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APPROXIMATION APPROACH FOR NONLINEAR FILTERING PROBLEM WITH TIME DEPENDENT NOISES

Part II: Stable Nonlinear Filters

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This second part is devoted to the design of a stable nonlinear filter conditionally optimal in the minimum mean square (MMS) sense. The technique used here is known as an inversion of a direct Lyapunov function method which suggests to find a filter in such a way that a Lyapunov function, calculated along the filter trajectory, will change according to some prescribed law. Some properties of the stable filter are investigated. Connections of a stable MMS filter with an MMS filter proposed in Part I is established. Stability of the proposed stable filter with respect to misspecification of the model error statistics as well as the parameter uncertainty will be also examined. Numerical examples and simulation study are given to illustrate the efficiency of the proposed stable filter.

1. INTRODUCTION

In [15] a new approximation approach is proposed to solve a nonlinear filtering problem

$$\begin{aligned} x(t+1) &= \phi_t[x(t), w(t)], & t = 0, 1, \ldots \\ z(t+1) &= h_{t+1}[x(t+1), v(t+1)], & t = 0, 1, 2, \ldots \end{aligned} \tag{1}$$

in which $x(t)$ denotes the nth dimensional system state, $z(t)$ is an observed vector of p dimension, $\phi_t(\cdot)$ and $h_t(\cdot)$ are known deterministic functions, $w(t)$, $v(t)$ are random vectors of corresponding dimensions which may not be white and mutually independent. This approach allows to construct a nonlinear recursive filter, optimal in the minimum mean square (MMS) sense, in members of a class of admissible nonlinear filters. A uniqueness w.p.1 (with probability 1) of the MMS-filter (denoted by MMSF) is established. Computational method for realization of the filter as well as its efficiency are also demonstrated numerically.

The purpose of this second part is to apply the method in [15] to design a stable MMSF (denoted by SMMSF). It is known (cf. [18] – [19]) that one of the most difficulties arising in practical implementation of any filter is concerned with its possible instability with respect to the different types of uncertainty available in specification of system parameters (for instance, *a priori* statistics for the initial state, model and observation errors...). As example, in the field of the data assimilation in meteorology and oceanography [29], [6], [8], specification of the statistics for the model error (i. e., the error resulting from descretization, linearization, missing physics, boundary conditions...) in the numerical model, is simply impossible. Unfortunately, this type of uncertainty, as shown in [21], can lead to large estimation error or even to divergence of the standard Kalman filter (KF). From practical point of view, any filter could be useful only if it can meet conditions ensuring the filter's stability [7].

Historically, the question on stability of the KF arised very early when Kalman and Bucy presented their first significant work related to the optimal filtering [18]. Even for the simplest case of linear filtering with white noises, an optimality of the KF does not imply its stability. This property has to be proved [7], [17], [19], [21]. The first results, related to stability of the KF, are obtained in [9], [10], [25] [27], [28] from which we know, for instance, that if the linear system is uniformly completely observable and uniformly completely controllable, then a KF is uniformly asymptotically stable, provided that the initial error covariance matrix (ECM) for system state $P(0)$ is symmetric nonnegative definitive. This fact is very important since in practice the initial statistics are never known exactly. Mention that to analyse a stability of the constructed filter, the so-called direct Lyapunov function (LF) method is of common use so far. This method proposes first to select a LF candidate and after, to demonstrate that for the filter under consideration this LF possesses the necessary properties (cf. [17], [19]). The difficulty, associated with the direct LF-method, is that there exists no universal tool to select the LF for particular system, and checking the required stability conditions represents in general a hard task. To overcome this difficulty, in present work, we will follow other approach which may be referred to as an *"inversion"* technique (cf. [3], [4], [5], [31]). The inversion method suggests to seek an optimal filter (cf. [14]) in members of a family of nonlinear filters of given structure, so that along the filter trajectory, the selected LF will change according to some prescribed law. In other words, according to this inversion approach, our task is not to establish the existence of LF for the designed filter, but inversely, the structure and parameters of LF are given *a priori* and the problem then reduces to the design of an optimal filter possessing this LF candidate.

The paper is organized as follows. In Section 2, the problem statement will be formulated in precise fashion. Solution to the defined stable filtering problem is presented in Section 3. Some properties of the stable filter are studied in Section 4. We show there that inversion of the direct LF method is an efficient tool for constructing an SMMSF and proof of its stability follows directly from the equation required for the selected LF. Application to a linear filtering problem as well as sensitivity analysis of the SMMSF and MMSF with respect to parameter uncertainty are shortly presented in Section 5. It is clear that the inversion technique allows to verify, in the simple fashion, stability of the linear filter when the noises may be correlated and whose covariance matrices may be singular. Numerical examples and simulation study, concerning application of the SMMSF to parameter and state estimation in linear and nonlinear dynamical systems, are presented in the Section 6. Conclusion and final remarks are given in Section 7.

2. PROBLEM STATEMENT

2.1. Definition of the filter stability

Consider a filtering problem for the system (1). For simplicity, without loss of generality, let $\{v(t)\}$ be a sequence of dependent values which is independent of $\{w(t)\}$ and has $F(v_t^1), v_t^1 := [v(1), \ldots, v(t)]$, where F is a distribution function (d.f.). Let $\{w(t)\}$ and $\{v(t)\}$ be independent of $x(0)$, and $\{w(t)\}$ be a white sequence with zero mean and the covariance matrix $Q(t)$. The variables $x(0)$ and $\{w(t)\}$ are supposed to be given with a known d. f. Following the method in [15], let us introduce the following class of nonlinear admissible filters

$$\hat{x}(t+1) = \delta_t \xi_t[\hat{x}(t-r), \ldots, \hat{x}(t), z(t-q), \ldots, z(t+1)] + \gamma_t, \quad t = 0, 1, \ldots \tag{2}$$

where r, q are integer numbers, $0 \leq r \leq t,\ -1 \leq q \leq t-1$. Subject to the constraint (2), the filtering problem for the system (1) reduces to the problem of finding an optimal (in some sense) filter in members of the class (2). In (2), the n_ξ-dimensional vector-function ξ_t is supposed to be known [15],[24] and $\delta_t \in R^{n \times n_\xi}, \gamma_t \in R^n$ are unknown functions to be determined as a solution of some optimization problem. The equation (2) is referred to as a (r, q)-model [15]. It is proved in [15] that an MMSF in the class (2) exists, is unique w.p.1 and the optimal parameters (ξ_t, γ_t) along with the error covariance matrix (ECM) $P(t+1)$ for the filtered estimate $\hat{x}(t+1)$ can be determined explicitly (Theorem 1 [15]). Let $tr(A)$ denote the operator "trace" of a matrix A. In present paper, a stable MMSF in the class (2) is defined as follows

Definition 1. A nonlinear filter in the class of filters (2) is *stable* if $\lim\{tr\, P(t)\} = tr\, P(\infty) < \infty$ as $t \to \infty$.

The fact of existence of $\lim\{tr\, P(t)\} < \infty$ is of great practical importance since if it holds, the estimate $\hat{x}(t)$ will track $x(t)$ with a finite error even $\sum_{k=1}^{n} m(t,k) \to \infty$ as $t \to \infty$ where $m(t,k) := E[\tilde{x}_k(t)]^2, \tilde{x}_k(t) := x_k(t) - \bar{x}_k(t), x_k(t)$ is the kth component of $x(t)$ [19].

2.2. Definition of an optimal SMMSF

The idea on possible inversion of the direct LF-method was first suggested by Zubov [31] for the optimal control problem. Some procedures related to the design of control algorithms by the inverse LF-method are presented by Crutko in the works [3], [4], [5]. In [14] this idea was proposed to design a stable nonlinear filter in the members of a class of nonlinear filters for nonlinear systems with dependent noises.

The key idea underlying the inversion of the direct LF-method, in the context of the filter design, is outlined as follows (cf. [14]): To estimate the state $x(t)$ of the system (1), let a class of the filters (2) be assumed to be selected *a priori*. Let $\beta \in (0,1)$ be some fixed value. Considering $\nu(t) := tr\{P(t)\}$ as a LF and one requires that along the trajectory of the filter (2), the function $\nu(t)$ must change according to the rule

$$\Delta\nu(t) + \beta\nu(t) = 0, \; \Delta\nu(t) := \nu(t+1) - \nu(t). \tag{3}$$

Evidently, if (3) holds, the filter (2) will be stable since $\Delta\nu(t) = -\beta\nu(t) < 0$. Let A^T denote the transpose of the matrix A.

Definition 2. Let $\beta \in (0,1)$. A filter in the class (2) is the SMMSF if it satisfies three following conditions:

(C1) unbiasedness: $E[\hat{x}(t)] = E[x(t)]$ where $E(\cdot)$ is the expectation operator;

(C2) $\nu(t)$, considered as a function of the time variable t, must change along the trajectory of the filter (2) according to the rule (3);

(C3) $J[\delta] := tr[\delta^T \delta] \to \min_\delta$.

2.3. Relationship between the SMMSF and MMSF

Two following questions naturally arise in the light of Definition 2: (1) what is the reason which motivates the need to introduce the SMMSF? (2) what is the relationship between the SMMSF and the MMSF defined in Part I [15]? One observes that major difference between these two definitions is concerned with condition (C2) which, as expected, is introduced to ensure stability of the filter. Really, that condition implies $\Delta\nu(t) = -\beta\nu(t) < 0$ or the sum of error variances of the estimate $\hat{x}(t)$ monotonically decreases and if $\nu(0) < \infty$ then $\nu(t) < \infty$ for all t. Stability of the filter (2) is thus established (Definition 1). As to the second question, since the condition (C1) is the same as (C1) in Definition 1 of [15], let us turn the attention to the condition (C3) and its relationship to the condition $C(2)$ in Definition 1 of [15]. One sees from the proof of Theorem 1 [15] that finding an MMSF is equivalent to solving the matrix equation (8) in [15] for δ. As the latter has a nonunique solution (uniqueness is provided only if K_η is nonsingular), the solution δ^0 with minimum norm (10) [15] is proposed to be used which is unique w.p.1 (Theorem 1 of [15]). The MMSF thus minimizes the criterion (C3) under the constraints (2) (C1) and (8) in [15].

3. SOLUTION TO THE SMMSF PROBLEM

For definiteness, let in (2) $\xi \in R^{n_\xi}, \delta \in R^{n \times n_\xi}$. For two random vectors χ and η, let $\bar{\chi} = E[\chi], K_\chi = E[\chi\chi^T], K_{\chi\eta} = E[\chi\eta^T]$. Introduce $||x|| := \left[\sum_{i=1}^n x_i^2\right]^{1/2}, ||A|| := [tr(AA^T)]^{1/2}$ where $x \in R^n$ and $A \in R^{n \times m}$ matrix.

3.1. Theorem on existence of the SMMSF. Scalar case

To present clearly the idea on inversion, consider first the case when $\xi(\cdot)$ in (2) is scalar.

Theorem 1. Suppose that the condition $K_{\chi\eta} \neq 0$ in (33) (Appendix) holds where $\chi := x(t+1) - \bar{x}(t+1), \eta := \xi - \bar{\xi}$. Then the optimal parameters (δ^*, γ^*) for the SMMSF, in the class (2), are given by

$$\begin{aligned} \gamma_t^* &= \bar{x}(t+1) - \delta^* \bar{\xi}_t, \delta_t^* = \kappa\delta^0, \delta^0 = K_{\chi\eta} K_\eta^{-1} \qquad (4) \\ \kappa &= 1 - \sqrt{1-\alpha}, \alpha := (\Pi + \beta - 1)\,[\Pi\rho^2]^{-1} \\ \Pi &:= \Pi(t+1/t) = tr(K_\chi)\,[tr[P(t)]]^{-1}, \rho^2 := \rho^2(t+1) = K_{\chi\eta}^T K_{\chi\eta} [K_\eta\, tr(K_\chi)]^{-1}. \end{aligned}$$

Under the condition

$$g := g(t+1/t) = (\rho^2 - 1)\Pi + 1 > 0 \qquad (5)$$

for β from the interval

$$g_0 < \beta < g, g_0 := \max(0, 1 - \Pi) \qquad (6)$$

the solution δ_t^* to the minimization problem (C3) (Definition 2) exists, is unique and which is given in (4). The ECM $P(t+1)$ for the filtered estimate $\hat{x}(t+1)$ is governed by (12). The LF $\nu(t)$, computed along the trajectory of the filter (2), will change according to the rule (3).

Proof. Condition (C1) (Definition 2) implies the formula for γ_t^* as shown in the proof of Theorem 1 [15]. According to Definition 2, finding the SMMSF reduces then to the minimization problem (C3) subject to the constraint (C2). Making the use of the multipliers Lagrange method leads to the following unconstrained minimization problem

$$\tilde{J}[\delta, \lambda] = J[\delta] + \lambda[\Delta\nu(t) + \beta\nu(t)] \rightarrow \min_{(\delta,\lambda)}, \delta := \delta_t. \qquad (7)$$

Substituting γ^* in (4) into (2) gives $\hat{x}(t+1) = \delta_t\eta + \bar{x}(t+1), e(t+1) := \hat{x}(t+1) - x(t+1) = \delta_t\eta - \chi$ hence $\Delta\nu(t) = E[e(t+1)^T e(t+1)] - E[e(t)^T e(t)] = tr\, E\{(\delta_t\eta - \chi)(\delta_t\eta - \chi)^T\} - tr\{P(t)\}$.
Inserting of the obtained relations into (7) reads

$$\tilde{J}[\delta, \lambda] = tr[\delta\delta^T + \lambda(\delta\delta^T K_\eta - 2\delta K_{\eta\chi}^T + A)] \rightarrow \min_{(\delta,\lambda)}, A := K_\chi + (\beta - 1)P(t). \qquad (8)$$

A necessary condition for the optimization problem (8) is

$$\nabla_\delta \tilde{J}[\delta, \lambda] = 0, \quad \nabla_\lambda \tilde{J}[\delta, \lambda] = \Delta\nu(t) + \beta\nu(t) = 0. \qquad (9)$$

Consider (9). Taking a derivative of $\tilde{J}[\delta, \lambda]$ with respective to δ yields $\nabla_\delta \tilde{J}[\delta, \lambda] = \delta(1 + \lambda K_\eta) - \lambda K_{\chi\eta} = 0$ from which follows

$$\delta^* = \lambda K_{\chi\eta}[1 + \lambda K_\eta]^{-1}. \qquad \mathbf{(10)}$$

The parameter λ is determined by inserting (10) into (9). This leads to equation

$$a\lambda^2 + 2b\lambda + c = 0, \; a := bK_\eta, \; b := cK_\eta - K_{\chi\eta}^T K_{\chi\eta}, \; c := tr[A] \qquad \mathbf{(11)}$$

with the matrix A defined in (8). For equation (11), the condition $D := b^2 - ac > 0$ implies that (11) has two solutions λ_1, λ_2 determined by (33) (Lemma A1, Appendix). From Lemma A1, the optimal value λ^* which minimizes $J[\delta^*(\lambda)]$ (condition (C3) in Definition 2) is $\lambda^* = \lambda_1 = \kappa[(1-\sqrt{\alpha})\,K_\eta]^{-1}$ here κ, α are defined in (4). From $D > 0$, subject to $K_{\chi\eta} \neq 0$ (Lemma A1), one comes to the requirement (6) where Π and ρ^2 are defined in (4).

It rests to show that under the conditions of the theorem, the function $\nu(t)$ indeed changes along the trajectory of (2) according to the rule (3). Really, setting λ^* into (10), after lengthy but trivial series of manipulations, one arrives to the formula for δ^* in (4). The ECM $P(t+1)$ for the estimate $\hat{x}(t+1)$ now is determined by

$$P(t+1) = K_\chi + (\kappa^2 - 2\kappa)\,\delta^0 K_{\eta\chi} \tag{12}$$

since $\alpha = 2\kappa - \kappa^2$. From (12) and (4),

$$\begin{aligned} P(t+1) &= K_\chi - (\Pi + \beta - 1)\,[\Pi\rho^2]^{-1} K_\eta^{-1} K_{\chi\eta} K_{\chi\eta}^T \\ &= K_\chi - (\Pi + \beta - 1)\,[\Pi\rho^2]^{-1} [K_\eta tr(K_\chi)]^{-1} K_{\chi\eta} K_{\chi\eta}^T tr(K_\chi) \end{aligned} \tag{13}$$

and taking the operator $tr(\cdot)$ for both sides of (13) gives

$$tr[P(t+1)] = tr[K_\chi] - (\Pi + \beta - 1)\,[\Pi\rho^2]^{-1}\rho^2 tr(K_\chi) = (1-\beta)\,tr[P(t)]. \tag{14}$$

Remembering $\nu(t) = tr[P(t)]$ by definition, one concludes that really along the trajectory of the filter, the LF $\nu(t)$ changes according to the rule (3), and as consequence, the constructed filter is stable. □

Comment.

(i) From Subsection 4.1, $g \leq 1$ therefore if (5) holds then β from (6) always satisfies $0 < \beta < 1$. Solution for an SMMSF is unique only up to a fixed value β from the interval (g_0, g). For β, ranging in (g_0, g), one obtains a whole family of the SMMSFs. The choice of β influences on performance of the filter (see Theorem 4).

(ii) Due to Lemma A1, A4, the condition $K_{\chi\eta} \neq 0$ (see (33)) is most important for existence of the solution.

(iii) From Lemma A1, condition for existence of λ should be $D \geq 0$. However, due to (33) $K_{\chi\eta} \neq 0, D = b(b - cK_\eta) = bK_{\chi\eta}^T K_{\chi\eta} = 0$ iff (if and only if) $b = 0$. The latter implies $a = 0$ and hence $c = 0$ (see (11)). Finally, $b = 0$ and $c = 0$ lead to $K_{\chi\eta} = 0$ which is in contradiction with the assumption (33).

3.2. Multidimensional case

Suppose now $\xi_t(\cdot)$ is an n_ξ-vector function. Consider the filter (2) with

$$\gamma_t^* = \bar{x}(t+1) - \delta_t^* \bar{\xi}_t, \quad \delta_t^* = \kappa\delta_t^0, \quad \delta_t^0 := K_{\chi\eta} K_\eta^+ \tag{15}$$

where K_η^+ is the Moore–Penrose pseudoinverse of K_η [1]. It is not hard to see that for thus defined (δ_t^*, γ_t^*) the estimator (2) is unbiased and its ECM $P(t+1)$ satisfies the equation

$$P(t+1) = K_\chi + (\kappa^2 - 2\kappa)\,\delta^0 K_{\eta\chi}. \tag{16}$$

In further, let κ be parametrized as

$$\kappa := 1 - \sqrt{1-\alpha}, \quad 0 < \alpha < 1. \tag{17}$$

Evidently, $\kappa(\alpha)$ is a strictly monotonically increasing function of α, $0 < \alpha < 1$. For (17), equation (16) takes the form

$$P(t+1) = K_X - \alpha\delta^0 K_{\eta X}. \tag{18}$$

Return to the filter (2), (15), (17), (18). Let α in (17), in its turn, be parametrized as

$$\left.\begin{array}{lll} \alpha(\beta) := & (\Pi + \beta - 1)\,[\Pi\rho^2]^{-1}, & \text{if } g > 0 \\ \alpha(\beta) := & \alpha_f, 0 < \alpha_f < 1, & \text{if } g \leq 0. \end{array}\right\} \tag{19}$$

In (19), α_f is a fixed value in $(0,1)$, g is given by (5), Π – in (4) and

$$\rho^2 := \rho^2(t+1) = tr[K_{\eta X}^T K_\eta^+ K_{\eta X}]\{tr[K_X]\}^{-1}. \tag{20}$$

From (19) for $g > 0$, $\alpha = \alpha(\beta)$ is also a strictly monotononical increasing function of β. The values of $\alpha(\beta)$ then can vary only from 0 to 1 since $g \leq 1$ (see Subsection 4.1).

Remark. Function g will be nonpositive if $\Pi \geq 1/[1-\rho^2]$ $(1-\rho^2 \neq 0$ since if $\rho^2 = 1$ then $g = 1 > 0$ (see Theorem 1)). For nonpositive g, there exists no possibility to make $tr[P(t+1)] < tr[P(t)]$ due to, for example, a very high uncertainty in the model error ($M(t+1) = K_X$ is large enough in comparison with $P(t)$; See the formula for Π).

Theorem 1'. Under the condition (5), for $\beta \in (g_0, g)$ the LF $\nu(t)$ computed along the trajectory of the filter (2), (15), (17), (18), will change according to the rule (3).

Proof of the theorem can be found in the Appendix.

4. SOME PROPERTIES OF THE SMMSF

4.1. Stability

From (C2) (Definition 2), for $\psi(t,0) := \prod_{\tau=0}^{t-1}(1-\beta_\tau)$ one sees $\nu(t) = \psi(t,0)\nu(0)$ and hence if the covariance matrix of the initial state $P(0)$ is finite, $\nu(0) < \infty$ and for $0 < \beta_\tau < 1$ we have $\nu(t) < \infty$ for all t or $\nu(\infty) < \infty$ which implies a stability of the designed filter. The fact that $0 < \beta_\tau < 1$ if $g > 0$ may be shown as follows: In view of Lemma A3, $0 \leq \rho^2 \leq 1$, and as $\Pi \geq 0$ the inequality $g \leq 1$ always takes place. The latter means that the choice of β_τ from interval (6) guarantees $0 < \beta_\tau < 1$.

4.2. Convergence

Due to the inequality $1 - x \leq e^{-x}$ for small positive x,

$$\psi(t,0) = \prod_{\tau=0}^{t-1}(1-\beta_\tau) \leq \prod_{\tau=0}^{t-1} e^{-\beta_\tau} = e^{-\sum_{\tau=0}^{t-1}\beta_\tau}. \tag{21}$$

Theorem 2. Suppose there exists a possibility to choose the sequence $\{\beta_\tau\}$ a such that $\lim_{t\to\infty}\sum_{\tau=0}^{t-1}\beta_\tau < \infty$. Then the sum of error variances of $\hat{x}(t)$ will tend to a finite value as $t \to \infty$. If $\lim_{t\to\infty}\sum_{\tau=0}^{t-1}\beta_\tau = \infty$ then $tr[P(t)]$ tends to 0 as $t \to \infty$

Corollary. Suppose that $g(t+1/t) \leq 0$ only finitely happen. Then the filter (2), (15), (17), (18) is stable. Otherwise, the filter (2), (15), (17), (18) is unstable.

4.3. Relationship between SMMSF and MMSF

A nonlinear MMSF in the class of filters (2) is defined in [15] as an unbiased (condition (C1) in Definition 2) filter which minimizes the penalty function

$$J[\delta] = tr[P(t)] \to \min_{\delta}. \tag{22}$$

This MMSF exists, is unique w.p.1 and its optimal parameters (δ^0, γ^0) are determined by

$$\gamma^0 = \bar{x}(t+1) - \delta^0\bar{\xi}_t, \delta^0 = K_{\chi\eta}K_\eta^+ \tag{23}$$
$$P^0(t+1) = M(t+1) - \delta^0 K_{\chi\eta}^T, M(t+1) := K_\chi.$$

Now we show that the SMMSF (2), (15), (17) (18) can approach the MMSF in mean square.

Theorem 3. Let the matrix of the second moments K_η exist. Then the MMSF (2), (23) is a limit (in mean square sense) of the SMMSF (2), (15), (17), (18) as $\kappa \to 1$.

Theorem 3 is valid since for $\hat{x} = \delta^*\eta + \bar{x}, \hat{x'} = \delta^0\eta + \bar{x}$ we have $\hat{x} - \hat{x'} = (\delta^0 - \delta^*)\eta$ therefore $E\{(\hat{x} - \hat{x'})(\hat{x} - \hat{x'})^T\} = (\delta^0 - \delta^*)K_\eta(\delta^0 - \delta^*)^T$. As $\kappa \to 1$ (for relationships between β, α, κ see Lemma A2), we have $\delta^0 - \delta^* = (1-\kappa)K_{\chi\eta}K_\eta^+$. One can conclude that $\lim tr[P(t)] = tr[P^0(t)]$ as $\kappa \to 1$.

Theorem 4 below, whose proof is evident, confirms that in members of the family of filters (2), the MMSF is the best one. In what follows, $P(t;\alpha)$ denotes the ECM $P(t)$ defined in (18) which depends on the parameter α.

Theorem 4. The function $tr[P(t;\alpha)]$ is a strictly monotonically decreasing function of α in (0,1), i.e. $tr[P(t;\alpha_2)] < tr[P(t;\alpha_1)]$ for $0 < \alpha_1 < \alpha_2 < 1$ and $\lim tr[P(t;\alpha)] = tr[P^0(t)]$ as $\alpha \to 1$.

Thus, the closer α to 1, the better performance of the SMMSF. This rule, however, does not hold in general if there exists a misspecification of the required parameters. For details, see Sections 5–6.

4.4. Practical realization of SMMSF

Theorem 3 shows that when all the parameters are known perfectly, in order to minimize the mean square error (MSE) of SMMSF one should guess α to be close to 1 independently on whether g is positive or negative. For $g > 0$, this strategy is equivalent to keeping β close to g (Lemma A2), and the corresponding value of β is found from $\beta = 1 + \Pi(\alpha\rho^2 - 1)$. Then $\nu(t) = tr[P(t)]$ may be computed simply from equation (3). One simple method is to put $\alpha = \alpha_f$ with α_f being a fixed positive value from (0,1) which is close to 1. Theorem 3 then guarantees that the performance of the SMMSF is almost the same as the performance of the MMSF when all the parameters are precisely given.

5. LINEAR SMMSF. SENSITIVITY ANALYSIS

5.1. Linear SMMSF (LSMMSF)

For linear system when in (1) $\phi_t[x(t), w(t)] := \Phi_t x(t) + w(t), h_t[x(t), v(t)] = H_t x(t) + v(t)$ one can introduce the class of linear filters $\hat{x}(t+1) = \delta_t \xi_t[\hat{x}(t), z(t+1)] + \gamma_t, \delta_t := (A_t, K_{t+1}), \xi_t^T[\hat{x}(t), z(t+1)] := \{[\Phi_t \hat{x}(t)]^T, z^T(t+1)\}$ where A_t, K_{t+1} are the matrices of appropriate dimensions. Introduce $\hat{x}(t+1/t) := \Phi_t \hat{x}(t), \hat{z}(t+1/t) := H_{t+1}\hat{x}(t+1/t), \chi := x(t+1) - \hat{x}(t+1/t), \eta := z(t+1) - \hat{z}(t+1/t)$. Evidently for the choice $A_t = (I - K_{t+1}H_{t+1})$ where I is a unit matrix, condition (C1) in Definition 2 is satisfied with $\gamma_t = 0$. Class of filters (2) is simplified as $\hat{x}(t+1) = \hat{x}(t+1/t) + K_{t+1}\eta$ and application of Theorem 1' to this model gives $K_{t+1}^* := \kappa K_{t+1}^0, K_{t+1}^0 := K_{\chi\eta}K_\eta^+$. This filter will be referred to as an LSMMSF. Denoting by D an $(n \times n)$ diagonal matrix $D := diag[\kappa, \kappa, \ldots, \kappa]$ one can write $K_{t+1}^* := DK_{t+1}^0$. This structure for the gain matrix is introduced in [12] for which one can ensure a stability for the filter under detectability condition. Subject to $\kappa = 1$, the LSMMSF reduces to the MMSF obtained in [23] for the linear system with correlated noises. In addition, if $\{w(t)\}, \{v(t)\}$ are white and mutually uncorrelated and uncorrelated with $x(0)$ the LSMMSF reduces to a KF.

5.2. Sensitivity of SMMSF to parameter uncertainties

For simplicity, we present here only results on filter's sensitivity for the scalar observation case ($p = 1$). Let θ denote the vector of all uncertain parameters. Suppose that instead of knowing the exact value of θ we are given only its approximation θ_c. Let K_c^* denote the value of the gain K^* which is computed through the formula for the gain K^* subject to θ_c. The same notations are introduced for $K^0, P(t;\alpha)$, ect. which are computed through the formulas of the optimal filter. Recall that $P^0(t+1) = P(t+1;\alpha = 1)$. Mention that $P_c(t;\alpha)$ is only a computed ECM but no longer the actual ECM. The actual ECM will be denoted by $P_c^a(t;\alpha)$. We reserve the notation $P^a(t;\alpha)$ for the actual ECM when $\theta_c = \theta$, i.e. when all the parameters are correctly specified.

From Theorem 4, for $0 < \alpha_1 < \alpha_2 < 1$

$$tr[P^a(t,\alpha_1)] > tr[P^a(t,\alpha_2)] > tr[P^{0,a}(t)]. \quad (24)$$

The question we are interested in is whether a relationship similar to (24) holds for $tr[P_c^a(t,\alpha_1)]$? In further, let $\Delta A := A_c - A$ where A_c is a computed value of some parameter A. Let $M := M(t+1/t) = E\{[\hat{x}(t+1/t)-x(t+1)][\hat{x}(t+1/t)-x(t+1)]^T\} = E(\chi\chi^T)$.

Theorem 5. Consider an LSMMSF and suppose that $0 < \alpha_1 < \alpha_2 < 1$. Let $\kappa_i := \kappa(\alpha_i)$ where $\kappa(\alpha)$ is defined by (17). Assume that $\hat{x}(t)$ is given. Then

$$tr[P_c^a(t,\alpha_1)] > tr[P_c^a(t,\alpha_2)] > tr[P_c^{0,a}(t)] \tag{25}$$

iff the following inequality holds

$$D := \|K_c^0\|^2 < \frac{2[K^{0,a}]^T K_c^0}{(\kappa_1+\kappa_2)}, K^{0,a} := K^{0,a}(t+1). \tag{26}$$

Otherwise

$$tr[P_c^a(t,\alpha_1)] < tr[P_c^a(t,\alpha_2)] < tr[P_c^{0,a}(t)]. \tag{27}$$

Theorem 5 is proved by using the following formula for the actual $P_c^a(t,\kappa)$

$$P_c^a(t+1;\kappa) = M^a + \kappa^2 K_c^0[K_c^0]^T K_\eta - \kappa K_c^0[K_{\chi\eta}]^T - \kappa K_{\chi\eta}[K_c^0]^T \tag{28}$$

and calculating $P_c^a(t+1;\kappa_1) - P_c^a(t+1;\kappa_2)$.

Comment. Theorem 4 is a consequence of Theorem 5 since if $\theta_c = \theta$ and $0 < \kappa_i < 1$, $i = 1,2$, then $D = \|K^{0,a}\|^2 < 2\|K^{0,a}\|^2/(\kappa_1+\kappa_2)$. That relation holds even if one of κ_i is equal to 1.

Statement 1. Consider a scalar filtering problem (i.e. $n = p = 1$) for the linear system in Subsection 5.1 with white noise sequences $\{w(t)\}, \{v(t)\}$ which are mutually uncorrelated and uncorrelated with $x(0)$. Let $P(0)$, $Q(t)$, $R(t)$ be covariance matrices of $x(0), w(t), v(t)$ respectively. Suppose that the class of filters is chosen as in Subsection 5.1 and instead of $P(0)$, $Q(t)$, $R(t)$ we are given only their approximations $P_c(0), Q_c(t), R_c(t)$. Then for the LSMMSF, under the condition that $\hat{x}(t)$ is given, $M^a := M^a(t+1/t) \leq M_c := M_c(t+1/t)$ and $\alpha_2 = 1$, the relation (25) is valid iff

$$\begin{aligned} M_{\min}(t+1/t) &< M^a(t+1/t;\kappa_1) \leq M_c(t+1/t;\kappa_1) \\ M_{\min}(t+1/t) &:= (1+\kappa_1) R M_c[2R_c + H^2 M_c(1-\kappa_1)]^{-1}. \end{aligned} \tag{29}$$

Formula (29) expresses the condition only under which the KF performs better than the SMMSF with a given value κ_1. In what follows suppose that all the parameters are exactly specified except that the model covariance Q is known unprecisely, $Q \leq Q_c$ (in practice of satellite altimetric data assimilation, if R may be assumed to be known precisely (indeed, R is very small), specification of Q is simply impossible, cf. [8]). From (29), if we keep κ_1 to be close to 1, the value $M_{\min}(t+1/t)$ will approach $M_c(t+1/t;\kappa_1)$ and "almost surely" $M^a(t+1/t;\kappa_1) < M_{\min}(t+1/t)$. In other words, the SMMSF will certainly perform better than the corresponding KF under misspecification of the model error variance. By keeping κ_1 to be close but not enough to 1, one can guarantee a good performance for the SMMSF in both cases of perfect and imperfect data. That is one of the most nice properties of the SMMSF (for numerical example, see Section 6). Using (28) one can prove

Theorem 6. Under conditions of Theorem 5, for a fixed value $\alpha \in (0,1)$ the inequality $tr[P_c^a(t;\alpha)] \le tr[P_c(t;\alpha)]$ holds iff

$$D = \|K_c^0\|^2 \ge \frac{2[K_c^0]^T \Delta K_{X\eta}}{\kappa \Delta K_\eta} - \frac{tr[\Delta M]}{\kappa^2 \Delta K_\eta}, \quad \Delta M := M_c(t/t-1) - M^a(t/t-1). \tag{30}$$

The following statement is an application of Theorem 6 whose proof is given in the Appendix.

Statement 2. Suppose that $\Delta M(t) := M_c(t/t-1) - M^a(t/t-1) \ge 0$. Then for the SMMSF the relation (30) always holds.

Remark 1. At instant t let (30) hold hence $\Delta P(t) \ge 0$ (Theorem 6). Evidently fo $Q_c \ge Q^a$ it follows $\Delta M(t+1) \ge 0$. Then as a consequence of Statement 2 we have $\Delta P(t+1) = P_c(t+1) - P^a(t+1) \ge 0$. Thus if (30) holds for t then it will be true for $t+1$. By choosing $P_c(0) \ge P(0), Q_c \ge Q$ the computed $P_c(t), M_c(t)$ can be considered as the upper bounds for the actual $P(t), M(t)$.

Remark 2. Consider the situation described in Remark 1. Since for all t, $\Delta M := M_c(t/t-1;\kappa) - M^a(t/t-1;\kappa) \ge 0$ from (30) it follows that for $Q_c \ge Q$ if we keep κ to be close to 1 then for all t almost surely the SMMSF performs better than the corresponding MMSF.

Statement 3. Suppose that the vector of uncertain parameters θ consists of elements of covariance matrices $P(0)$, Q, R. Assume that $P_c(0) \ge P(0), Q_c \ge Q, R_c \ge R$. Then for the LSMMSF $tr[P_c^a(t,\alpha] \le tr[P_c(t,\alpha)]$.

Statement 3 implies that although actual value of θ (for instance, $P(0), Q, R$) can be given unprecisely, the computed "variance" in the LSMMSF, as in the KF case (cf. [17]), can be considered as an upper bound for the actual error variance, and to determine whether the conservative estimates of θ give the satisfactory filter performance. More importantly, in view of Statement 1, overestimation of the error variances should be considered as a preferred strategy (in comparison with underestimation). By appropriately choosing the parameter α such strategy can yield the SMMSF which performs certainly better than the corresponding MMSF, for all t, as shown in Remark 2.

6. NUMERICAL EXAMPLES. SIMULATION

6.1. Numerical examples

(a^0) Consider the problem of estimation of the state of the linear model in Subsection 5.1.. For simplicity let $\Phi = I, \Gamma = 0$. That problem is led to a problem of estimation of the unknown n-vector $x(t+1) = x(t) = x$. Suppose that $H \neq 0, P(0) > 0$. The SMMSF subject to $\alpha = 1$ constitutes the recursive MLS (Mean Least Squares) algorithm (cf. [20]) for estimation of x. Since whatever is $\alpha \in (0,1)$, $M(t+1/t) = P(t)$, from (4) $\Pi = 1, g_0 = 0, g = \rho^2 > 0$ ($\rho^2 > 0$ due to $P(t;\alpha) > 0$ and $H \neq 0$; The fact $P(t;\alpha) > 0$ is guaranteed by Lemma A5 with the assumption $P(0) > 0$). It means that $\nu(t)$ satisfies (3) for all t. Then $\nu(t)$ is a strictly monotonically decreasing function of t which is bounded from below by 0. Thus the sequence $\nu(t)$ must have a finite limit, i. e. the filter SMMSF is stable. Mention that no assumption concerning a whiteness of the observational noise sequence $\{v(t)\}$ was required to establish a stability of the filter.

(b^0) Consider the model in a^0 with the difference that $\Phi \neq I, \Phi$ is nonsingular. Suppose that all the singular values of Φ are no larger than 1. We state then that the SMMSF is stable. Really, let T be the orthonormal matrix diagonalizing $\Phi^T\Phi$. From $T^TT = I$ and $tr[TP(t)T^T] = tr[P(t)]$ it implies $tr[M(t+1/t)] = tr[\Phi P(t)\Phi^T] = tr[P(t)\Phi^T\Phi] = tr[T^TP(t)TT^T\Phi^T\Phi T] = tr[T^TP(t)TD_\Phi^2]$ where D_Φ^2 is the diagonal matrix whose diagonal elements $\lambda_i^2, (i = 1,\ldots,n)$ are the eigenvalues of $\Phi^T\Phi$. Direct computation yields $tr[T^TP(t)TD_\Phi^2] = \sum_{i=1}^n p'_{ii}(t)\lambda_i^2$ where $p'_{ii}(t)$ is the (ii) element of $T^TP(t)T$. Due to $P(t) := P(t;\alpha) > 0$ (proof is similar to that of Lemma A5 subject to $P(0) > 0$ and Φ is nonsingular), all $p'_{ii}(t)$ are positive, and since $\lambda_i^2 \leq 1$ the following estimate is valid

$$tr[M(t+1/t)] = tr[T^TP(t)TD_\Phi^2] = \sum_{i=1}^n p'_{ii}(t)\lambda_i^2 \leq tr[T^TP(t)T] = tr[P(t)]$$

from which $\Pi = tr[M(t+1/t)]/tr[P(t)] \leq 1$ therefore $g > 0$ for all t (this fact follows from $\rho^2 \neq 0$ due to $P(t;\alpha) > 0, H \neq 0$ and $\rho^2 \neq 1$). Equation (3) then holds for all t which means that the SMMSF is stable. If the transition matrix Φ_t is time-varying which is unstable only at time instants $t = t_1,\ldots,t_N$ where N is finite then the SMMSF remains stable due to Corollary 2.1. It is not hard to obtain the stability conditions for the case of singular Φ. As in a^0 the sequence $\{v(t)\}$ may be correlated. For the case when there exists the model error $\Gamma \neq 0$ the analysis of filter stability may be carried out in the same manner and it depends mostly on whether the events $\Pi > 1/[1-\rho^2]$ happen finitely or infinitely (see Remark in Subsection 3.2).

(c^0) Let in the linear SMMSF $n = p = 1$ and consider the sensivity of SMMSF and KF to the model error uncertainty. Suppose $\Phi = H = R = R_c = 1,\ Q = 3,\ Q_c = 5$ and all $x(0), \{w(t)\}, \{v(t)\}$ are uncorrelated. For simplicity, at instant t, let $P(t) = 1$. Equation (29) yields $M_{\min} = (1+\kappa_1)M_c/[2 + M_c(1-\kappa_1)]$ where $M_c = \Phi^2P(t) + Q_c = 6$, $M^a = \Phi^2P(t) + Q = 4$. In Figure 1 the curve labelled MMIN shows $M_{\min}$ as a function of κ_1. Two curves MCOMPUTED and MACTUAL denote values of M_c and M^a respectively. It is seen that for $\kappa_1 \in (0.87, 1)$ the constraint $M^a > M_{\min}$ (see

(29)) cannot be satisfied and by Statement 1 the SMMSF must perform better than the KF. This fact is confirmed numerically in Figure 2 where the curve PKALMAN denotes the actual error variance of the estimate produced by the KF, and the curve labelled PSMMSF expresses the actual error variance of the SMMSF estimator which depends on the parameter κ_1. Computations for PKALMAN and PSMMSF were carried out on the basis of the formula (28). The best performance for SMMSF is attained at about $\kappa_1 \in (0.93, 0.94)$. When κ_1 tends to 1, two filters yield the same estimate. The choice of κ_1 to optimize the performance of SMMSF can be considered as a problem of regularization (cf. [12]). Under parameter uncertainty, one possible way is to select κ_1 to minimize the prediction error as done in the adaptive filtering [16] under the constraint, for example, $\kappa_1 \in [0.6, 0.95]$ (see also next subsection).

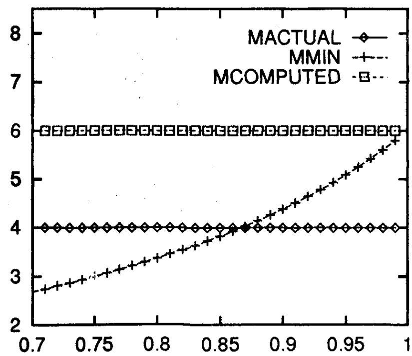

Fig. 1. Behaviour of $M_{\min}(\kappa_1)$ under model error uncertainty Q_c.

6.2. Simulation

The SMMSF and KF are used to test their effectiveness in estimation of the solution of the nonlinear reaction-diffusion process described by the following system of PDEs (cf. Sewell [26]) in the domain $\Omega := [0,1] \times [0,1]$:

$$\begin{aligned} &u_t - u_{xx} - u_{yy} - v^2 + 3u = 0, v_t - 4v_{xx} - 4v_{yy} + 2v^2 - 6u = 0 \qquad (31)\\ &u(t,x,y=0) = v(t,x,y=0) = u(t,x=0,y)\\ &\quad = v(t,x=0,u) = u(t,x=1,y) = v(t,x=1,y) = 1\\ &\frac{\partial u}{\partial y}(t,x,y=1) = \frac{\partial v}{\partial y}(t,x,y=1) = 0, u(t=0,x,y) = v(t=0,x,y) = 1. \end{aligned}$$

System (31) is integrated by a simple finite difference scheme, centered in space, and by the Euler scheme in time, with $\Delta x = \Delta y = 1/20$, $\Delta t = 1/3200$. The resulting discrete-time nonlinear numerical model in state-space form is $x'(t_k+1) = F[x'(t_k)], t_k := k * \Delta t$ whose full state $x'(t_k)$ is of dimension $n = 882$ (441 grid points for two components u, v).

The "*true*" state $x(t_k), t_k := k * \Delta t$ is numerically obtained by integration of the numerical model for $x'(t_k)$ from t_0 to t_{200} in which a spatially and temporally uncorrelated Gaussian noise w, with zero mean and variance $K_w = 0.5 * I$, is added at each time instant t_k to simulate the model error. At each t_k, 12 observations for each component of the velocity vector (u, v) are given, which are homogeneously distributed in the domain (see crosses in Fig. 3a). Observations are contaminated by a Gaussian noise with zero mean and variance $K_v = 0.1 * I$ (spatially and temporally uncorrelated).

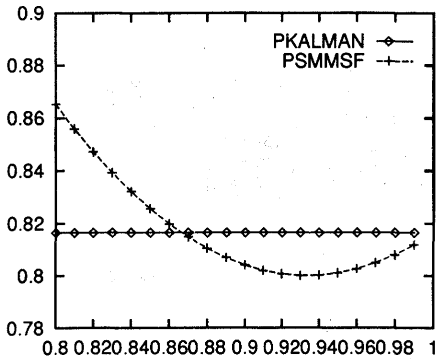

Fig. 2. MSE of stable MMSF and standard KF under model error uncertainty Q_c.

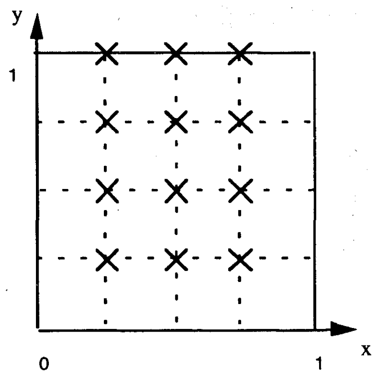

Fig. 3a. Locations of 12 points (x_i, y_j) of observations for each component (u, v) of the velocity vector the reduced-space consists of 24 elements $u(x_i, y_j)$ and $v(x_i, y_j)$. ($\times$:= points of observations for each velocity component)

Due to high dimension of the system state (typical systems, arising in the field of data assimilation in meteorology and oceanography [8], have the system state of dimension of order $10^6 - 10^7$), a reduced-order filtering approach in Hoang et al [11] is used. The reduced state x_r is defined as the vector whose components are the values of (u, v) at the points where observations are available. Then the dimension of the reduced state is $n_r = p = 24$. One defines L_r as an operator giving the values of u, v at the observational points, i.e. $x_r = L_r x$. The operator $P_r := L_r^+$, where

L_r^+ is the pseudoinversion of L_r. The class of reduced-order filters is supposed to be of the form [11]

$$\begin{aligned} \hat{x}_r(t_k+1) &= F[P_r(\hat{x}_r(t_k))] + K_r\{z(t_k+1) - HF[P_r\hat{x}_r(t_k)]\}, \\ \hat{x}(t_k+1) &= P_r\hat{x}_r(t_k+1) \end{aligned} \tag{32}$$

where H is an observation operator. Thus $\hat{x}_r$ is the estimate for the reduced state x_r and $\hat{x}$ is the estimate for the full state x. Using the extended KF approach one can write down immediately the formula for the gain $K_r := K_r^{KF}$ by linearizing $F[\hat{x}_r(t_k)]$ at each time instant t_k around the current estimate $\hat{x}_r(t_k)$ and solving the associated Riccati equation. The SMMSF (or stable KF) is of the form of extended KF except that its gain matrix K_r^{SF} is of the structure $K_r^{SF} = \kappa K_r^{KF}$. Due to nonlinearity, in fact we cannot specify exactly covariances for the model and observation errors. That is why in the experiment we used only K_w, K_v for their approximations. Parameter κ is estimated adaptively by the method [11] to achieve a minimum prediction error (MPE) subject to the constraint $\kappa \in [0.6, 0.95]$. To have the idea on how the SMMSF work in comparison with other filters, in Figure 3b we present the MSE for the filtered estimates produced by 5 filters: (i) SMMSF; (ii) extended KF; (iii) Diagonal adaptive filter (AF), i.e. the AF in which diagonal elements of the gain matrix are tuned adaptively to achieve a MPE [11]; (iv) Adaptive filter (i.e. the AF with all elements of the gain matrix tuned adaptively); (v) Newton relaxation (nudging method). The Newtonian relaxation is the method widely used up to now in data assimilation in oceanography [30].

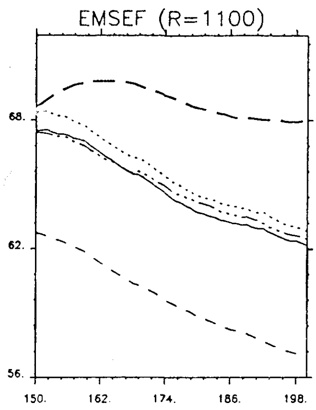

Fig. 3b. MSE for the estimates produced by 5 filters.
— —: Nudging, ____: Adaptive Filter, ... : Diagonal Adaptive Filter, ...–...–: Kalman Filter, - - - -: Stable Kalman Filter

This method consists in introducing, at the points where observations are available (denoted by *), a term of the form $-R(u-u^*)$ (and $-R(v-v^*)$ respectively) in the first equation of system (31) (and in the second one respectively). It is assumed here that the parameter R is constant in time and in space, and that the value of R is equal to 1100. Initial values for both velocity components, used in all the filters, are equal to 0.5 at every point in the modeling domain, except at the boundaries at which the boundary conditions are assigned. Mention that *true* system is initialized by 1 for all values of u, v. The boundary conditions, described in (31), are assumed to be exactly known.

Figure 3b presents the MSE expressed in the term of Empirical Mean Square Error for the Filtered estimate (EMSEF $:= [1/(200-k)]\sum_{k=100}^{200}\|e(k)\|^2, e(k) := \hat{x}(t_k) - x(t_k)$), for all the filters. In SMMSF and EKF the values of K_w, K_v are exactly specified. Note that no information on K_w, K_v is used in the AF and diagonal AF [11]. It is clear that, the EKF and other AFs produce nearly the same results. Worse efficiency is observed for the Newton relaxation. The best performance for the filtered estimate is obtained by the SMMSF. This result can be explained by the fact that due to nonlinearity and involving the model reduction, the EKF is no longer optimal. In contrast, in the SMMSF, which is of the optimal structure of the EKF with correctly specified statistics of the model and observation errors, the parameter κ is used as additional degree of freedom to optimize the filter performance. This factor is of importance to compensate the lack of information on the model noise statistics we have in the practice of data assimilation.

7. CONCLUSION

The inversion approach, presented in this paper, is developed for the design of a stable filter. This technique proposes to find the conditionally optimal filter in such a way that along the filter trajectory, the LF-candidate will change according to some prescribed law. Examples for checking the stability conditions are illustrated in Subsection 6.1. By introducing the Definition 2, the stable optimal filter in fact is only a slight modification of the MMSF, obtained in the Part I. It appears that the optimal matrix δ^* (or the filter gain K in the linear case) in the SMMSF differs from the corresponding optimal parameter δ^0 in the MMSF only by a positive parameter κ, $0 < \kappa < 1$. This makes the problem of computational realization for the SMMSF to be of the same order as that of the MMSF. The inversion approach, as we see, is not an alternative to the classical MMSF approach. Sensitivity analysis of the filtered error with respect to system parameters shows a possible large degradation of the MMSF performance under parameter uncertainties which, however, can be avoided in the SMMSF. The SMMSF thus can be regarded as an extension of the MMSF (or of a KF in particular) to deal with parameter uncertainties. Simulation study for the nonlinear filtering problem in Section 6 supports this encouraging fact. Under parameter uncertainties, one simple way to find an optimal value of κ is to tune it adaptively to achieve a MPE [11] as done in Subsection 6.2. In general, κ should be chosen as a function of the parameter uncertainty [12]. Derivation of the theoretical results related to this regularization problem, is of practical and theoretical interest, and requires the further study.

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APPENDIX

Lemma A1. Let $\chi \in R^n$, $\eta \in R^1$ be random variables. Further, let

$$K_{\chi\eta} \neq 0, K(\lambda) := \lambda K_{\chi\eta}(1 + \lambda K_\eta)^{-1}$$
$$\lambda_1 = [-b - \sqrt{D}][bK_\eta]^{-1}, \quad \lambda_2 = [-b + \sqrt{D}][bK_\eta]^{-1}$$
$$D := b^2 - ac > 0, \quad a := bK_\eta, \quad b < 0, \ c > 0. \tag{33}$$

Under conditions (33), the following strict inequality holds

$$||K(\lambda_1)|| < ||K(\lambda_2)||. \tag{34}$$

Proof. To prove lemma, let us show the equivalent version

$$e := ||K(\lambda_1)||^2 - ||K(\lambda_1)||^2 < 0. \tag{35}$$

Since $||K(\lambda)||^2 = \lambda^2 ||K_{\chi\eta}||^2(1 + \lambda K_\eta)^{-2}$ then under the condition (33), the inequality (35), in turn, is equivalent to $e_1 := \lambda_1^2(1 + \lambda_1 K_\eta)^{-2} - \lambda_2^2(1 + \lambda_2 K_\eta)^{-2} < 0$. After some manipulations for the left of the last inequality, one sees that $e_1 < 0$ iff $[\lambda_1 - \lambda_2](\lambda_1 + \lambda_2 + 2\lambda_1\lambda_2 K_\eta) < 0$. However, due to $\lambda_1 - \lambda_2 = -(2\sqrt{D})[bK_\eta]^{-1} > 0$ since $b < 0$, the last requirement holds iff $e_2 := \lambda_1 + \lambda_2 + 2\lambda_1\lambda_2 K_\eta < 0$. Under condition $D > 0$ we have $e_3 := \lambda_1 + \lambda_2 = -2\sqrt{D} < 0, e_4 := \lambda_1\lambda_2 = ac[bK_\eta]^{-2} < 0$. H nce $e_2 = e_3 + 2e_4 K_\eta < 0$ which proves the lemma. □

Comment. In (33) we implicitly assumed $K_\eta \neq 0$. This condition holds under assumption $K_{\chi\eta} \neq 0$ (see Lemma A2 below). Clearly, the formula for $K(\lambda)$ is correct only if $\lambda \neq -1/K_\eta$. For λ equal to λ_1 or λ_2 this requirement is always satisfied. It is not hard to see that the formula for K_λ may be represented as

$$K(\lambda) = K(\epsilon) = K_{\chi\eta}[\epsilon + K_\eta]^{-1}, \quad \epsilon := 1/\lambda. \tag{36}$$

In the form (36) the parameter ϵ is introduced in the works [20], [12] as a regularization parameter to provide a stability of the filter with respect to parameter uncertainties.

Lemma A2. Let $\beta \to 0$. Then for α defined in (4), $\alpha \to (\Pi - 1)(\Pi\rho^2), \kappa \to 1 - \sqrt{g/(\Pi\rho^2)}$. For $\beta \to g$ we have $\alpha \to 1, \kappa \to 1, \epsilon \to 0$.

Lemma A2 is proved by direct checking.

Lemma A3. For ρ^2 defined by (4), the following inequalities hold: $0 \leq \rho^2 \leq 1$.

Proof. For $\alpha = 1$, $P^0(t+1)$ in (23) is the ECM for the estimate obtained by the MMSF. Thus, $P^0(t+1)$ must be nonnegative definitive, $P^0(t+1) \geq 0$. Applying the operator $tr(\cdot)$ to both sides of the equation for $P^0(t+1)$ in (23) leads to $tr[P^0(t+1)] = tr[K_\chi] - tr[K_{\chi\eta}K_\eta^+ K_{\chi\eta}^T] \geq 0$ or $tr[K_{\chi\eta}K_\eta^+ K_{\chi\eta}^T] \leq tr[K_\chi]$ or $\rho^2 = tr[K_{\chi\eta}K_\eta^+ K_{\chi\eta}^T]\{tr[K_\chi]\}^{-1} \leq 1$. □

Lemma A4. Let χ, η be any two random vectors variables. Then $K_{\chi\eta} \neq 0$ implies $K_\chi \neq 0,\ K_\eta \neq 0$.

Proof. First we prove

$$tr[K_{\chi\eta}^T K_{\chi\eta}] \leq tr[K_\chi]\, tr[K_\eta]. \tag{37}$$

Write $\eta = (\eta_1, \dots, \eta_{n_\eta})^T$ evidently $K_{\chi\eta}^T = (K_{\chi\eta_1}^T, \dots, K_{\chi\eta_{n_\eta}}^T)$ and $tr[K_{\chi\eta}^T K_{\chi\eta}] = \sum_{i=1}^{n_\eta} tr[K_{\chi\eta_i}^T K_{\chi\eta_i}]$. Due to Lemma A3 $K_{\chi\eta_i}^T K_{\chi\eta_i} \leq tr[K_\chi] K_{\eta_i}$ therefore $\sum_{i=1}^{n_\eta} tr[K_{\chi\eta_i}^T K_{\chi\eta_i}] \leq tr[K_\chi] \sum_{i=1}^{n_\eta} K_{\eta_i} = tr[K_\chi]\, tr[K_\eta]$ from which follows (37). Now, for instance, if we suppose that $K_\eta = 0$ then $tr[K_\eta] = 0$ and (37) implies $tr[K_{\chi\eta}^T K_{\chi\eta}] = 0$ which contradicts the condition of lemma. □

Proof of Theorem 1'. Consider the filter (2).(4). Let the condition (5) hold. By definition (19), $\alpha(\beta)$ varies only in the interval $(0,1)$ and the values of $\kappa = \kappa[\alpha(\beta)]$ belong to $(0,1)$ too. From (19), $\alpha(\beta)$ is defined in (4) since $g > 0$. Substituting this $\alpha(\beta)$ into (18), one can check, as in the proof of the Theorem 1, that indeed the equation (3) holds. □

Proof of Statement 1. Using the linear SMMSF in the Subsection 5.1, the inequality (26) is equivalent to $M^a[H^2M_c + R_c]/[H^2M^a + R]M_c > (1+\kappa_1)/2$ from which follows $M^a[2R_c + (1-\kappa_1)H^2M_c] > (1+\kappa_1)\, RM_c$ or the left of (29). The right inequality of (29) is evident from assumption made in statement. □

Proof of Statement 2. Using the linear SMMSF we have to check (30) if $\Delta M \geq 0$, or

$$\Delta M + \kappa^2 \frac{H^2 \Delta M (HM_c)^2}{[H^2M_c + R]^2} \geq 2\kappa \frac{H\Delta M(HM_c)}{[H^2M_c + R]}$$

which is equivalent to $[H^2M_c{+}R]^2\Delta M{+}\kappa^2 H^2\Delta M(HM_c)^2 \geq 2\kappa H\Delta M(HM_c)\,[H^2M_c{+}R]$. Due to $\Delta M \geq 0$ the last inequality reduces to $[H^2M_c + R]^2 + \kappa^2H^2(HM_c)^2 - 2\kappa H^2M_c[H^2M_c + R] \geq 0$ or $aM_c^2 + 2bM_c + c \geq 0$ where $a := (1-\alpha)H^4, b := (1-\kappa)H^2R, c := R^2$. Here we used again the fact that $\alpha = 2\kappa - \kappa^2$. Considering the left of $aM_c^2 + 2bM_c + c \geq 0$ as a function of M_c, one can check that since $b^2 - ac = -2\alpha < 0$, the function $aM_c^2 + 2bM_c + c$ is positive for all M_c since $a > 0$ which proves statement. □

Lemma A5. Let in (1) $\phi_t[x(t), w(t)] = \phi_t[x(t)] + w(t)$ and the covariance matrix $Q(t)$ for the model error $w(t)$ be nonsingular. Then for all t, the ECM $P(t;\alpha)$, defined in (19), is positive definitive, i.e. for all t, $P(t;\alpha) > 0, \forall\, \alpha \in (0,1)$ where

$$P(t+1;\alpha) = M(t+1) - \delta_t^* K_{\chi\eta}^T = M(t+1) - \alpha\delta_t^0 K_{\chi\eta}^T. \tag{38}$$

Proof. From (1) with $\phi_t[x(t), w(t)] = \phi_t[x(t)] + w(t)$ it is not hard to see that $M(t+1) = E\{[x(t+1) - \bar{x}(t+1)][x(t+1) - \bar{x}(t+1)]^T\} > 0$ due to independence of $\{w(t)\}$ on $x(0), \{v(t)\}$ and $Q(t)$ is nonsingular. Consider the ECM $P^0(t+1)$ (cf.

(23)) for the MMS estimator. Evidently $P^0(t+1) \geq 0$. It means that $x^T P^0(t+1)x \geq 0, \forall\, x \in R^n, x \neq 0$. Let $R^n \backslash \{0\} = R_1 \bigcup R_2$ where $R_1 := \{x : x^T P^0(t+1)x = 0, x \neq 0\}, R_2 := \{x : x^T P^0(t+1)x > 0, x \neq 0\}$. Now lemma will be proved if we can show that $x^T P x > 0, P := P(t+1;\alpha), \forall\, x \in R^n, x \neq 0$. Let $M := M(t+1), A := \delta_t^0 K_{\chi\eta}^T$. For $x \in R_1, x^T M x = x^T A x$ therefore from $M > 0$ it follows $x^T A x > 0$. We have: $x^T P x = x^T M x - \alpha x^T A x = x^T M x - x^T A x + x^T A x - \alpha x^T A x = (1-\alpha) x^T A x > 0$ since $1 - \alpha > 0$. It remains to show $x^T P x > 0, \forall\, x \in R_2$. But for $x \in R_2, x^T M x > x^T A x$ and $\forall\, \alpha \in (0,1)$ evidently $x^T M x > \alpha x^T A x$ or $x^T M x - \alpha x^T A x = x^T P x > 0$. □

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REFERENCES

[1] R. Albert: Regression and the Moore–Penrose Pseudoinverse. Academic Press, New York 1972.

[2] A. Bryson and Ho–Yu–Chi: Applied Optimal Control. Ginn and Co., Waltham, Mass. 1969.

[3] P. D. Crutko: The Lyapunov Functions in Inverse Problems for Dynamical Controlled Systems. Scalar Models. Izv. Akad. Nauk SSSR Tekhn. Kibernet. (1983), No. 4.

[4] P. D. Crutko: The Lyapunov Functions in Inverse Problems for Dynamical Controlled Systems. Multidimensional models. Izv. Akad. Nauk SSSR Tekhn. Kibernet. (1984), No. 4.

[5] P. D. Crutko: Invesion of Direct Lyapunov Method in Control Problems for Dynamical Systems. Izv. Akad. Nauk SSSR Tekhn. Kibernet. (1989), No. 3.

[6] D. P. Dee: Simplification of the Kalman filter for meteorological data assimilation. Quart. J. Roy. Meteorol. Soc. *117* (1991), 365–384.

[7] R. J. Fitzgerald: Divergence of the Kalman filter. IEEE Trans. Automat. Control *AC-16* (1971), 736–747.

[8] M. Ghil and P. Malanotte–Rizzoli: Data assimilation in meteorology and oceanography. Adv. in Geophysics *33* (1991), 141–266.

[9] R. E. Griffin and A. P. Sage: Sensitivity analysis of discrete filtering and smoothing algorithms. In: AIAA Guidance, Control and Flight Dynamics Conf., Pasadena, California 1988, Paper No. 68–824.

[10] H. Heffes: The effect of erroneous models on the Kalman filter response. IEEE Trans. Automat. Control *AC-11* (1966), 541–543.

[11] H. S. Hoang, P. De Mey and O. Talagrand: A simple algorithm of stochastic approximation type for system parameter and state estimation. In: 33rd IEEE CDC, Florida 1994, pp. 447-452.

[12] H. S. Hoang, R. Baraille, O. Talagrand, P. De Mey and X. Carton: On the design of an stable adaptive fiilter. In: Proc. 35th IEEE CDC, Kobe 1996, Vol. 3, pp. 3543–3544.

[13] H. S. Hoang, P. De Mey, O. Talagrand and R. Baraille: A new reduced–order adaptive filter for high dimensional systems. In: Proc. of IFAC Internat. Symp. Adaptive Systems in Control and Signal Processing (Cs. Banyasz, ed.), Budapest 1995, pp. 153–158. Also: Automatica *33* (1997), 8, 1475–1498.

[14] H. S. Hoang and T. L. Nguyen: Time–stable non–linear filters: Stochastic Lyapunov function approach. In: Recent Advances in Mathematical Theory of Systems and Control, Networks and Signal Processing (H. Kimura and S. Kodama, eds.), Mita Press, Tokyo 1992, pp. 653–658.

[15] S. Hoang, L. Nguyen, R. Baraille and O. Talagrand: Approximation approach for nonlinear filtering problem with time dependent noises. Part I: Conditionally optimal filter in the minimum mean square sense. Kybernetika *33* (1997), 4, 409–425.

[16] H. S. Hoang and O. Talagrand: On regularization approach to parameter estimation and application to design of stable filters. In: IFAC 12th World Congress, V-4, Sydney 1993, pp. 213–218.
[17] A. H. Jazwinski: Stochastic Processes and Filtering Theory. Academic Press, New York 1970.
[18] R. E. Kalman and R. S. Bucy: New results in linear filtering and prediction theory. In: Trans. ASME, J. Basic Eng., 1961, 83D, pp. 95–108.
[19] R. S. Liptser and A. N. Shiryaev: Statistics of Random Processes. Nauka, Moscow 1974.
[20] L. Ljung and T. Sodestrom: Theory and Practice of Recursive Identification. Academic Press, New York 1983.
[21] R. K. Mehra: On the identification of variances and adaptive Kalman filtering. IEEE Trans. Automat. Control *AC-15* (1970), 175–184.
[22] V. I. Meleshko and S. S. Sekt: Regularized estimates in problems with singular variance matrices. Automat. Remote Control *3* (1988), 293–297.
[23] T. L. Nguyen and H. S. Hoang: On solution of ill-posed optimal linear filtering problem with correlated noises. Automat. Remote Control *4* (1983), 1, 453–466.
[24] V. S. Pugachev: Recursive estimation of variables and characteristics in the stochastic systems described by the difference equations. Dokl. Acad. Nauk USSR *243* (1976), 5.
[25] S. Safonov and M. Athans: On stability theory. In: Proc. IEEE CDC, San Diego 1979.
[26] G. Sewell: The numerical Solution of Ordinary and Partial Differential Equations. Academic Press, New York 1988.
[27] L. M. Silverman: Discrete Riccati equations: Algorithms, asymptotic properties and system theory interpretation. In: Filtering and Stochastic Control in Dynamical Systems (C. T. Leondes, ed.), Mir, Moscow 1980.
[28] H. W. Sorenson: On the error behavior in linear minimum variance estimation problems. IEEE Trans. Automat. Control *AC-12* (1967), 557–562.
[29] O. Talagrand and P. Courtier: Variational assimilation of meteorological observations with the Aajoint vorticity equations. I. Theory. Quart. J. Roy. Meteorol. Soc. *113* (1987), 1311–1328.
[30] J. Verron: Nudging satellite altimetry data into quasi-geostrophic models. J. Geophys. Research *97* (1992), 7479–7491.
[31] V. I. Zubov: On theory of analytic design of regulators. Automat. Remote Control (1963), No. 8.

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