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NON ADDITIVE ORDINAL RELATIONS REPRESENTABLE BY LOWER OR UPPER PROBABILITIES

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We characterize (in terms of necessary and sufficient conditions) binary relations representable by a lower probability. Such relations can be non-additive (as the relations representable by a probability) and also not "partially monotone" (as the relations representable by a belief function).

Moreover we characterize relations representable by upper probabilities and those representable by plausibility. In fact the conditions characterizing these relations are not immediately deducible by means of "dual" conditions given on the contrary events, like in the numerical case.

1. INTRODUCTION

A problem that often occurs in Artificial Intelligence is the following: the field expert (a doctor, for instance) is not actually able to give a reliable numerical evaluation of the degree of uncertainty on the relevant statements concerning a given problem. In this case one merely may state his degree of belief on a set of propositions (events) without exact quantification, but only by a suitable ordering relation. The main problem relating to ordinal relations, expressing a comparative degree of belief, is the restatement of a rule system assuring coherence of a relation, with respect to the idea translated by it (such as "not less probable than", "not less believable than" and so on). Usually such a problem is associated to the consistency of the ordinal relation with some (numerical) theoretical model. More precisely, given a numerical framework (probability, belief functions, capacity and so on), one seeks the necessary and sufficient properties for the existence of a such numerical assessment (related to the chosen framework) agreeing with the ordinal relation (see Section 2).

In the literature ordinal relations representable by probabilities ([1, 5]), belief functions ([7, 10]) and possibility functions ([3]) have been characterized.

In this paper we give a characterization (in terms of necessary and sufficient conditions) of relations agreeing with a coherent lower probability, that is a function which can be obtained as lower envelope of some sets of (de Finetti) coherent probabilities.

Contrary to the numerical context (see [9]), relations agreeing with a lower probability coincide with those agreeing with a 0-monotone function (see Proposition 2 and Theorem 1), they strictly contain relations agreeing with a belief function (see Example 1), but the latter coincide with those agreeing with 2-monotone function (see Proposition 1 and Theorem 4 in [10]). Moreover we characterize relations agreeing with an upper probability (a dual of a lower probability), those agreeing with a plausibility function (a dual of a belief function) and those agreeing with a necessity measure (a dual of a possibility measure).

We note that the comparative context is different from the numerical one with respect to the dual framework. In fact, the rules characterizing relations representable by a function (lower probability, belief, possibility and so on) can not be directly deduced by those characterizing the relations representable by the respective dual function. Indeed, the request that the relations on the contrary events satisfy some rules is not informative. Moreover, we underline that if we have some comparative relations (given for instance by a field expert), we need to test if there is some function agreeing with them, without transforming the relation into an other (completely changed with respect to the previous one).

2. CHARACTERIZING AXIOMS

Let $\mathcal{A}$ be a set of events containing the impossible event $\emptyset$ and the sure event Ω. If $\preceq$ is a binary relation defined on $\mathcal{A}$, and f is a function from $\mathcal{A}$ to $I\!R$, we say that f represents (or is agreeing with) $\preceq$ if

$$A \preceq B \Leftrightarrow f(A) \leq f(B).$$

We first introduce for a binary relation $\preceq$ the basic three axioms necessary for the existence of a capacity (not negative real function monotone with respect to $\subseteq$) representing it.

(A_0) the relation $\preceq$ is a total preorder, (that is: $\preceq$ is reflexive, transitive and defined for every pair $A, B \in \mathcal{A}$);

(A_1) $\emptyset \prec \Omega$;

(A_2) $A \subseteq B \Rightarrow A \preceq B$.

We note that the function f, which summarizes the framework chosen to manage the uncertainty (probability, lower probability, belief function and so on), has a property that characterizes it (additivity, 2-monotonicity, n-monotonicity and so on).

If we consider the ordinal relation induced on a set of events by one of these functions by putting

$$f(A) < f(B) \Rightarrow A \prec B$$

$$f(A) = f(B) \Rightarrow A \sim B$$

then we can learn that it satisfies a qualitative property which is the "comparative translation" of that characterizing the (numerical) function. In literature it is well known the additivity axiom (P) (introduced by de Finetti in [4]), which characterizes in the above sense the comparative probability

(P) $\forall A, B, C \in \mathcal{A} : A \wedge C = \emptyset = B \wedge C$ we have $A \preceq B \Leftrightarrow A \vee C \preceq B \vee C$.

We note that axioms (P), (A_0) and (A_1) imply (A_2), in the case that $\mathcal{A}$ is an algebra.

We recall that (A_0), (A_1) and (P) are not sufficient to assure the representability of $\preceq$ with a probability (as proved in [6]). In fact the necessary and sufficient condition for a relation $\preceq$, defined on any finite set of events $\mathcal{F}$, is the following (CP) (see [1]), which is the comparative version of de Finetti coherence condition (if $\mathcal{A}$ is an algebra then (CP) is equivalent to the Scott condition [8])

(CP) for any $n \in I\!N$ and for any $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{F}$, with $B_i \preceq A_i$, if, for some $r_1, \ldots, r_n > 0$

$$\sup \sum_{i=1}^{n} r_i(a_i - b_i) \leq 0 \quad \text{then it must be} \quad A_i \sim B_i \ \text{ for every } i = 1, \ldots, n$$

where a_i, b_i are the indicator functions of A_i, B_i respectively.

Indeed condition (P) does not characterize comparative belief relations, as proved in [10], where the following partial monotonicity condition (B) is introduced as characterizing axiom

(B) $\forall A, B, C \in \mathcal{A} : A \subset B$ and $B \wedge C = \emptyset$ we have $A \prec B \Rightarrow A \vee C \prec B \vee C$.

In the same paper it is proved that axioms (A_0), (A_1), (A_2), (B) are also sufficient for the existence of a belief function agreeing with $\preceq$.

We recall that a function f is called 2-monotone if it satisfies the following condition:

$$f(A \vee B) \geq f(A) + f(B) - f(A \wedge B).$$

Proposition 1. Let $\mathcal{A}$ be an algebra of events and $\preceq$ a relation on $\mathcal{A}$ induced by a 2-monotone function $f : \mathcal{A} \to I\!R$. Then $\preceq$ satisfies condition (B).

Proof. To prove the above assertion it is sufficient to consider that if f is a 2-monotone function representing $\preceq$, then we have $f(B) - f(A) > 0$ and

$$f(B \vee C) = f(A \vee B \vee C) \geq f(A \vee C) + f(B) - f(A). \qquad \square$$

Condition (B) is not indeed necessary for the existence of a lower probability representing $\preceq$, as the following Example 1 shows.

Example 1. Let $\mathcal{A}$ be the algebra generated by elementary events A, B, C, D and let us consider in $\mathcal{A}$ the following ordinal relation (elements in the same group are assessed equivalent)

$$\begin{matrix} \emptyset \\ C \\ D \\ (C \vee D) \end{matrix} \prec \begin{matrix} A \\ B \\ (B \vee D) \end{matrix} \prec \begin{matrix} (A \vee C) \\ (A \vee D) \\ (A \vee C \vee D) \end{matrix} \prec \begin{matrix} (B \vee C) \\ (B \vee C \vee D) \end{matrix} \prec \begin{matrix} (A \vee B) \\ (A \vee B \vee C) \\ (A \vee B \vee D) \end{matrix} \prec \Omega.$$

The above relation does not satisfy condition (B): in fact we have $A \subset A \vee C$ and $A \prec (A \vee C)$, but $(A \vee B) \sim (A \vee B \vee C)$.

On the contrary, it is agreeing with a lower probability $\underline{P}$ defined by putting

$$\underline{P}(\emptyset) = \underline{P}(C) = \underline{P}(D) = \underline{P}(C \vee D) = 0$$
$$\underline{P}(A) = \underline{P}(B) = \underline{P}(B \vee D) = 0.1$$
$$\underline{P}(A \vee C) = \underline{P}(A \vee D) = \underline{P}(A \vee C \vee D) = 0.2$$
$$\underline{P}(B \vee C) = \underline{P}(B \vee C \vee D) = 0.3$$
$$\underline{P}(A \vee B) = \underline{P}(A \vee B \vee C) = \underline{P}(A \vee B \vee D) = 0.8$$
$$\underline{P}(\Omega) = 1.$$

We note that the function $\underline{P}$ is a lower probability, in fact it can be obtained as lower envelope of the set of probabilities induced by the following distributions P_i

	A	B	C	D
P_1	0.2	0.8	0	0
P_2	0.1	0.7	0.1	0.1
P_3	0.7	0.2	0.1	0
P_4	0.7	0.1	0.2	0
P_5	0.4	0.4	0	0.2

We propose the following condition (L) as a characterizing axiom for comparative lower probability on an algebra $\mathcal{A}$ of events

(L) $\forall A, B \in \mathcal{A} : \emptyset \prec A$ and $A \wedge B = \emptyset$ we have $B \prec A \vee B$.

We recall that a function f is said 0-monotone if satisfies the following condition:

$$\text{if } A \wedge B = \emptyset, \quad \text{then } f(A \vee B) \geq f(A) + f(B).$$

It is immediate to prove the following Proposition 2.

Proposition 2. Let $\mathcal{A}$ be an algebra of events and $\preceq$ a relation on $\mathcal{A}$ induced by a 0-monotone function $f : \mathcal{A} \to \mathbb{R}$. Then $\preceq$ satisfies condition (L).

We note that, as discussed in the previous section, it is not immediate to derive the condition characterizing the comparative plausibility (upper probability) from that characterizing the comparative belief (lower probability), like it happens in the numerical case. In fact, if we take, following suggestion of Wong [10], the definition of comparative upper probability (plausibility) $\preceq_*$ as the dual relation of a comparative lower probability (belief) $\preceq$, we obtain

$$A \preceq_* B \Leftrightarrow B^c \preceq A^c.$$

Nevertheless these conditions do not give any explicit information about the conditions characterizing the comparative upper probability (plausibility).

Therefore it should be very difficult to test if a binary relation is a comparative upper probability (plausibility).

We introduce now the characterizing axioms for the comparative plausibility and comparative upper probability.

(PL) $\forall A, B, C \in \mathcal{A} : A \subseteq B$ and $A \sim B, B \wedge C = \emptyset$ we have $A \vee C \sim B \vee C$.

A function f is said "2-alternanting" if it satisfies the following condition

$$f(A \vee B) \leq f(A) + f(B) - f(A \wedge B).$$

Proposition 3. Let $\mathcal{A}$ be an algebra of events and $\preceq$ a relation on $\mathcal{A}$ induced by an 2-alternanting function $f : \mathcal{A} \rightarrow \mathbb{R}$. Then $\preceq$ satisfies condition (PL).

Proof. If f is an 2-alternanting function representing $\preceq$, then, for A, B, C as in condition (PL), we have

$$\begin{aligned} & f(A \vee C) \leq f(B \vee C) = f(B \vee (A \vee C)) \\ \leq \; & f(B) + f(A \vee C) - f(B \wedge (A \vee C)) = f(B) + f(A \vee C) - f(A). \end{aligned}$$

By noting that $A \sim B$ and so $f(A) = f(B)$, we have $f(A \vee C) = f(B \vee C)$. □

The previous condition (PL) does not characterize a comparative upper probability, as the Example 2 shows.

Example 2. Let $\mathcal{A}$ be the algebra generated by elementary events A, B, C, D and E, and let us consider the following relation in $\mathcal{A}$:

$$\begin{array}{c} \emptyset \\ E \end{array} \prec \begin{array}{l} A \\ D \\ A \vee E \\ D \vee E \end{array} \prec \begin{array}{l} B \\ A \vee B \\ B \vee E \\ A \vee B \vee E \end{array} \prec \begin{array}{l} C \\ C \vee E \end{array} \prec \begin{array}{l} A \vee D \\ A \vee D \vee E \end{array}$$

$$\prec \begin{array}{l} B \vee C \\ C \vee D \\ A \vee B \vee D \\ B \vee C \vee E \\ B \vee D \vee E \end{array} \quad \begin{array}{l} B \vee D \\ A \vee C \\ A \vee B \vee D \vee E \\ A \vee C \vee E \\ C \vee D \vee E \end{array} \prec \begin{array}{l} A \vee B \vee C \\ A \vee B \vee C \vee E \end{array}$$

$$\prec \begin{array}{l} A \vee C \vee D \\ A \vee C \vee D \vee E \end{array} \prec \begin{array}{l} B \vee C \vee D \\ B \vee C \vee D \vee E \\ A \vee B \vee C \vee D \\ \Omega \end{array} .$$

This relation does not satisfy the condition (PL): in fact the event B is included in $A \vee B$ and $B \sim (A \vee B)$, but $(B \vee C) \prec (A \vee B \vee C)$.

On the contrary it is agreeing with the upper probability $\overline{P}$ defined by putting

$\overline{P}(\emptyset) = \overline{P}(E) = 0$
$\overline{P}(A) = \overline{P}(A \vee E) = \overline{P}(D) = \overline{P}(D \vee E) = 0.3$
$\overline{P}(B) = \overline{P}(B \vee E) = \overline{P}(A \vee B) = \overline{P}(A \vee B \vee E) = 0.4$
$\overline{P}(C) = \overline{P}(C \vee E) = 0.5$
$\overline{P}(A \vee D) = \overline{P}(A \vee D \vee E) = 0.6$
$\overline{P}(B \vee C) = \overline{P}(B \vee C \vee E) = \overline{P}(A \vee C) = \overline{P}(A \vee C \vee E) = \overline{P}(B \vee D) = \overline{P}(C \vee D)$
$= \overline{P}(B \vee D \vee E) = \overline{P}(C \vee D \vee E) = \overline{P}(A \vee B \vee D) = \overline{P}(A \vee B \vee D \vee E) = 0.7$
$\overline{P}(A \vee B \vee C) = \overline{P}(A \vee B \vee C \vee E) = 0.8$
$\overline{P}(A \vee C \vee D) = \overline{P}(A \vee C \vee D \vee E) = 0.9$
$\overline{P}(B \vee C \vee D) = \overline{P}(B \vee C \vee D \vee E) = \overline{P}(A \vee B \vee C \vee D) = \overline{P}(\Omega) = 1.$

Note that the function $\overline{P}$ is actually an upper probability: in fact we can obtain it as upper envelope of the set of probabilities defined by the following distributions P_i

	A	B	C	D	E
P_1	0.1	0.2	0.5	0.2	0
P_2	0.3	0.1	0.3	0.3	0
P_3	0	0.4	0.3	0.3	0
P_4	0.3	0.1	0.4	0.2	0

We propose the following condition (U) as characterizing axiom of comparative upper probability

(U) $\forall A, B \in \mathcal{A} : \emptyset \sim A$ we have $B \sim A \vee B$.

A function is said "0-alternanting" if it satisfies the following condition:

$$\text{if } A \wedge B = \emptyset, \quad \text{then } f(A \vee B) \leq f(A) + f(B).$$

It is immediate to prove the following Proposition 4.

Proposition 4. Let $\mathcal{A}$ be an algebra of events and $\preceq$ a relation on $\mathcal{A}$ induced by an 0-alternanting function $f : \mathcal{A} \to \mathbb{R}$. Then $\preceq$ satisfies condition (U).

A separate description is needed for the necessity and possibility theory since they have not a characterizing property like additivity, n-monotonicity and so on. The possibility relation is usually determined by the following axiom (PO) (see for example [3])

(PO) $\forall A, B, C \in \mathcal{A}$ we have $A \preceq B \Rightarrow A \vee C \preceq B \vee C$.

Its numerical counterpart, that is a particular case of plausibility function, is understood as degree of possibility by means of

$$\Pi(A) = \sup_{u \in A} \pi(u)$$

where π is a possibility distribution (membership).

In literature the following axiom (N) characterizing the dual relation (necessity) is also given (see [3])

(N) $\forall A, B, C \in \mathcal{A}$ we have $A \preceq B \Rightarrow A \wedge C \preceq B \wedge C$.

We note that, like for comparative probability, in the case that $\mathcal{A}$ is an algebra, axioms (A_0), (A_1) and (PO) (or (N)) imply (A_2). Contrary to the probability (A_0), (A_1) and (PO) (or (N)) are also sufficient to assure representability of $\preceq$, with possibility (necessity) function on a finite algebra of events (see [3]).

We introduce here an equivalent axiom (N') involving logical sums, which permits an easier comparison with the previous characterizing properties (in particular with (P), (B) and (L))

(N') $\forall A, B, H, K \in \mathcal{A} : (A \vee B) \wedge (H \vee K) = \emptyset$ we have

$$A \prec B \Rightarrow A \vee H \prec B \vee K.$$

It is easy to check that a numerical counterpart of necessity relation is

$$N(A) = \inf_{u \not\subseteq A} (1 - \pi(u))$$

and it represents a particular case of belief function.

3. REPRESENTATION THEOREMS

The following theorems prove that conditions (L), (PL), (U) are also sufficient to represent a binary relation satisfying conditions (A_0), (A_1), (A_2) respectively by comparative lower probability, comparative plausibility, comparative upper probability.

Before to show the theorems, we introduce a preliminary lemma.

Lemma 1. Let $\mathcal{A}$ be an algebra of events and let $\preceq$ be a comparative lower probability on $\mathcal{A}$. Then the following condition holds: for every pair $A_i, A_j \in \mathcal{A}$ such that $A_i \subseteq A_j$ and $A_i \sim A_j$, for every $A \subseteq A_i^c \wedge A_j$ we have $\emptyset \sim A$.

Proof. If there are A_i, A_j such that $A_i \subseteq A_j$ and $A \subseteq A_i^c \wedge A_j$ with $\emptyset \prec A$, by condition (L) and monotonicity we have that $A_i \prec A_j$. Therefore, if $A_i \subseteq A_j$ and $A_i \sim A_j$ then, for every $A \subseteq A_i^c \wedge A_j$ we have $\emptyset \sim A$. □

Theorem 1. Let $\mathcal{F}$ be a finite set of events, and $\preceq^*$ a binary relation on $\mathcal{F}$. The following conditions are equivalent:

i) there exists a binary relation $\preceq$ on the algebra $\mathcal{A}$ spanned by $\mathcal{F}$, extending $\preceq^*$, that satisfies (A_0), (A_1), (A_2), (L)

ii) there exist a coherent lower probability $\underline{P} : \mathcal{F} \to I\!R$ representing $\preceq^*$.

Proof. The proof of the implication ii) $\Rightarrow$ i) is in the Proposition 2.

We have to prove the implication i) $\Rightarrow$ ii). Let $\mathcal{F} = \{A_1, \ldots, A_m\}$ and let $\{C_1, \ldots, C_k\}$ be the set of the elementary events generated by $\mathcal{F}$, that is all possible logical products between the events and their negations. Consider the algebra $\mathcal{A} = \{A_1, \ldots, A_n\} \supset \mathcal{F}$, obtained by making all the finite logical sums of C_i.

Let $\preceq$ be a binary relation on $\mathcal{A}$, extension of $\preceq^*$ and satisfying (A_0), (A_1), (A_2), (L). By (A_0) the symmetrical part $\sim$ of $\preceq$ is an equivalence relation, then we can consider the equivalence classes of $(\mathcal{A}, \preceq)$.

Let $\mathcal{E}_0, \ldots, \mathcal{E}_l$ be these classes, where the indexes are such that if $s < k$ then for all $A_i \in \mathcal{E}_s$ and $A_j \in \mathcal{E}_k$ (so in particular $A_i \sim \emptyset$ if $A_i \in \mathcal{E}_0$, and $A_j \sim \Omega$ if $A_j \in \mathcal{E}_l$) we have $A_i \prec A_j$. By condition (A_1) follows that $l \geq 1$.

Now we can build a numerical lower probability agreeing with $\preceq$, obtained as lower envelope of a set $\{P^i\}_{i=1\ldots n}$ of probabilities.

For every $A_i \in \mathcal{A}$ we will define the probability P^i_r, where r is the index of the equivalence class of A_i. If $A_i \in \mathcal{E}_0$ we define P^i_0 as follows:

$$\begin{cases} P^i_0(A_i) & = 0 \\ P^i_0(A_h) & = \dfrac{1}{k - s_i} \sum_{C_j \subseteq A^c_i \wedge A_h} \chi_j \end{cases}$$

where k is the total number of elementary events, s_i is the number of $C_j \subseteq A_i$ and χ is the characteristic function. If $A_i \in \mathcal{E}_l$ we put

$$\begin{cases} P^i_l(A_i) & = 1 \\ P^i_l(A_h) & = \dfrac{1}{s_i} \sum_{C_j \subseteq A_i \wedge A_h} \chi_j . \end{cases}$$

If $A_i \in \mathcal{E}_r$ with $r = 1, \ldots, l-1$ we define

$$\begin{cases} P^i_r(A_i) & = \dfrac{1}{2^{(2n-r+1)n}} & & \\ P^i_r(A_h) & = 0 & & \forall A_h \in \mathcal{E}_0 \\ P^i_r(A_h) & \geq \dfrac{1}{2^n} & & \forall A_h \not\subseteq A_i \text{ and } A_h \notin \mathcal{E}_0 \\ \dfrac{1}{2^{(2n-r+2)n}} & \leq P^i_r(A_h) & \leq \dfrac{1}{2^{(2n-r+1)n}} & \forall A_h \subset A_i \text{ and } A_h \notin \mathcal{E}_r . \end{cases}$$

We will verify that the functions P_r^i are (coherent) probability. The proof is trivial for P_0^i and P_l^i. Suppose $r = 1, \ldots, l-1$. The quantity $\frac{1}{2^{(2n-r+1)n}}$ has to be distributed among elementary events $C_j \subseteq A_h \subset A_i$ so as to respect the constraints imposed. That is possible since the number of incompatible events A_h is less than $n-1$ and

$$\sum_{h=1}^{n-1} \frac{1}{2^{(2n-r+2)n}} = \frac{n-1}{2^{(2n-r+2)n}} < \frac{2^n}{2^{(2n-r+2)n}} = \frac{1}{2^{(2n-r+1)n}}.$$

Now the quantity $1 - P_r^i(A_i)$ can be distributed on the $C_j \not\subset A_i$, $C_j \notin \mathcal{E}_0$, in a way that $P_r^i(A_j) \geq \frac{1}{2^n}$, for every $A_j \not\subseteq A_i$. Since the maximum number of incompatible events satisfying the above condition, is $n-1$, then we have

$$\frac{1 - P_r^i(A_i)}{n-1} = \frac{1 - \frac{1}{2^{(2n-r+1)n}}}{n-1} > \frac{1}{2(n-1)} > \frac{1}{2^n}.$$

We will show now that, if $\underline{P} = \inf_i\{P_r^i\}$, then for every $A_i \in \mathcal{A}$

$$\underline{P}(A_i) = P_r^i(A_i)$$

for every r such that $A_i \in \mathcal{E}_r$. To show this, it is sufficient to prove that, if $A_i \in \mathcal{E}_r$, then for every $s \neq r$ we have $P_s^j(A_i) \geq P_r^i(A_i)$ for every j.

For $A_i \in \mathcal{E}_0$ the proof is trivial, since $P_0^i(A_i) = 0$. For $A_i \in \mathcal{E}_l$ it is sufficient to note that the Lemma 1 implies $\emptyset \sim A_i^c$. Therefore, since by definition $P_r^j(A_i^c) = 0$, for every P_r^j, then $P_r^j(A_i) = 1$.

For events $A_i \in \mathcal{E}_r$ with $r = 1, \ldots, l-1$, it is sufficient to prove that, for $A_h \sim A_i$ the inequality $P_r^i(A_h) \geq P_r^h(A_h)$ holds. In fact if $A_h \not\subset A_i$ then we have

$$P_r^i(A_h) \geq \frac{1}{2^n} > \frac{1}{2^{(2n-r+1)n}} = P_r^h(A_h).$$

While if $A_h \subset A_i$ by Lemma 1 we have $\emptyset \sim A_i \wedge A_h^c$ and so

$$P_r^i(A_h) = P_r^i(A_i) = \frac{1}{2^{(2n-r+1)n}} = P_r^h(A_h).$$

By monotonicity of P_r^i, with respect to the index r, it follows that the lower envelope $\underline{P}$ agrees with the binary relation. □

Lemma 2. Let $\mathcal{A}$ be an algebra of events and let $\preceq$ be a comparative upper probability on $\mathcal{A}$. Then the following condition holds: for every pair A_i, $A_j \in \mathcal{A}$ such that $A_i \subset A_j$ and $A_i \prec A_j$ there exists an event A such that $A \subseteq A_i^c \wedge A_j$ and $\emptyset \prec A$.

Proof. If there exists a pair of events A_i, A_j such that $A_i \subset A_j$, $A_i \prec A_j$ and for every event $A \subseteq A_i^c \wedge A_j$ it holds $\emptyset \sim A$, then by condition (U) we have a contradiction, that is $\emptyset \sim A_i^c \wedge A_j$, so $A_i \sim A_j$. □

Theorem 2. Let $\mathcal{F}$ be a finite set of events, $\preceq^*$ a binary relation on $\mathcal{F}$. The following conditions are equivalent:

i) there exists a binary relation $\preceq$ on the algebra $\mathcal{A}$ spanned by $\mathcal{F}$, extending $\preceq^*$, that satisfies (A_0), (A_1), (A_2), (U)

ii) there exist a coherent upper probability $\overline{P} : \mathcal{F} \longrightarrow \mathbb{R}$ representing $\preceq^*$.

Proof. The proof of the implication ii) $\Rightarrow$ i) is in Proposition 4.

We have to prove the implication $i) \Rightarrow ii)$. The line of the proof is similar to that one of Theorem 1, regarding the comparative lower probability. In fact we build a numerical upper probability agreeing $\preceq^*$ as upper envelope of a class of probability $\{P^i\}_{i=1,\dots,n}$.

Let $\{C_i\}_{i=1,\dots,k}$, $\mathcal{F}$, $\mathcal{E}_i$ $(i = 0, \dots, l)$ and $\mathcal{A}$ as in the proof of Theorem 1.

Let $\preceq$ be a binary relation on $\mathcal{A}$, extending $\preceq^*$ and satisfying (A_0), (A_1), (A_2), (U).

First of all we put $P^i_r(A_j) = 0$ if $A_j \in \mathcal{E}_0$ for every $r = 0, \dots, l$ and $i = 1, \dots, n$.

If $A_i \in \mathcal{E}_0$ (so $r = 0$) for any $A_h \notin \mathcal{E}_0$ we define P^i_0 as follows:

$$P^i_0(A_h) = \frac{1}{s} \sum\nolimits_{\substack{C_j \subseteq A_h \\ C_j \notin \mathcal{E}_0}} \chi_j$$

where s is the number of elementary events $C_j \notin \mathcal{E}_0$.

If $A_i \in \mathcal{E}_l$ we define

$$\begin{cases} P^i_l(A_h) & = 0 \qquad\qquad\qquad\qquad \text{if } A_h \wedge A_i = \emptyset \\ P^i_l(A_h) & = \dfrac{1}{s_i} \sum\nolimits_{\substack{C_j \subseteq A_i \wedge A_h \\ C_j \notin \mathcal{E}_0}} \chi_j \quad \text{if } A_h \wedge A_i \neq \emptyset \end{cases}$$

where s_i is the number of elementary events $C_j \notin \mathcal{E}_0$ and $C_j \subseteq A_i$.

It is trivial to prove that P^i_0 and P^i_l are probability distribution.

If $A_i \in \mathcal{E}_r$ with $r = 1, \dots, l - 1$

$$\begin{cases} P^i_r(A_i) & = 1 - \dfrac{1}{2^{(n+r)}} \\ P^i_r(A_h) & \leq \alpha \left(1 - \dfrac{1}{2^{(n+r)}}\right) \qquad \text{if } A_h \subset A_i \text{ and } A_h \notin \mathcal{E}_r \\ P^i_r(A_h) & = \dfrac{1}{2^{n+r}} \dfrac{1}{s - s_i} \sum\nolimits_{\substack{C_j \subseteq A_h \\ C_j \notin \mathcal{E}_0}} \chi_j \qquad \text{if } A_h \wedge A_i = \emptyset \text{ and } A_h \notin \mathcal{E}_r \end{cases}$$

where α is a number such that $\frac{1}{2} < \alpha < \frac{n-1}{n}$.

For the events A_h such that $A_h \wedge A_i = A_{h_1} \neq \emptyset$ and $A_h \wedge A_i^c = A_{h_2} \neq \emptyset$, $P^i_r(A_h)$ is given as the sum of $P^i_r(A_{h_1})$ and $P^i_r(A_{h_2})$ according the previous definition. This is well defined by Lemma 2 that ensures the existence of $C_j \subseteq A_{h_2}$ with $C_j \notin \mathcal{E}_0$.

We will show now that, if $\overline{P} = \sup_i\{P^i_r\}$, then for every $A_i \in \mathcal{A}$ it is $\overline{P}(A_i) = P^i_r(A_i)$ for r such that $A_i \in \mathcal{E}_r$. To show this, it is sufficient to prove that if $A_i \in \mathcal{E}_r$

for every $s \neq r$ we have $P_s^j(A_i) \leq P_r^i(A_i)$ for every j. The proof is trivial for $A_i \in \mathcal{E}_0$ and for $A_i \in \mathcal{E}_l$.

Suppose that $A_i \in \mathcal{E}_r$ $(r = 1, \ldots, l-1)$ and $A_h \in \mathcal{E}_s$ with $r \neq s$.

If $A_i \wedge A_h = \emptyset$ then we have

$$P_r^i(A_i) = 1 - \frac{1}{2^{n+r}} > \frac{1}{2^n} > \frac{1}{2^{n+s}} \frac{1}{s - s_h} \sum_{\substack{C_j \subseteq A_i \\ C_j \notin \mathcal{E}_0}} \chi_j = P_s^h(A_i).$$

If $A_i \subset A_h$, taking into account that $\frac{1}{2^n}\left(\frac{1}{2^r} - \frac{1}{2^s}\right) < 1 - \alpha$, we have

$$\alpha\left(1 - \frac{1}{2^{n+s}}\right) < 1 - \frac{1}{2^{n+r}}.$$

If $A_i \wedge A_h \neq \emptyset$ and $A_h \wedge A_i^c \neq \emptyset$ then to prove the inequality it is sufficient to prove this relation

$$\begin{aligned} P_r^i(A_i) &= 1 - \frac{1}{2^{n+r}} > P_s^h(A_i \wedge A_h) + P_s^h(A_i \wedge A_h^c) \\ &= \alpha\left(1 - \frac{1}{2^{n+s}}\right) + \frac{1}{2^{n+s}} \frac{1}{s - s_h} \sum_{\substack{C_j \subseteq A_i \wedge A_h^c \\ C_j \notin \mathcal{E}_0}} \chi_j \end{aligned}$$

which is equivalent to the following one:

$$\begin{aligned} 1 - \alpha &> \frac{1}{2^{n+r}} - \frac{\alpha}{2^{n+s}} + \frac{1}{2^{n+s}} \frac{1}{s - s_h} \sum_{\substack{C_j \subseteq A_i \wedge A_h^c \\ C_j \notin \mathcal{E}_0}} \chi_j \\ &= \frac{1}{2^n}\left(\frac{1}{2^r} - \frac{\alpha}{2^s} + \frac{1}{2^s} \frac{1}{s - s_h} \sum_{\substack{C_j \subseteq A_i \wedge A_h^c \\ C_j \notin \mathcal{E}_0}} \chi_j\right). \end{aligned}$$

The last inequality is verified since the second term is less than $\frac{1}{2^n}$ and $1 - \alpha \geq \frac{1}{2^n}$.

The monotonicity of P_r^i with respect to index r assures that $\overline{P}$ is agreeing with $\preceq$ and so $\overline{P}\,|_{\mathcal{F}}$ is agreeing with $\preceq^*$. □

For comparative plausibilities we can prove a theorem analogous to Theorem 2.

Theorem 3. Let $\mathcal{F}$ be a finite set of events, an $\preceq^*$ a binary relation on $\mathcal{F}$. The following conditions are equivalent:

i) there exists a binary relation $\preceq$ on the algebra $\mathcal{A}$ spanned by $\mathcal{F}$, extending $\preceq^*$, that satisfies (A_0), (A_1), (A_2), (PL).

ii) there exist a coherent plausibility $Pl : \mathcal{F} \to I\!R$ representing $\preceq^*$.

P r o o f. To prove ii) ⇒ i) it is sufficient to note that if $A_i \subset A_j$ and $A_i \sim A_j$ then the subadditivity property of plausibility implies that for all A_r such that $A_j \wedge A_r = \emptyset$ we have

$$Pl(A_i \vee A_r) \leq Pl(A_j \vee A_r) \leq Pl(A_j) + Pl(A_i \vee A_r) - Pl(A_i) = Pl(A_i \vee A_r).$$

The proof i) ⇒ ii) follows the line of that one made by Wong in [10] for comparative belief, taking into account the following Proposition 5. □

Proposition 5. Let $\mathcal{A}$ be an algebra of events.

i) If $A \subset B$ and $A \sim B$ then for all $D \subset A^c \wedge B$ we have $D^c \sim \Omega$.

ii) If $A \subset B$, $A \sim B$ and $D = H \vee K$ with $H \subset A^c \wedge B$ and $K \subset B^c$ then we have $D^c \sim D^c \vee H$.

Proof. Let A, B be in the condition of proposition. If $D \subset A^c \wedge B$ then we have $A \subset A \vee (B \wedge D^c) \subset B$ and $A \vee (B \wedge D^c) \sim B$.

It implies the thesis 1 of the proposition, because $D^c = A \vee (B \wedge D^c) \vee B^c \sim \Omega$.

Moreover note that if D is in the condition of part 2 then we have, since $A \subset B \wedge H^c$, that $B \wedge H^c \sim B$, so $D^c = (B \wedge H^c) \vee (B^c \wedge K^c) \sim B \vee (B^c \wedge K^c) = D^c \vee H$. □

4. CONCLUSION

As a consequence of the results shown in this paper, we could stress that some numerical uncertainty measures "collapse" in the same comparative structure (i.e. they are characterized by the same axioms).

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