapid oscillation in u, described properly in the limit as a weakly measurable mapping $\nu : x\mapsto\nu_x :$ $\Omega\to\mathrm{rpm}(\mathbb{R}^{nm})$, a so-called Young measure [22], with $\mathrm{rpm}(\cdot)$ standing for the set of all regular probability measures on the domain indicated. Let us denote the set of all Young measures by $\mathcal{Y}(\Omega;\mathbb{R}^n)$, and by $\mathcal{Y}^p(\Omega;\mathbb{R}^n)$ the set of all Young measures ν attainable by sequences bounded in $L^p(\Omega;\mathbb{R}^n)$; alternatively we may also write $\mathcal{Y}^p(\Omega;\mathbb{R}^n)=\{\nu\in\mathcal{Y}(\Omega;\mathbb{R}^n);\ (x\mapsto\int_{\mathbb{R}^n}|s|^p\nu_x(\mathrm{d}s))\in L^1(\Omega)\}$; cf. [13] for details. In continuum mechanics [3, 4, 5], these Young-measure solutions are interpreted as fine structures (called also microstructures). In order to have a chance for existence of solutions, the original problem (VP) is naturally (= continuously) relaxed by means of these Young measures, which gives the relaxed variational problem:

$$(\mathsf{RVP}_0)\qquad\left\{\begin{array}{ll}\text{minimize} & \displaystyle\int_\Omega\left[\int_{\mathbb{R}^n}\varphi(x,y(x),s)\,\nu_x(\mathrm{d}s)\right]\mathrm{d}x,\\[2ex] \text{subject to} & \displaystyle\int_{\mathbb{R}^n}s\,\nu_x(\mathrm{d}s)=\nabla y(x)\quad\text{for a. a. } x\in\Omega,\\[2ex] & y\in W_0^{1,p}(\Omega),\quad \nu=\{\nu_x\}_{x\in\Omega}\in\mathcal{Y}^p(\Omega;\mathbb{R}^n).\end{array}\right.$$

Also, we can naturally (= continuously) extend the Euler–Lagrange equation (1.1) – (1.2):

$$\operatorname{div}\lambda(x)=\int_{\mathbb{R}^n}\varphi_r'(x,y(x),s)\,\nu_x(\mathrm{d}s)\quad\text{ in the sense of } W^{-1,p/(p-1)}(\Omega) \tag{1.4}$$

and

$$\lambda(x)=\int_{\mathbb{R}^n}\varphi_s'(x,y(x),s)\,\nu_x(\mathrm{d}s)\qquad\text{in the sense of } L^{p/(p-1)}(\Omega;\mathbb{R}^n). \tag{1.5}$$

While (RVP_0) can be (under appropriate conditions) a natural extension of (VP) (cf. Proposition 2.1 together with Remark 3.1 below), this surprisingly cannot be said about the corresponding differential equation (1.1) – (1.2) (in the weak formulation) extended continuously to (1.4) – (1.5). Indeed, it was shown in [16, 18] that the set of the solutions to (1.4) – (1.5) may be very large and possibly unbounded even if (RVP_0) has a unique solution. Also, we can extend the Weierstrass condition (1.3), obtaining

$$\varphi(x,y(x),s)-\int_{\mathbb{R}^n}\varphi(x,y(x),\tilde{s})\,\nu_x(\mathrm{d}\tilde{s})\geq\lambda(x)\cdot\left(s-\int_{\mathbb{R}^n}\tilde{s}\,\nu_x(\mathrm{d}\tilde{s})\right). \tag{1.6}$$

However, the extended Weierstrass condition (1.5)–(1.6) has a little selectivity as well, because so has already the original Weierstrass condition (1.2)–(1.3), as pointed out by Ioffe and Tihomirov [9, p. 115]. (The original setting of the Weierstrass condition was, however, a bit different and had a greater selectivity, cf. [20, Chap.22].)

Therefore, it arises immediately a question how the correct (=selective) optimality conditions for (RVP_0) look like. It could be roughly said that the answer is represented by a combination of one part from the extended Euler–Lagrange equation (namely (1.4)) and of one part from the Weierstrass condition (namely (1.6)); cf. (3.5) and (3.7) with Remark 3.1. The resting part (1.5) is not involved and the momentum $\lambda \in L^{p/(p-1)}(\Omega;\mathbb{R}^n)$ is merely claimed to exist – in particular, φ'_s does not appear in these conditions at all.

The results derived in this paper basically generalize the results by McShane [14] for $n = 1$ and by Young [21] for the case that the energy density $\varphi(x,r,s)$ is independent of r; besides, [14, 21] delt factually only with Young measures from $\mathcal{Y}^\infty(\Omega;\mathbb{R}^n)$, which is however not much realistic. Our Euler/Weierstrass conditions for Young-measure relaxation can be used to establish uniqueness of the solution to the relaxed problem [8] or to estimate a number of atoms of a solution to the relaxed problem, which can be advantageously exploit within numerical implementation of the relaxed problems.

It should be also emphasized that, in contrast to the scalar case where the relaxation theory presented here is fairly complete, the relaxation theory in the vectorial case (i.e. if $y \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ for $m > 1$ and $n > 1$) is much more difficult and still exhibits many serious open problems. Also we will focuse our attentition to oscillation effects so that througout this paper we suppose $p > 1$, though some considerations would work also for $p = 1$ (i.e. the nonparametric-minimal-hypersurface-like problem) where the oscillation effects can be accompanied by the concentration ones.

2. A SUITABLE RELAXATION OF VARIATIONAL PROBLEMS

The set of Young measures $\mathcal{Y}^p(\Omega;\mathbb{R}^n)$ we used in (RVP_0) is not suitable for the analysis of the relaxed problem because, though being convex, it is not closed in an appropriate locally convex space and also it is not apriori clear that we can use it correctly for integrands with precisely p-growth, e.g. for φ itself. Therefore, we use here a finer convex hull of the Lebesgue space $L^p(\Omega;\mathbb{R}^n)$, employing the convex-compactification theory; cf. also [15, 17, 19].

We must briefly introduce a few definitions and notations. Let us denote by $\mathrm{Car}^p(\Omega,\mathbb{R}^n)$ the linear space of all Carathéodory functions $h : \Omega \times \mathbb{R}^n \to \mathbb{R}$ (that means $h(\cdot,s)$ is measurable and $h(x,\cdot)$ is continuous) such that $|h(x,s)| \le a_h(x) + b_h|s|^p$ for some $a_h \in L^1(\Omega)$ and $b_h < +\infty$. We can introduce a natural (semi)norm on $\mathrm{Car}^p(\Omega,\mathbb{R}^n)$ by $\|h\|_{\mathrm{Car}^p(\Omega,\mathbb{R}^n)} = \inf\{\|a\|_{L^1(\Omega)} + b;\ |h(x,s)| \le a(x) + b|s|^p\}$. Let us now consider a linear subpace H of $\mathrm{Car}^p(\Omega,\mathbb{R}^n)$, normed again be the norm of $\mathrm{Car}^p(\Omega,\mathbb{R}^n)$. Then the dual space H^* is a Banach space. We can imbed $L^p(\Omega;\mathbb{R}^n)$ into H^* by the imbedding i_H defined by $\langle i_H(u), h\rangle = \int_\Omega h(x,u(x))\,dx$. By the well-known properties of the Nemytskiĭ operators, i_H is (norm,weak*)-continuous.

The reason that we did not simply put $H = \mathrm{Car}^p(\Omega,\mathbb{R}^n)$ is that such a choice would yield a non-metrizable weak* topology on bounded sets in H^*, and thus we prefer to take some proper subspace H which may be separable. On the other hand, we will take H sufficiently rich in the sense that:

$$L^{p/(p-1)}(\Omega) \otimes (\mathbb{R}^n)^* \subset H, \tag{2.1a}$$

$$H \text{ contains } h_{\mathrm{N}}(x,s) = |s|^p, \tag{2.1b}$$

$$H \text{ contains all functions } \varphi \circ y \text{ with } y \in W_0^{1,p}(\Omega), \tag{2.1c}$$

$$H \text{ contains all functions } (\varphi_r' \circ y) \cdot \tilde{y} \text{ with } y, \tilde{y} \in W_0^{1,p}(\Omega), \tag{2.1d}$$

where $L^{p/(p-1)}(\Omega) \otimes (\mathbb{R}^n)^*$ is the linear hull of the set of functions $g \otimes v$ with $g \in L^{p/(p-1)}(\Omega)$ and $v : \mathbb{R}^n \to \mathbb{R}$ linear, with $g \otimes v$ defined as usual by $[g \otimes v](x,s) = g(x)\,v(s)$. Moreover, $[\varphi \circ y](x,s) = \varphi(x, y(x), s)$ in (2.1c), and similarly $[\varphi_r' \circ y](x,s) = \varphi_r'(x, y(x), s) \in \mathbb{R}^n$ in (2.1d). Note that (2.1a) ensures that i_H is injective and (2.1b) ensures that every net in $L^p(\Omega; \mathbb{R}^n)$, whose image via i_H is weakly* convergent, is eventually bounded in $L^p(\Omega; \mathbb{R}^n)$. Then it is reasonable to put

$$Y_H^p(\Omega; \mathbb{R}^n) = \text{w*-cl}\; i_H\left(L^p(\Omega; \mathbb{R}^n)\right). \tag{2.2}$$

The set $Y_H^p(\Omega; \mathbb{R}^n)$ endowed by the weak* topology is convex, closed, locally compact subset of H^* into which the original space $L^p(\Omega; \mathbb{R}^n)$ is imbedded homeomorphically and densely. Thus $Y_H^p(\Omega; \mathbb{R}^n)$ is a very natural hull of the original Lebesgue space $L^p(\Omega; \mathbb{R}^n)$, indeed. Likewise the classical Young measures can be considered as linear continuous functionals on H if $H = L^1(\Omega; C_0(\mathbb{R}^n))$, it is natural here to address the elements of $Y_H^p(\Omega; \mathbb{R}^n)$ as generalized Young functionals if $H \subset \mathrm{Car}^p(\Omega, \mathbb{R}^n)$ is general.

We say that a particular generalized Young functional $\eta \in Y_H^p(\Omega; \mathbb{R}^n)$ is p-nonconcentrating if there is a net $\{u_\xi\}_{\xi \in \Xi}$ bounded in $L^p(\Omega; \mathbb{R}^n)$ such that w*-$\lim_{\xi \in \Xi} i_H(u_\xi) = \eta$ and the set $\{|u_\xi|^p;\ \xi \in \Xi\}$ is relatively weakly compact in $L^1(\Omega)$. In $\eta \in Y_H^p(\Omega; \mathbb{R}^n)$ is p-nonconcentrating, there exists a Young measure $\nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^n)$ (not determined uniquely in general) such that

$$\forall h \in H: \quad \langle \eta, h \rangle = \int_\Omega \int_{\mathbb{R}^n} h(x,s)\, \nu_x(\mathrm{d}s)\, \mathrm{d}x. \tag{2.3}$$

For η, $\mathring{\eta} \in Y_H^p(\Omega; \mathbb{R}^n)$, we say that $\mathring{\eta}$ is a p-nonconcentrating modification of η if $\mathring{\eta}$ is p-nonconcentrating and $\langle \mathring{\eta}, h \rangle = \langle \eta, h \rangle$ for any $h \in H$ such that $|h(x,s)| \le a(x) + o(|s|^p)$ with some $a \in L^1(\Omega)$ and $o : \mathbb{R}^+ \to \mathbb{R}$ such that $\lim_{r \to \infty} o(r)/r = 0$. If H is separable, then every $\eta \in Y_H^p(\Omega; \mathbb{R}^n)$ admits its (uniquely determined) p-nonconcentrating modification; cf. [17].

We will need a generalization of the standard construction "substitution of a Young measure $\{\nu_x\}_{x \in \Omega}$ into a Carathéodory function $h = h(x,s)$", which gives the function $x \mapsto \int_{\mathbb{R}^n} h(x,s)\, \nu_x(\mathrm{d}s)$. However, our generalized substitution will result to a measure on $\bar{\Omega}$ (=the closure of Ω) because of possible concentration effects. Let us denote this generalized substitution operation by "$\bullet$". Let us suppose that H is invariant under multiplication by functions $\Omega \to \mathbb{R}$ that admit continuous extension on $\bar{\Omega}$ in the sense that

$$\forall h \in H \;\; \forall g \in C(\bar{\Omega}): \qquad gh \in H. \tag{2.4}$$

For $h = (h_1, \dots, h_k) \in H^k$ and for $\eta \in H^*$, let us define $h \bullet \eta \in \mathcal{M}(\Omega; \mathbb{R}^k) \cong C(\bar{\Omega}; \mathbb{R}^k)^*$ by the relation

$$\langle h \bullet \eta, g \rangle = \langle \eta, g \cdot h \rangle \qquad \text{for all } g \in C(\bar{\Omega}; \mathbb{R}^k). \tag{2.5}$$

This definition actually determines $h \bullet \eta$ as a Radon measure from $\mathcal{M}(\Omega; \mathbb{R}^k)$. Indeed, $g \mapsto \langle \eta, g \cdot h \rangle : C(\bar{\Omega}; \mathbb{R}^k) \to \mathbb{R}$ is obviously linear, and its continuity follows from the continuity of $\eta : H \to \mathbb{R}$ and from the obvious estimate $\|g \cdot h\|_{\mathrm{Car}^p(\Omega, \mathbb{R}^n)} \leq \|g\|_{C(\bar{\Omega}; \mathbb{R}^k)} \|h\|_{\mathrm{Car}^p(\Omega, \mathbb{R}^n)}$. Besides, for any $u \in L^p(\Omega; \mathbb{R}^n)$, we have obviously $h \bullet i_H(u) \in L^1(\Omega; \mathbb{R}^k)$ and $[h \bullet i_H(u)](x) = h(x, u(x))$ holds for a. a. $x \in \Omega$, therefore the mapping $\eta \mapsto h \bullet \eta$ can be understood as the extension of the Nemytskiĭ mapping $L^p(\Omega; \mathbb{R}^n) \to L^1(\Omega; \mathbb{R}^k)$ generated by h. Note that the extended operator is linear with respect to the geometry of H^* while the original Nemytskiĭ mapping was generally nonlinear with respect to the "usual" geometry of $L^p(\Omega; \mathbb{R}^n)$. Moreover, the following regularity will be useful: if $\eta \in Y^p_H(\Omega; \mathbb{R}^n)$ and $h \in \mathrm{Car}^q(\Omega; \mathbb{R}^n)^k$ for some $1 \leq q < p$, then $h \bullet \eta \in L^{p/q}(\Omega; \mathbb{R}^k)$.

Now, we may define the relaxed variational problem, denoted by (RVP), as follows:

$$\text{(RVP)} \qquad \begin{cases} \text{minimize } \ \langle \eta, \varphi \circ y \rangle, \\ \text{subject to } (1 \otimes \mathrm{id}) \bullet \eta \ = \nabla y, \\ \qquad\qquad y \in W^{1,p}_0(\Omega), \quad \eta \in Y^p_H(\Omega; \mathbb{R}^n), \end{cases}$$

where $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ denotes the identity on $\mathbb{R}^n$ so that, by (2.1a), $(1 \otimes \mathrm{id}) \in H^n$ and therefore $(1 \otimes \mathrm{id}) \bullet \eta$ is well defined in $L^p(\Omega; \mathbb{R}^n)$.

Let $q \geq 1$ be arbitrary if $p > n$ and $q < np/(n - p)$ if $p \leq n$, which guarantees the compact imbedding of $W^{1,p}_0(\Omega)$ into $L^q(\Omega)$.

Obviously, (2.1c) together with $H \subset \mathrm{Car}^p(\Omega, \mathbb{R}^n)$ represents a certain restriction on φ, which forces us to suppose $\varphi : \Omega \times (\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R}$ to be a Carathéodory function such that

$$|\varphi(x, r, s)| \ \leq \ a(x) + b|r|^q + c|s|^p \tag{2.6}$$

for some $a \in L^1(\Omega)$, b, $c \in \mathbb{R}$, and q as specified above. Then $\varphi \circ u$ just belongs to $\mathrm{Car}^p(\Omega, \mathbb{R}^n)$ whenever $u \in L^q(\Omega)$. Also we suppose

$$|\varphi(x, r, s) - \varphi(x, \tilde{r}, s)| \ \leq \ (a(x) + b|r|^{q-1} + b|\tilde{r}|^{q-1} + c|s|^{p(q-1)/q})|r - \tilde{r}| \tag{2.7}$$

for some $a \in L^{q/(q-1)}(\Omega)$, and b, $c \in \mathbb{R}$. Moreover, we suppose a coercivity of φ in the sense:

$$\varphi(x, r, s) \geq a(x) + c|s|^p - b|r|^\beta \tag{2.8}$$

with some c positive, $0 \leq \beta < p$, $b \in \mathbb{R}$, and $a \in L^1(\Omega)$. Besides, $\beta < q$ may and will be supposed without loss of generality.

The following assertion verifies that (RVP) is actually a correct relaxation of (VP). It is based on the following essential assertion, derived (in a far more general vectorial form) basically by Kinderlehrer and Pedregal [10, 12]: if $\eta \in Y^p_H(\Omega;\mathbb{R}^n)$ is p-nonconcentrating and $(1\otimes\mathrm{id})\bullet\eta = \nabla y$ for some $y \in W^{1,p}_0(\Omega)$, then η can be attained by gradients in the sense that there is a sequence $\{y_k\}_{k\in\mathbb{N}} \subset W^{1,p}_0(\Omega)$ such that w*-$\lim_{k\to\infty} i_H(\nabla y_k) = \eta$. In fact, in [10, 12] this assertion is formulated in terms of Young measures, but we mentioned that every $\eta \in Y^p_H(\Omega;\mathbb{R}^n)$ which is p-nonconcentrating admits the Young-measure representation. Also, [10, 12] does not impose any trace condition, but the modification for zero-trace functions we used here is quite obvious.

Proposition 2.1. Let (2.1),(2.4), (2.6)−(2.8) be satisfied and H be separable. Then:

(i) (RVP) possesses a solution and inf(VP) = min(RVP).

(ii) If $(y,\eta) \in$ (RVP) is a solution to (RVP), then η is p-nonconcentrating.

(iii) If $\{y_k\}_{k\in\mathbb{N}}$ is a minimizing sequence for (VP), then $\{(y_k, i_H(\nabla y_k))\}_{k\in\mathbb{N}}$ has a weak* cluster point in $W^{1,p}_0(\Omega)\times H^*$ and every such a cluster point solves (RVP).

(iv) Conversely, every solution (y,η) of (RVP) can be attained by some sequence $\{(y_k, i_H(\nabla y_k))\}_{k\in\mathbb{N}}$ such that $\{y_k\}_{k\in\mathbb{N}}$ is a minimizing sequence for (VP).

Proof. First, let us define the functional $j : W^{1,p}_0(\Omega)\times Y^p_H(\Omega;\mathbb{R}^n) \to \mathbb{R}$ and the mapping $A : W^{1,p}_0(\Omega)\times Y^p_H(\Omega;\mathbb{R}^n) \to L^p(\Omega;\mathbb{R}^n)$ respectively by

$$j(y,\eta) = \langle \eta, \varphi\circ y\rangle, \tag{2.9}$$

$$A(y,\eta) = (1\otimes\mathrm{id})\bullet\eta - \nabla y. \tag{2.10}$$

By (2.6) and (2.7), the functional j is, if restricted on bounded subsets, continuous with respect to the norm topology on $L^q(\Omega)$ and the weak* topology on H^*; note that (2.7) ensures $\|\varphi\circ y_1 - \varphi\circ y_2\|_{\mathrm{Car}^p(\Omega,\mathbb{R}^n)} \le C\|y_1-y_2\|_{L^q(\Omega)}$ with C depending on $\max(\|y_1\|_{L^q(\Omega)}, \|y_2\|_{L^q(\Omega)})$. Taking into account the compactness of the imbedding $W^{1,p}_0(\Omega)\subset L^q(\Omega)$ we can see that j, restricted on bounded subsets, is (weak×weak*)-continuous. Moreover, by (2.1a), A is weakly* continuous too.

The coercivity (2.8) ensures the coercivity of the functional j on the set of admissible pairs $\{(y,\eta)\in W^{1,p}_0(\Omega)\times Y^p_H(\Omega;\mathbb{R}^n);\ A(y,\eta)=0\}$ because of the obvious estimate $j(y,\eta) \ge \|a\|_{L^1(\Omega)} + c\langle\eta, h_{\mathrm{N}}\rangle - bC\|y\|^\beta_{W^{1,p}_0(\Omega)}$ where a, b, c, and β come from (2.8), h_{N} from (2.1b), and C is the norm of the continuous imbedding $W^{1,p}_0(\Omega)\subset L^\beta(\Omega)$. As both j and A are continuous, the existence of a solution (y,η) to (RVP) is ensured by the standard compactness arguments.

By the definition of j and A, it holds $j(y, i_H(\nabla y)) = \int_\Omega \varphi(x, y(x), \nabla y(x))\,\mathrm{d}x$ and also $A(y, i_H(\nabla y)) = 0$, which shows immediately that min(RVP) $\le$ inf(VP).

Let us further show the point (ii). Suppose, for a moment, the contrary, i.e. $\mathring{\eta} \ne \eta$, where $\mathring{\eta} \in Y^p_H(\Omega;\mathbb{R}^n)$ denotes the p-nonconcentrating modification of η with

(y, η) being a solution to (RVP); here we used the assumption that H is separable. Since the integrand $1 \otimes \mathrm{id}$ has lesser growth than p (remind that $p > 1$ is supposed throughout the whole paper), we have $(1 \otimes \mathrm{id}) \bullet \mathring{\eta} = (1 \otimes \mathrm{id}) \bullet \eta$. Therefore, the couple $(y, \mathring{\eta})$ is admissible for (RVP). Besides, as the integrand $\varphi \circ y$ is coercive, we have also $(\varphi \circ y) \bullet \mathring{\eta} < (\varphi \circ y) \bullet \eta$; cf. [17] for details. It contradicts the assumption that (y, η) is a minimizer of (RVP). Thus we showed that η is inevitably p-nonconcentrating.

As mentioned previously, by [12], η can be attained by a sequence $i_H(\nabla y_k)$ with $y_k \in W_0^{1,p}(\Omega)$. Then also $\nabla y_k = (1 \otimes \mathrm{id}) \bullet i_H(\nabla y_k) \to (1 \otimes \mathrm{id}) \bullet \eta = \nabla y$ weakly in $L^p(\Omega; \mathbb{R}^n)$. As $y \in W_0^{1,p}(\Omega)$, we get also $y_k \to y$ weakly in $W_0^{1,p}(\Omega)$, hence also strongly in $L^q(\Omega)$. Altogether, we have $j(y, \eta) = \langle \eta, \varphi \circ y \rangle = \lim_{k\to\infty} \langle i_H(\nabla y_k), \varphi \circ y_k \rangle = \lim_{k\to\infty} \int_\Omega \varphi(x, y_k(x), \nabla y_k(x))\, \mathrm{d}x$, which shows that the sequence $\{y_k\}_{k\in\mathbb{N}}$ is minimizing for (RVP) because $\min(\mathsf{RVP}) \le \inf(\mathsf{VP})$.

Conversely, let us take some sequence $\{y_k\}_{k\in\mathbb{N}}$ which is minimizing for (VP). By the coercivity (2.8), this sequence must be bounded in $W_0^{1,p}(\Omega)$, and then also $\{i_H(\nabla y_k)\}_{k\in\mathbb{N}}$ is bounded in H^*. Therefore, the sequence $\{(y_k, i_H(\nabla y_k))\}_{k\in\mathbb{N}}$ must have a weak* cluster point $(y, \eta) \in W_0^{1,p}(\Omega) \times H^*$. Apparently, $\eta \in Y_H^p(\Omega; \mathbb{R}^n)$ and $A(y, \eta) = \text{w-}\lim_{k\to\infty} A(y_k, i_H(\nabla y_k)) = 0$, so that the pair (y, η) is admissible for (RVP). By the continuity of j we have also $j(y, \eta) = \lim_{k\to\infty} j(y_k, i_H(\nabla y_k)) = \inf(\mathsf{VP})$. If $j(y, \eta) > \min(\mathsf{RVP})$, we could take some solution to (RVP) and construct, as above, another sequence in $W_0^{1,p}(\Omega)$ which would reach a strictly lower value of the energy functional in (VP), which contradicts the hypothesis that $\{y_k\}_{k\in\mathbb{N}}$ is minimizing for (VP). In particular, we showed also that $\min(\mathsf{RVP}) \ge \inf(\mathsf{VP})$. □

Remark 2.1. Let us remind the recent result by Kinderlehrer and Pedregal [11, Corollary 1.4] which says in particular that, if a sequence $\{y_k\}_{k\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ is minimizing for (VP) with $\varphi(x, r, s)$ independent of r and satisfying (2.8), then the set $\{|\nabla y_k|^p;\ k \in \mathbb{N}\}$ is weakly relatively compact in $L^1(\Omega)$. Using Proposition 2.1 (iii)–(iv), this would be equivalent with the point (ii) for such a special case of φ. Nevertheless, this conclusion expects first that the situation $\min(\mathsf{RVP}) < \inf(\mathsf{VP})$ is apriori excluded, for which we just needed a bit stronger result, namely $(\varphi \circ y) \bullet \mathring{\eta} < (\varphi \circ y) \bullet \eta$ derived in [17].

3. OPTIMALITY CONDITIONS

The relaxation developed in the previous section is well fitted to isolate selective and enough informative optimality conditions for the relaxed problem. To do this, let us notice that (RVP) has got the form of an abstract optimization problem

$$
(\mathsf{P}) \qquad \begin{cases} \text{minimize } \ j(y, z), \\ \text{subject to } A(y, z) = 0, \\ \qquad\qquad y \in Y, \quad z \in K, \end{cases}
$$

where $j : Y \times Z \to \mathbb{R}$ is (under certain further data qualification) Gâteaux differentiable, $A : Y \times Z \to \Lambda$ is continuous and linear, K is closed convex subset of Z, and Y, Z, and Λ are Banach spaces. If $0 \in \operatorname{int} A(Y \times K)$, we can simply derive the necessary optimality conditions for (P). Indeed, if (y,z) is optimal for (P), then inevitably

$$\begin{aligned}\nabla j(y,z) &\in -N_{\operatorname{Ker} A \cap (Y\times K)}(y,z)\\ &= \operatorname{Range} A^* - N_{Y\times K}(y,z) = \operatorname{Range} A^* - \{0\} \times N_K(z),\end{aligned}$$

where $N_K(z) \subset Z^*$ denotes the normal cone to $K \subset Z$ at the point $z \in K$; cf. also [1, p. 175]. In other words, we can equally say that there is some $\lambda \in \Lambda^*$ such that

$$\nabla_y j(y,z) - A_y^* \lambda = 0, \tag{3.1}$$

$$\nabla_z j(y,z) - A_z^* \lambda \in -N_K(z), \tag{3.2}$$

where $A^* = (A_y^*, A_z^*) : \Lambda^* \to Y^* \times Z^*$ and $\nabla j = (\nabla_y j, \nabla_z j) : Y \times Z \to Y^* \times Z^*$. Of course, in our case $Y = W_0^{1,p}(\Omega)$, $Z = H^*$, $\Lambda = L^p(\Omega; \mathbb{R}^n)$, $K = Y_H^p(\Omega;\mathbb{R}^n)$, and j and A are defined respectively by (2.9) and (2.10); thus we will write again "η" instead of "z".

We already mentioned that $K = Y_H^p(\Omega;\mathbb{R}^n)$ is convex and closed. Our further task is to verify the above required properties of j and A and to establish the particular form of ∇j, A^*, and N_K. Obviously, (2.1d) together with $H \subset \operatorname{Car}^p(\Omega,\mathbb{R}^n)$ is a requirement on φ_r', which forces us to suppose $\varphi_r' : \Omega \times (\mathbb{R}\times\mathbb{R}^n) \to \mathbb{R}$ to be a Carathéodory function such that

$$|\varphi_r'(x,r,s)| \le a(x) + b|r|^{q-1} + c|s|^{p(q-1)/q} \tag{3.3}$$

for some $a \in L^{q/(q-1)}(\Omega)$ and $b,\ c < +\infty$. Note that (3.3) is designed just to guarantee $(\varphi_r' \circ y)\cdot\tilde{y} \in \operatorname{Car}^p(\Omega,\mathbb{R}^n)$ whenever $y, \tilde{y} \in L^q(\Omega)$. Moreover, for $q \ge 2$ we will suppose that

$$|\varphi_r'(x,r,s) - \varphi_r'(x,\tilde{r},s)| \le \left(a(x) + b|r|^{q-2} + b|\tilde{r}|^{q-2} + |s|^{p(q-2)/q}\right)|r - \tilde{r}| \tag{3.4}$$

with some $a \in L^{q/(q-2)}(\Omega)$ and $b,\ c < +\infty$. Note that no smoothness of $\varphi(x,r,\cdot)$ is required. Taking into account our choice of $q < np/(n-p)$, the requirement $q \ge 2$ represents factually a mild restriction on p provided $n \ge 3$, namely $p > 2n/(n+2)$.

Lemma 3.1. Let (2.1) be valid. Then $A : W_0^{1,p}(\Omega) \times H^* \to L^p(\Omega;\mathbb{R}^n)$ defined by (2.10) is linear, continuous, and surjective in the sense $A(W_0^{1,p}(\Omega) \times Y_H^p(\Omega;\mathbb{R}^n)) = L^p(\Omega;\mathbb{R}^n)$ and, for $\lambda \in L^{p/(p-1)}(\Omega;\mathbb{R}^n)$, it holds

$$A^*\lambda \equiv (A_y^*\lambda, A_\eta^*\lambda) = (\operatorname{div}\lambda, \lambda \otimes \mathrm{id}),$$

where the divergence of λ is understood in the sense of $W^{-1,p/(p-1)}(\Omega) \cong W_0^{1,p}(\Omega)^*$ and $[\lambda \otimes \mathrm{id}](x,s) = \lambda(x)\cdot s$. Furthermore, for every $h \in H$ and $\eta \in Y_H^p(\Omega;\mathbb{R}^n)$,

$$h \in N_{Y_H^p(\Omega;\mathbb{R}^n)}(\eta) \Longleftrightarrow \langle \eta, h\rangle = \sup_{u \in L^p(\Omega;\mathbb{R}^n)} \int_\Omega h(x,u(x))\,\mathrm{d}x.$$

Finally, if also (2.4), (2.6), (2.7), (3.3), and (3.4) are valid, then j is Gâteaux differentiable with

$$\nabla j(y,\eta) \equiv (\nabla j_y(y,\eta), \nabla j_\eta(y,\eta)) \;=\; ((\varphi_r' \circ y) \bullet \eta, \varphi \circ y) \in W^{-1,p/(p-1)}(\Omega) \times H.$$

Proof. By (2.1a), we can easily verify the estimate $\|g \otimes \mathrm{id}\|_{\mathrm{Car}^p(\Omega,\mathbb{R}^n)} \le c\|g\|_{L^{p/(p-1)}(\Omega;\mathbb{R}^n)}$, which makes the mapping $\eta \mapsto (1 \otimes \mathrm{id}) \bullet \eta$ defined and (weak*, weak)-continuous from H^* to $L^p(\Omega;\mathbb{R}^n)$. Thus A is well defined on $W_0^{1,p}(\Omega) \times H^*$, too. In view of (2.10) and by the Green formula, we have

$$\langle A^*\lambda, (y,\eta)\rangle = \langle \lambda, A(y,\eta)\rangle = \langle \lambda, (1 \otimes \mathrm{id}) \bullet \eta\rangle - \langle \lambda, \nabla y\rangle = \langle \eta, \lambda \otimes \mathrm{id}\rangle + \langle \mathrm{div}\,\lambda, y\rangle$$

for all $(y,\eta) \in W_0^{1,p}(\Omega) \times H^*$ and $\lambda \in L^{p/(p-1)}(\Omega;\mathbb{R}^n)$. This means just $A^*\lambda = (\mathrm{div}\,\lambda, \lambda \otimes \mathrm{id})$. The surjectivity of A is obvious because for $\eta = i_H(u)$ with $u \in L^p(\Omega;\mathbb{R}^n)$ arbitrary we have $A(0,\eta) = (1 \otimes \mathrm{id}) \bullet \eta = u$.

The condition $h \in N_{Y_H^p(\Omega;\mathbb{R}^n)}(\eta)$ means just $\langle \tilde\eta - \eta, h\rangle \le 0$ for every $\tilde\eta \in Y_H^p(\Omega;\mathbb{R}^n)$. By the definition of $Y_H^p(\Omega;\mathbb{R}^n)$, this is equivalent with $\langle \eta, h\rangle \ge \langle i_H(u), h\rangle = \int_\Omega h(x,u(x))\,dx$ for every $u \in L^p(\Omega;\mathbb{R}^n)$, hence also with $\langle \eta, h\rangle \ge \sup_{u\in L^p(\Omega;\mathbb{R}^n)} \int_\Omega h(x,u(x))\,dx$. This means basically the equality because $\eta \in Y_H^p(\Omega;\mathbb{R}^n)$ is weakly* attainable by some net $\{i_H(u_\xi)\}$ so that, for any $\varepsilon > 0$, there is some ξ_ε such that $\langle \eta, h\rangle - \varepsilon \le \langle i_H(u_{\xi_\varepsilon}), h\rangle = \int_\Omega h(x,u_{\xi_\varepsilon}(x))\,dx \le \sup_{u\in L^p(\Omega;\mathbb{R}^n)} \int_\Omega h(x,u(x))\,dx$.

Let us calculate the Gâteaux differential of j at (y,η). Thanks to (3.3) and (3.4), for any $\tilde y \in W_0^{1,p}(\Omega)$, one can after short calculations obtain the estimate $|\varepsilon^{-1}[\varphi \circ (y+\varepsilon\tilde y) - \varphi \circ y] - (\varphi_r' \circ y)\cdot\tilde y| \le \frac{1}{2}|\varepsilon|(a(x) + (q-2)b(|y|^{q-2} + |\tilde y|^{q-2}) + c|s|^{p(q-2)/q})|\tilde y|^2$ with a, b, and c from (3.4), from which one gets

$$\left\| \frac{\varphi \circ (y+\varepsilon\tilde y) - \varphi \circ y}{\varepsilon} - (\varphi_r' \circ y)\cdot \tilde y \right\|_{\mathrm{Car}^p(\Omega,\mathbb{R}^n)} \le C|\varepsilon|$$

with a suitable C depending, in particular, on $\tilde y$ but not on ε. Then, for $\tilde\eta \in H^*$, we can simply calculate:

$$\begin{aligned}
\langle \nabla j(y,\eta), (\tilde y, \tilde\eta)\rangle &= \lim_{\varepsilon\to 0} \frac{j(y+\varepsilon\tilde y, \eta+\varepsilon\tilde\eta) - j(y,\eta)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \left[\left\langle \eta, \frac{\varphi \circ (y+\varepsilon\tilde y) - \varphi \circ y}{\varepsilon} \right\rangle + \langle \tilde\eta, \varphi \circ (y+\varepsilon\tilde y)\rangle \right] \\
&= \langle \eta, (\varphi_r' \circ y)\cdot\tilde y\rangle \;+\; \langle \tilde\eta, \varphi \circ y\rangle,
\end{aligned}$$

where we also used the continuity of the mapping $y \mapsto \varphi \circ y : L^q(\Omega) \to \mathrm{Car}^p(\Omega,\mathbb{R}^n)$ which is guaranteed by (2.7). By the definition (2.5), employing also (2.4), we get eventually $\nabla j_y(y,\eta) = (\varphi_r' \circ y) \bullet \eta$ from the first term; note that (3.3) implies the mapping $\tilde y \mapsto (\varphi_r' \circ y)\tilde y : W_0^{1,p}(\Omega) \mapsto H$ continuous, which eventually guarantees $(\varphi_r' \circ y) \bullet \eta \in W^{-1,p/(p-1)}(\Omega)$. The second term gives immediately $\nabla j_\eta(y,\eta) = \varphi \circ y$. □

Now we can readily formulate the first-order optimality conditions for the relaxed variational problem (RVP). They include the so-called integral maximum principle, known also from optimal control theory, which involves here the "Hamiltonian" $\mathcal{H}_{y,\lambda} \in H$ defined by $\mathcal{H}_{y,\lambda} = -\varphi \circ y + \lambda \otimes \mathrm{id}$, that means

$$\mathcal{H}_{y,\lambda}(x,s) = -\varphi(x,y(x),s) + \lambda(x)\cdot s.$$

Note that (2.1a) and (2.1c) just guarantee $\mathcal{H}_{y,\lambda} \in H$ for any $y \in W_0^{1,p}(\Omega)$ and $\lambda \in L^{p/(p-1)}(\Omega;\mathbb{R}^n)$.

Proposition 3.1. Let $q \geq 2$, and (2.1), (2.4), (2.6), (2.7), (3.3), and (3.4) be valid. If (y,η) solves (RVP), then there is $\lambda \in L^{p/(p-1)}(\Omega;\mathbb{R}^n)$ such that

$$\operatorname{div}\lambda \;-\; (\varphi_r' \circ y)\bullet\eta = 0 \qquad \text{in the sense of } W^{-1,p/(p-1)}(\Omega), \tag{3.5}$$

and

$$\langle \eta, \mathcal{H}_{y,\lambda}\rangle = \sup_{u\in L^p(\Omega;\mathbb{R}^n)} \int_\Omega \mathcal{H}_{y,\lambda}(x,u(x))\,\mathrm{d}x. \tag{3.6}$$

Conversely, if j is convex and $(y,\eta) \in \left(W_0^{1,p}(\Omega)\times Y_H^p(\Omega;\mathbb{R}^n)\right) \cap \operatorname{Ker} A$ and $\lambda \in L^{p/(p-1)}(\Omega;\mathbb{R}^n)$ satisfy (3.5)–(3.6), then (y,η) solves (RVP).

Proof. It suffices just to observe that (3.5) and (3.6) are obtained respectively when the results from Lemma 3.1 are put into (3.1) and (3.2). The converse implication follows from the sufficiency of the optimality conditions (3.1) and (3.2) in case j is convex. □

Let us still remark that j is convex with respect to the geometry of $W_0^{1,p}(\Omega)\times H^*$ if, e. g., $\varphi(x,r,s) = \varphi_0(x,r)+\varphi_1(x,s)$ with $\varphi_0(x,\cdot)$ convex. The convexity of j by no means represents any requirement on a convexity of $\varphi(x,r,\cdot)$. On the other hand, convexity of $\varphi(x,\cdot,s)$ itself does not guarantee the convexity of j.

Our next task is to localize the integral maximum principle (3.6) into the particular space instances, which will basically result to the extended Weierstrass condition. This procedure is parallel to what is done in optimal control theory within the passage from the integral maximum principle to the so-called Pontryagin maximum principle; cf., e. g., [7].

Proposition 3.2. Let (2.1), (2.6) and (2.8) be valid and η be p-nonconcentrating (which holds, in particular, for the solutions to (RVP)). Then (3.6) is equivalent with

$$[\mathcal{H}_{y,\lambda}\bullet\eta](x) = \max_{\mathbb{R}^m}\mathcal{H}_{y,\lambda}(x,\cdot) \qquad \text{for a. a. } x\in\Omega. \tag{3.7}$$

Moreover, (3.7) can be also understood as an equality of two functions in $L^1(\Omega)$.

Proof. By (2.8), $\varphi(x,s,\cdot)$ has a superlinear growth and thus $-\mathcal{H}_{y,\lambda}$ is coercive, so that the supremum of $\mathcal{H}_{y,\lambda}$ is actually attained, which authorizes us to write "max" in (3.7).

Let us prove the implication (3.7) $\Rightarrow$ (3.6). As η is supposed p-nonconcentrating, the left-hand side of (3.7) lives in $L^1(\Omega)$ and we can integrate it over Ω, which gives $\int_\Omega [\mathcal{H}_{y,\lambda} \bullet \eta](x)\,dx = \int_\Omega \max_{\mathbb{R}^m} \mathcal{H}_{y,\lambda}(x,\cdot)\,dx \geq \sup_{u\in L^p(\Omega;\mathbb{R}^n)} \int_\Omega \mathcal{H}_{y,\lambda}(x,u(x))\,dx$. Furthermore, taking a net $\{u_\xi\}$ such that $i_H(y_\xi) \to \eta$ weakly* in H^* and passing to the limit in the obvious estimate

$$\sup_{u\in L^p(\Omega;\mathbb{R}^n)} \int_\Omega \mathcal{H}_{y,\lambda}(x,u(x))\,dx \geq \int_\Omega \mathcal{H}_{y,\lambda}(x,u_\xi(x))\,dx,$$

we obtain the converse inequality

$$\sup_{u\in L^p(\Omega;\mathbb{R}^n)} \int_\Omega \mathcal{H}_{y,\lambda}(x,u(x))\,dx \geq \lim_{\xi} \int_\Omega \mathcal{H}_{y,\lambda}(x,u_\xi(x))\,dx = \int_\Omega [\mathcal{H}_{y,\lambda} \bullet \eta](x)\,dx.$$

Thus (3.6) has been proved.

Let us prove the converse implication, supposing that (3.7) does not hold. This means $f(x) \equiv [\mathcal{H}_{y,\lambda} \bullet \eta](x) < \max_{\mathbb{R}^m} \mathcal{H}_{y,\lambda}(x,\cdot) \equiv g(x)$ for $x \in \Omega^+ \subset \Omega$ with $\operatorname{meas}(\Omega^+) > 0$. Let $\varepsilon > 0$ be arbitrary, and define the set-valued mapping $S_\varepsilon : \Omega \rightrightarrows \mathbb{R}^n$ by $S_\varepsilon(x) = \{s \in \mathbb{R}^n;\ \mathcal{H}_{y,\lambda}(x,s) \geq g(x) - \varepsilon\}$. It is obvious that S_ε has non-empty and closed values. By using [2, Theorems 8.1.3 and 8.1.4 with Lemma 8.2.6], we can see that S_ε is measurable and possesses some measurable (single-valued) selection $u_\varepsilon : \Omega \to \mathbb{R}^n$, i.e. $u_\varepsilon(x) \in S_\varepsilon(x)$. Every such selection must fulfil $\varphi(x,y,u_\varepsilon) \leq \lambda(x)\cdot u_\varepsilon - g(x) + \varepsilon$. By (2.8), it is easy to estimate

$$\begin{aligned} c|u_\varepsilon|^p &\leq -a(x) + b|y|^\beta + \lambda(x)\cdot u_\varepsilon - g(x) + \varepsilon \\ &\leq -a(x) + b|y|^\beta + C|\lambda(x)|^{p/(p-1)} + \frac{c}{2}|u_\varepsilon|^p + \varphi(x,y,0) + \varepsilon \end{aligned}$$

with a, b, c, and β from (2.8), and C large enough in dependence on p and c. In particular, $u_\varepsilon \in L^p(\Omega;\mathbb{R}^n)$ because $c > 0$, $\lambda \in L^{p/(p-1)}(\Omega;\mathbb{R}^n)$, $|y|^\beta \in L^{q/\beta}(\Omega) \subset L^1(\Omega)$, and $[\varphi \circ y](\cdot,0) \in L^1(\Omega)$. It is easy to see that $\int_\Omega \mathcal{H}_{y,\lambda}(x,u_\varepsilon(x))\,dx \geq \int_\Omega f(x)\,dx + \int_\Omega [g-f](x)\,dx - \varepsilon\operatorname{meas}(\Omega)$. Since $\int_\Omega [g-f](x)\,dx = \int_{\Omega^+} [g-f](x)\,dx$ is positive and ε can be made arbitrarily small, we get eventually $\langle \eta, \mathcal{H}_{y,\lambda}\rangle < \int_\Omega \mathcal{H}_{y,\lambda}(x,u_\varepsilon(x))\,dx$, and thus (3.6) cannot be valid – a contradiction. □

Remark 3.1. Note that, by Proposition 2.1 (ii), every solution (y,η) to (RVP) has the component η p-nonconcentrating, and therefore η admit the Young-measure representation $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^n)$ in the sense (2.3). Putting such representation to (RVP), one just gets (RVP_0). Likewise, (3.5) and (3.7) result respectively to (1.4) and (1.6). This justifies the notation of Section 1. Also, (1.6) means that $\operatorname{supp}(\nu_x) \subset \operatorname{Arg\,max}\mathcal{H}_{y,\lambda}(x,\cdot)$ for a. a. $x \in \Omega$, which makes sometimes possible to estimate the number of atoms of ν_x.

Remark 3.2. The requirements (2.1ab) and (2.4) are not much restrictive. Note that $H = \operatorname{Car}^p(\Omega,\mathbb{R}^n)$ apparently fulfils all of them but such a large H is not separable, which does not fit with the assumptions of Proposition 2.1. Therefore, a better example might be $H \subset \mathrm{Car}^p(\Omega, \mathbb{R}^n)$ consisting of functions of the form $h(x,z) = h_0(x,z)(1+|z|^p) + h_1(x, z/|z|)|z|^p$ with $h_0(x,\cdot)$ vanishing at infinity and $h_1(x,\cdot) \in C(S^{nm-1})$ for a. a. $x \in \Omega$, where S^{nm-1} denotes the unit sphere in $\mathbb{R}^n$. Such test functions have been essentially used by DiPerna and Majda [6]. Of course, we could find many other subspaces of $\mathrm{Car}^p(\Omega, \mathbb{R}^n)$ that would satisfy these requirements but smaller subspaces impose via (2.1cd) a stronger restriction on the generality of φ. Also, H cannot consist only of functions with growth strictly less than p because (2.8) with (2.1c) could not be fulfilled simultaneously; besides, (2.1b) would not be valid, either.

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REFERENCES

[1] J.-P. Aubin and I. Ekeland: Applied Nonlinear Analysis. Wiley, New York 1984.

[2] J.-P. Aubin and H. Frankowska: Set-valued Analysis. Birkhäuser 1990.

[3] J. M. Ball and R. D. James: Fine phase mixtures as minimizers of energy. Arch. Rational Mech. Anal. *100* (1988), 13–52.

[4] J. M. Ball and R. D. James: Proposed experimental tests of a theory of fine microstructure and the two-well problem. Philos. Trans. Roy. Soc. London Ser. A *338* (1992), 389–450.

[5] M. Chipot and D. Kinderlehrer: Equilibrium configurations of crystals. Arch. Rational Mech. Anal. *103* (1988), 237–277.

[6] R. J. DiPerna and A. J. Majda: Oscillations and concentrations in weak solutions of the incompressible fluid equations. Commun. Math. Physics *108* (1987), 667–689.

[7] R. V. Gamkrelidze: Principles of Optimal Control Theory (in Russian). Tbilisi Univ. Press, Tbilisi 1977. (English Translation: Plenum Press, New York 1978).

[8] K.-H. Hoffmann and T. Roubíček: Optimal control of a fine structure. Appl. Math. Optim. *30* (1994), 113–126.

[9] A. D. Ioffe and V. M. Tihomirov: Extensions of variational problems. Trans. Moscow Math. Soc. *18* (1968), 207–273.

[10] D. Kinderlehrer and P. Pedregal: Characterizations of Young measures generated by gradients. Arch. Rational Mech. Anal. *115* (1991), 329–365.

[11] D. Kinderlehrer and P. Pedregal: Weak convergence of integrands and the Young measure representation. SIAM J. Math. Anal. *23* (1992), 1–19.

[12] D. Kinderlehrer and P. Pedregal: Gradient Young measures generated by sequences in Sobolev spaces. J. Geom. Anal. *4* (1994), 59–90.

[13] M. Kružík and T. Roubíček: Explicit characterization of L^p-Young measures. J. Math. Anal. Appl. *198* (1996), 830–843.

[14] E. J. McShane: Necessary conditions in the generalized-curve problems of the calculus of variations. Duke Math. J. 7 (1940), 1–27.

[15] T. Roubíček: Convex compactifications and special extensions of optimization problems. Nonlinear Anal. *16* (1991), 1117–1126.
[16] T. Roubíček: A note about optimality conditions for variational problems with rapidly oscillating solutions. In: Progress in Partial Differential Equations: Calculus of Variations, Applications (C. Bandle et al, eds., Pitman Res. Notes in Math. Sci. 267), Longmann, Harlow, Essex 1992, pp. 312–314.
[17] T. Roubíček: Nonconcentrating generalized Young functionals. Comment. Math. Univ. Carolin. *38* (1997), 91–99.
[18] T. Roubíček and K.-H. Hoffmann: About the concept of measure–valued solutions to distributed parameter systems. Math. Methods Appl. Sci. *18* (1995), 671–685.
[19] T. Roubíček and K.-H. Hoffmann: Theory of convex local compactifications with applications to Lebesgue spaces. Nonlinear Anal. *25* (1995), 607–628.
[20] K. Weierstraß: Mathematische Werke, Teil VII "Vorlesungen über Variationsrechnung". (Nachdr. d. Ausg., Leipzig, 1927) Georg Olms, Verlagsbuchhandlung, Hildesheim 1967.
[21] L. C. Young: Necessary conditions in the calculus of variations. Acta Math. *69* (1938), 239–258.
[22] L. C. Young: Lectures on the Calculus of Variations and Optimal Control Theory. W. B. Saunders, Philadelphia 1969.
[23] E. Zeidler: Nonlinear Functional Analysis and its Applications III. Variational Methods and Optimization. Springer, New York 1985.

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