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AGGREGATION OF NETWORKS

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We investigate a problem how to reduce the size of large scale networks which arise as models in project management studies. We propose a method for aggregation of nodes and arcs preserving all characteristics important for finding an optimal project management scheme.

1. INTRODUCTION

Networks modelling has proved to be very efficient tool for project management. Unfortunately investment projects include so much activities that the analysis of extensive networks or sets of mutually linked networks with numerous nodes and arcs and extensive amount of information posses some difficulties. Some project analysis should preferably use the networks with fewer arcs and nodes. These aggregated networks (with fewer arcs and nodes) need to be found to replace the original ones.

The networks aggregation proposed in the paper is based on a decomposition of the building project (network) into specific closed units – the set of activities forming one complete part. These complete parts have to be defined for each project at the same time as the network itself and each node has to be assigned to a complete part. Each complete part is replaced by a new activity in the aggregated network. Links between complete parts have to be newly defined.

2. METHOD OF AGGREGATION

Let us have a network $G = \{V, E\}$ where $V = \{v_1, v_2, \dots, v_n\}$ is a set of vertexes (nodes), E is a set of of arcs (activities, edges). G is a connected direct noncyclic graph arcs weighted by $h_{ij} \geq 0$, $(v_i, v_j) \in E$ (time duration of the activity (v_i, v_j)).

Definition. $T = \{t_{ij}; (v_i, v_j) \in E\}$ is called a feasible solution of G if it holds

$$t_{ij} + h_{jk} \leq t_{jk} \tag{1}$$

for each two adjacent arcs $(v_i, v_j), (v_j, v_k) \in E$. The number t_{ij} will be called the start term of the activity (v_i, v_j).

We shall construct a graph $G^a = \{V^a, E^a)$ called aggregated graph. It is based on:

1. the decomposition of the set V into disjunct subsets $\{V_r\}$, $r = 1, 2, \ldots, R$, $\cup V_r = V$ and $V_r \cap V_s = \emptyset$, $r \neq s$,
2. given feasible solution $T^0 = \{t^0_{ij}\}$.

First we denote partial subgraphs $G_r = \{V_r, E_r\}$, where

$$E_r = \{(v_i, v_j) \in E; v_i, v_j \in E_r\},$$

G_r will be denoted as the complete part of G. We determine initial node v_{p_r} and final node v_{q_r} of G such that $v_{p_r}, v_{q_r} \in V_r$ and indegree $v_{p_r} = 0$ and outdegree $v_{q_r} = 0$ on G_r.

We shall suppose that for each G_r:

1. there is no arc $(v_{p_r}, v_j) \in E$ such that $v_j \notin V_r$,
2. there is no arc $(v_i, v_{q_r}) \in E$ such that $v_i \notin V_r$,
3. $v_{p_r} \neq v_{q_r}$, $\quad r = 1, 2, \ldots, R$.

These conditions can be fulfilled in each network (without loss of generality) if we use certain modifications of the network.

Further we denote $t^0_i = \min t^0_{ij}$, $(v_i, v_j) \in E$ the earliest start term of all activities which origin in v_i (in the solution T^0) and $\bar{t}^0_i = \max t^0_{ki} + h_{ki}$, $(v_k, v_i) \in E$ the last term of finishing all activities ending in the node v_i. From (1) we get

$$\bar{t}^0_i \leq t^0_i \quad \text{for } i = 1, 2, \ldots, n. \tag{2}$$

Definition. The aggregated graph $G^a = \{V^a, E^a\}$, where $V^a = \{v_{p_r}, v_{q_r}; r = 1, 2, \ldots, R\}$, $E^a = E^a_1 \cup E^a_2$, $E^a_1 = \{(v_{p_r}, v_{q_r}); r = 1, 2, \ldots, R\}$, $E^a_2 = \{(v_{q_r}, v_{p_s})$; if it exists the arc $(v_i, v_j) \in E$ such that $v_i \in V_r$ and $v_j \in V_s$, $r \neq s\}$. We shall call E^a_1 complete parts and E^a_2 linkages between complete parts.

The set V^a contains only initial and end nodes of all complete parts of G.

Now we determine the weights of arcs of E^a:
if $(v_{p_r}, v_{q_r}) \in E^a_1$ the weight is defined as

$$d_r = \bar{t}^0_{p_r} - t^0_{q_r} \tag{3}$$

if $(v_{q_r}, v_{p_s}) \in E^a_2$ the weight is defined as

$$w^{ij}_{rs} = \bar{t}^0_{p_s} - t^0_{q_r} - s_{ij} \tag{4}$$

where s_{ij} is a slack of the arc (v_i, v_j) in the solution T^0 defined as

$$s_{ij} = \bar{t}^0_j - t^0_i - h_{ij}. \tag{5}$$

The weight d_r means the time duration of the G_r in time solution T^0, the weight w^{ij}_{rs} is the time distance between the start term of G_s and the finish term of G_r in T^0 which can be shorted on the slack of the associated arc $(v_i, v_j) \in E$.

3. PROPERTIES OF THE AGGREGATED GRAPH

The weighted graph G^a has the following properties.

Lemma 1. The graph G^a is connected.

Proof. The proof follows directly from connectivity of G. □

Lemma 2. The graph G^a can contain a cycle but each cycle has a nonpositive length.

Proof. We denote a cycle in G^a by

$$(v_{p_1}, v_{q_1}), (v_{q_1}, v_{p_2}), (v_{p_2}, v_{q_2}), \ldots, (v_{p_m}, v_{q_m}), (v_q, v_{p_1}).$$

Along the cycle arcs must alternate – complete parts and distance arcs – linkages between two complete parts. This follows from the fact that the sets V_r, $r = 1, 2, \ldots, R$ are disjunct. The length of the cycle is

$$L = d_1 + w_{12} + d_2 + \ldots + d_m + w_{m,1}. \tag{6}$$

Substituting to (6) from (3) and (4) we obtain

$$\begin{aligned} L &= (\bar{t}^0_{q_1} - t^0_{p_1}) + (t^0_{p_2} - \bar{t}^0_{q1} - s_{12}) + (\bar{t}^0_{q_2} - t^0_{p_2}) + \ldots \\ &\quad \ldots + (\bar{t}^0_{q_m} - t^0_{q_m}) + (t^0_{p_1} - \bar{t}^0_{q_m} - s_{m,1}) \\ &= -s_{12} - s_{23} - \ldots - s_{m,1} \le 0 \end{aligned}$$

where $s_{ij} \ge 0$ is a slack defined in (5). □

Lemma 3. Put $t^a_{p_r q_r} = t^0_{p_r}$ and $t^a_{q_r p_s} = \bar{t}^0_{q_r}$, $r = 1, 2 \ldots, R$. Then $T^a = \{t^a_{ij}; (v_i, v_j) \in E^a\}$ is a feasible solution of G^a. In short: the feasible solution T^0 of G is also a feasible solution of the aggregated graph G^a.

Proof. We must prove the inequality (1) for each two adjacent arcs from $E^a = E^a_1 \cup E^a_2$. There are two cases:

a) $(v_{p_r}, v_{q_r}), (v_{q_r}, v_{p_s}) \in E^a$, $r \ne s$. Then from (3) it follows

$$t^a_{p_r, q_r} + d_r = t^0_{p_r} + (\bar{t}^0_{q_r} - t^0_{p_r}) = \bar{t}^0_{q_r} = t^a_{q_r, p_s},$$

b) $(v_{q_r}, v_{p_s}), (v_{p_s}, v_{q_s}) \in E^a$. In this case we get from (3) and (4)

$$t^a_{q_r, p_s} + w_{rs} = \bar{t}^0_{q_r} + (t^0_{p_s} - \bar{t}^0_{q_r} - s_{ij}) \le t^0_{p_s} = t^a_{p_s, q_s}.$$ □

Now, let us consider a feasible solution $T^a = \{t^a_{ij}; (v_i, v_j) \in E^a\}$ of the aggregated graph G^a. A question is if there exists an feasible solution associated to G^a of the former network G. Let us denote $\tau_r = t^a_{p_r q_r}$ (start term of G_r in T^a) and $\tau^0_r = t^0_{p_r}$ (start term of G_r in T^0). Further for each arc $(v_i, v_j) \in E$ we define

$$t_{ij} = t^0_i + (\tau_r - \tau^0_r), \tag{7}$$

where r must satisfy $v_i \in V_r$. Now we can form the following property.

Lemma 4. $T = \{t_{ij}; (v_i, v_j) \in E\}$ defined in (7) is a feasible solution of the former graph G.

Proof. We must prove the inequality (1) for each two adjacent arcs $(v_i, v_j), (v_j, v_k) \in E$; i.e. the inequality

$$t_{jk} - t_{ij} - h_{ij} \geq 0. \tag{8}$$

We have to examine two cases:

a) $v_i, v_j \in V_r$ and

b) $v_i \in V_r,\ v_j \in V_s,\ r \neq s$.

In the case a) we can substitute (7) for (8) and we get

$$t_{jk} - t_{ij} - h_{ij} = t_j^0 + (\tau_r - \tau_r^0) - [t_i^0 - (\tau_r - \tau_r^0)] - h_{ij} = t_j^0 - t_i^0 - h_{ij} \geq 0.$$

In the case b) after substituting (7) for (8) we obtain

$$\begin{aligned} & t_{jk} - t_{ij} - h_{ij} \\ = \ & t_j^0 + (\tau_s - \tau_s^0) - [t_i^0 + (\tau_r - \tau_r^0)] - h_{ij} \\ = \ & (t_j^0 - t_i^0 - h_{ij}) + (\tau_s - \tau_s^0) - (\tau_r - \tau_r^0) \\ = \ & (t_j^0 - t_i^0 - h_{ij}) + (\tau_s - \tau_r) - (\tau_s^0 - \tau_r^0) \\ \geq \ & (t_j^0 - t_i^0 - h_{ij}) + d_r + w_{rs} - (t_{p_s}^0 - t_{p_r}^0) \\ = \ & (t_j^0 - t_i^0 - h_{ij}) + (\bar{t}_{q_r}^0 - t_{p_s}^0) + (t_{p_s}^0 - \bar{t}_{q_r}^0 - s_{ij}) - (t_{p_s}^0 - t_{q_r}^0) \\ = \ & (t_j^0 - t_i^0 - h_{ij}) - s_{ij} = (t_j^0 - t_i^0 - h_{ij}) - (\bar{t}_j^0 - t_i^0 - h_{ij}) = t_j^0 - \bar{t}_j^0 \geq 0. \quad \square \end{aligned}$$

Remark. If there are two arcs in the G in the form $(v_i, v_j) \in E$ $(v_k, v_l) \in E$ such that $v_i, v_k \in V_r, v_j, v_l \in V_s,\ r \neq s$, and $w_{rs}^{ij} \geq w_{rs}^{kl}$, then we can form only stronger linkage between G_r and G_s, i.e. the arc (v_{p_r}, v_{q_s}) with weight w_{rs}^{ij}.

The method proposed in our study was programmed on the computer and serves for the aggregation of set of construction works. The results have proved a considerable decrease in the quantity of processed information (such as arcs and nodes) when exploiting this method. The results acquired from the aggregation of set of networks can be further used in operative management of the whole set projects and in a study of their mutual links, especially those concerning materials and capacity.

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