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ON FUZZY β–COMPACT SPACES AND FUZZY β–EXTREMALLY DISCONNECTED SPACES

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The concept of fuzzy β-open set is introduced. Using fuzzy β-open sets the concepts of fuzzy β-compact spaces and fuzzy β-extremally disconnected spaces are introduced and some interesting properties of these spaces are investigated.

1. INTRODUCTION

Pre open sets were introduced by Mashour [5]. And using fuzzy sets the above concept is introduced and studied in fuzzy setting by Bin Shahna [4]. The concept of β-open sets was introduced in [1] and studied also by Allam and El Hakeim [2]. In this paper we introduce and study this concept in fuzzy setting.

2. PRELIMINARIES

A fuzzy set λ in a fuzzy topological space X is called fuzzy semi open [4] if for some fuzzy open set ν we have $\nu \le \lambda \le \mathrm{cl}\,(\nu)$ and the complement of a fuzzy semiopen set is called a fuzzy semiclosed set in X. A fuzzy set λ is called preopen if $\lambda \le \mathrm{Int}\,\mathrm{cl}\,\lambda$ and the complement of a fuzzy preopen set is called fuzzy preclosed set. A fuzzy set λ is called fuzzy α-open [4] if $\lambda \le \mathrm{Int}\,\mathrm{cl}\,\mathrm{Int}\,\lambda$.

A fuzzy topological space X is product related [4] to a fuzzy topological space Y if for any fuzzy set ν in X and $\mathcal{C}$ in Y whenever $\lambda'\ (= 1-\lambda) \not\ge \nu$ and $\mu'\ (= 1-\mu) \not\ge \mathcal{C}$ imply $\lambda' \times 1 \vee 1 \times \mu' \ge \nu \times \mathcal{C}$, where λ is a fuzzy open set in X and μ is a fuzzy open set in Y, there exist a fuzzy open set λ_1 in X and a fuzzy open set μ_1 in Y such that

$$\lambda_1' \ge \nu \quad \text{or} \quad \mu_1' \ge \mathcal{C} \quad \text{and} \quad \lambda_1' \times 1 \vee 1 \times \mu_1' = \lambda' \times 1 \vee 1 \times \mu'.$$

For two mappings $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, we define the product $f_1 \times f_2$ of f_1 and f_2 to be a mapping from $X_1 \times X_2$ to $Y_1 \times Y_2$ sending (x_1, x_2) in $X_1 \times X_2$ to $(f_1(x_1), f_2(x_2))$.

A function f from a fuzzy topological space X to a fuzzy topological space Y is said to be fuzzy β-continuous if the inverse image of each fuzzy open set in Y is fuzzy β-open in X. f is said to be M-β-fuzzy continuous if the inverse image of each fuzzy β-open set in Y is fuzzy β-open in X. Also f is called M-β-fuzzy open if the image of each fuzzy β-open set in X is fuzzy β-open in Y. f is called fuzzy precontinuous [4] if $f^{-1}(\lambda)$ is fuzzy preopen set in X whenever λ is a fuzzy open set in Y.

3. FUZZY β-OPEN SETS

Definition. Let X be a fuzzy topological space. A fuzzy set λ of X is called fuzzy β-open if $\lambda \leq \operatorname{cl}\operatorname{Int}\operatorname{cl}(\lambda)$. The complement of a fuzzy β-open set is called fuzzy β-closed.

The family of all fuzzy β-open sets of X is denoted by $F\beta 0(X)$. The fuzzy β-closure of λ will be denoted by $F\beta - \operatorname{cl}(\lambda)$.

The following are the properties of fuzzy β-open sets and fuzzy β-continuous maps.

1. Arbitrary union of fuzzy β-open sets is a fuzzy β-open set.

Proof. Follows from

$$(\vee\lambda_i) \leq \vee \operatorname{cl}\operatorname{Int}\operatorname{cl}(\lambda_i) \leq \operatorname{cl}\operatorname{Int}\operatorname{cl}(\vee\lambda_i). \qquad \square$$

2. Arbitrary intersection of fuzzy β-closed sets is fuzzy β-closed.
3. The implications contained in the following diagram are true.

Fuzzy open (Fuzzy closed)
↓
Fuzzy preopen (Fuzzy preclosed)
↓
Fuzzy β-open (Fuzzy β-closed)

The following example [2] shows that the reverse need not be true.

Example. Let $I = [0,1]$ and define fuzzy sets on I as

$$\mu_1(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} \\ 2x-1 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & 0 \leq x \leq \frac{1}{4} \\ 2-4x & \frac{1}{4} \leq x \leq \frac{1}{2} \\ 0 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$= \begin{cases} 0 & 0 \leq x \leq \frac{1}{4} \\ \frac{1}{3}(4x-1) & \frac{1}{4} \leq x \leq 1. \end{cases}$$

Put $\tau = \{0, \mu_3, 1\}$; $\sigma = \{0, \mu_1, \mu_2, \mu_1 \vee \mu_2, 1\}$. Then μ_1 in (I,τ) is fuzzy preopen but not fuzzy open and μ_3 in (I,σ) is not fuzzy preopen but it is fuzzy β-open.

4. Suppose λ is fuzzy β-open in X and μ is fuzzy β-open in Y. Then $\lambda \times \mu$ is fuzzy β-open in $X \times Y$ if X is product related to Y [4].
5. Let μ be a fuzzy set in X and λ is a fuzzy preopen set such that $\lambda \leq \mu \leq \operatorname{cl}\operatorname{Int}\lambda$. Then μ is a fuzzy β-open set.

Proof. Since λ is a fuzzy preopen set we have $\lambda \leq \operatorname{Int}\operatorname{cl}(\lambda)$. Then

$$\mu \leq \operatorname{cl}\operatorname{Int}\lambda \leq \operatorname{cl}\operatorname{Int}[\operatorname{Int}\operatorname{cl}\lambda] = \operatorname{cl}\operatorname{Int}\operatorname{cl}\lambda \leq \operatorname{cl}\operatorname{Int}\operatorname{cl}(\mu).$$ □

6. Let X_1, X_2, Y_1 and Y_2 be fuzzy topological spaces such that X_1 is product related to X_2 and $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ be mappings. If f_1 and f_2 are fuzzy β-continuous, then so is $f_1 \times f_2$.

Proof. Let $\lambda = \bigvee_{i,j}(\lambda_i \times \mu_j)$ where λ_i and μ_j are fuzzy open sets in Y_1 and Y_2 respectively, be a fuzzy open set in $Y_1 \times Y_2$. Now

$$(f_1 \times f_2)^{-1}(\lambda) = \bigvee (f_1 \times f_2)^{-1}(\lambda_i \times \mu_j) = \bigvee f_1^{-1}(\lambda_i) \times f_2^{-1}(\mu_j).$$

since f_1 and f_2 are fuzzy β-continuous $f_1^{-1}(\lambda_i)$ and $f_2^{-1}(\mu_j)$ are fuzzy β-open. And so $(f_1 \times f_2)^{-1}(\lambda)$ is fuzzy β-open by (1) and (3). That is $f_1 \times f_2$ is fuzzy β-continuous. □

7 Let X, X_1 and X_2 be fuzzy topological spaces and $p_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection mappings. If $f : X \to X_1 \times X_2$ is fuzzy β-continuous, then so is $p_i \circ f$.

Proof. This follows because projection maps are fuzzy continuous. □

8. The implications contained in the following diagram are true:

fuzzy continuity
↓
fuzzy precontinuity
↓
fuzzy β-continuity.

The following example shows that the reverse need not be true.
Define $f : (I, \tau') \to (I, \sigma)$ by $f(x) = \frac{x}{2}$, where $\tau' = \{0, \mu_3', 1\}$. Then f is fuzzy precontinuous but not fuzzy continuous.

4. FUZZY β–COMPACT SPACES

Definition 1. A space X is called fuzzy β-compact (Lindelöf) if every fuzzy β-open cover of X has a finite (countable) subcover.

If (X, T) is a fuzzy topological space, then T_β stands for the fuzzy topology on X having $F\beta 0(X, T)$ as a subbase.

Proposition 1. (X,T) is a fuzzy β-compact $\leftrightarrow (X,T_\beta)$ is fuzzy compact.

Proof. If (X,T_β) is fuzzy compact, then (X,T) is fuzzy β-compact since $F\beta 0(X,T) \subset T_\beta$. The converse is a consequence of the famous Alexander's subbase theorem for fuzzy topological spaces. □

Definition 2. A function $f:(X,T) \to (Y,S)$ is called ϕ_β-fuzzy continuous (ϕ'_β-continuous) if $f:(X,T_\beta) \to (Y,S)$ $(f:(X,T_\beta) \to (Y,S_\beta))$ is fuzzy continuous.

Example 1. T_β-fuzzy open $\not\Rightarrow$ β-fuzzy open.

Let $X = \{a,b,c\}$; $T = \{0_X, 1_X, g\}$ where $g: X \to [0,1]$ is such that $g(a) = g(b) = 1$; $g(c) = 0$. Let $f: X \to [0,1]$ be such that $f(a) = f(b) = 0$; $f(c) = 1$.

Then f is T_β-fuzzy open and f is not β-fuzzy open.

The following proposition follows from the definitions.

Proposition 2. If $f:(X,T) \to (Y,S)$ is fuzzy β-continuous then f is ϕ_β-fuzzy continuous.

Example 2. The converse of the above proposition is not true. Let

$$\begin{aligned} X &= \{a,b,c\} \\ T_1 &= \{0_X, 1_X, f\} \text{ where } f: X \to I \text{ is such that } f(a) = f(b) = 1;\ f(c) = 0 \\ T_2 &= \{0_X, 1_X, f, g\} \text{ where } g: X \to I \text{ is such that } g(a) = g(b) = 0;\ g(c) = 1. \end{aligned}$$

Let $i:(X,T_{1\beta}) \to (X,T_2)$ be the identity mapping. Then since $T_{1\beta}$ is the discrete fuzzy topology, i is fuzzy continuous; but i is not fuzzy β-continuous since $g \in T_2$, $i^{-1}(g) = g$ and g is not fuzzy β-open in X.

Proposition 3. If $f:(X,T) \to (Y,S)$ is M-β-fuzzy continuous, then f is ϕ'_β-fuzzy continuous.

Proof. Follows from the definitions of M-β-fuzzy continuity and ϕ'_β-fuzzy continuity. □

Example 3. The converse of the above proposition is not true. In Example 2, f is ϕ'_β-fuzzy continuous but f is not M-β-fuzzy continuous.

Example 4. ϕ_β-fuzzy continuity $\not\Rightarrow$ ϕ'_β-continuity. Let $X = \{a,b,c\}$. Define fuzzy topologies T_1 and T_2 on X as follows:

$$\begin{aligned} T_1 &= \{0_X, 1_X, \lambda_1\} \text{ where } \lambda_1: X \to [0,1] \text{ is such that } \lambda_1(b) = \lambda_1(c) = 0;\ \lambda_1(a) = 1 \\ T_2 &= \{0_X, 1_X, \lambda_2\} \text{ where } \lambda_2: X \to [0,1] \text{ is such that } \lambda_2(a) = \lambda_2(b) = 1;\ \lambda_2(c) = 0. \end{aligned}$$

Let $i:(X,T_{1\beta}) \to (X,T_2)$ be the identity function. Then i is fuzzy continuous. That is i is ϕ_β-fuzzy continuous. But $i:(X,T_{1\beta}) \to (X,T_{2\beta})$ is not fuzzy continuous. Since $\lambda_3: X \to I$ is such that $\lambda_3(b) = \lambda_3(c) = 1$; $\lambda_3(a) = 0$ belongs to $T_{2\beta}$ but $i^{-1}(\lambda_3) = \lambda_3 \in T_{1\beta}$.

Proposition 4. If $f : (X,T) \to (Y,S)$ is a ϕ_β-fuzzy continuous surjective function and (X,T) is fuzzy β-compact, then (Y,S) is fuzzy compact.

Proposition 5. For a fuzzy topological space X, the following are equivalent.

(i) X is fuzzy β-compact

(ii) For any family of fuzzy β-closed sets $\{\lambda_i\}_{i\in J}$ with the property that $\bigwedge_{j\in F}\lambda_j \neq 0$ for any finite subset F of J, we have $\bigwedge_{i\in J}\lambda_i \neq 0$.

Proposition 6. A fuzzy β-closed subset of a fuzzy β-compact space is fuzzy β-compact.

Proposition 7. If $f : (X,T) \to (Y,S)$ is M-β-fuzzy continuous and λ is fuzzy β-compact, then $f(\lambda)$ is fuzzy β-compact.

Proof. Let $\mathcal{B}$ be a fuzzy β-open cover of $f(\lambda)$. Then $f(\lambda) \leq \bigvee_{\mu\in\mathcal{B}}\mu$. And

$$\lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}(\vee\mu) = \bigvee f^{-1}(\mu).$$

As f is M-β-fuzzy continuous $f^{-1}(\mu)$ is fuzzy β-open for all $\mu \in \mathcal{B}$. As λ is fuzzy β-compact $f^{-1}\left(\bigvee_{\mu\in\mathcal{F}}\mu\right) \geq \lambda$ where $\mathcal{F}$ is a finite subcollection of $\mathcal{B}$. Hence $f(\lambda) \leq \bigvee_{\mu\in\mathcal{F}}\mu$. That is $f(\lambda)$ is a fuzzy β-compact. □

Proposition 8. Let $f : (X,T) \to (Y,S)$ be an M-β-fuzzy continuous surjective function of a fuzzy β-compact space X onto a space Y. Then Y is fuzzy β-compact.

Proposition 9. Let $f : (X,T) \to (Y,S)$ be an M-β-open bijective function and Y be a fuzzy β-compact space. Then X is fuzzy β-compact.

Remarks. In view of Proposition 1, Proposition 5 and Proposition 6 (Proposition 7 and Proposition 8) remain valid if fuzzy β-closed (M-β-fuzzy continuous) is replaced by T_β-fuzzy closed (ϕ'_β-fuzzy continuous). Also Proposition 9 remains valid if M-β-fuzzy open is replaced by ϕ'_β-fuzzy open.

Proposition 10. Let X be a fuzzy β-compact space, Y be a fuzzy Hausdorff space [3] and $f : (X,T) \to (Y,S)$ be a ϕ_β-fuzzy continuous function, then the image of each T_β-fuzzy closed set in X is fuzzy closed in Y.

Proposition 11. Let $U \subset (X,T)$ be such that χ_U is fuzzy α-open. Let λ be a fuzzy β-open in X. Then $\lambda \wedge \chi_U$ is fuzzy β-open in $(U,\ T/U)$.

Proposition 12. Let $U \subset (X,T)$ be such that χ_U is a fuzzy α-open in (X,T). Then χ_U is fuzzy β-compact in $(X,T) \Leftrightarrow (U,\ T/U)$ is fuzzy β-compact.

5. FUZZY β-EXTREMALLY DISCONNECTEDNESS

Definition. Let (X,T) be any fuzzy topological space. X is called fuzzy β-extremally disconnected if the β-closure of a fuzzy β-open set is fuzzy β-open.

The following proposition gives several characterizations of fuzzy β-extremally disconnected spaces.

Proposition 13. For any fuzzy topological space the following are equivalent.

(a) X is fuzzy β-extremally disconnected.

(b) For each fuzzy closed set λ, $\beta - \mathrm{Int}\,(\lambda)$ is fuzzy β-closed.

(c) For each fuzzy open set λ, we have $\beta - \mathrm{cl}\,(\lambda) + \beta - \mathrm{cl}\,(1 - \beta - \mathrm{cl}\,(\lambda)) = 1$.

(d) For every pair of fuzzy open sets λ, μ, in X with $\beta - \mathrm{cl}\,(\lambda) + \mu = 1$, we have $\beta - \mathrm{cl}\,(\lambda) + \beta - \mathrm{cl}\,(\mu) = 1$.

Proof. (a) $\Rightarrow$ (b). Let λ be any fuzzy closed set. Now $1 - \beta - \mathrm{Int}\,(\lambda) = \beta - \mathrm{cl}\,(1 - \lambda)$. Since λ is fuzzy closed, $1 - \lambda$ is fuzzy open and therefore $1 - \lambda$ is fuzzy β-open. By (a) $\beta - \mathrm{cl}\,(1 - \lambda)$ is fuzzy β-open. That is $\beta - \mathrm{Int}\,(\lambda)$ is β-closed.

(b) $\Rightarrow$ (c). Let λ be any fuzzy open set. Then

$$\begin{aligned} & \beta - \mathrm{cl}\,(\lambda) + \beta - \mathrm{cl}\,(1 - \beta - \mathrm{cl}\,(\lambda)) = \beta - \mathrm{cl}\,(\lambda) + \beta - \mathrm{cl}\,(\beta - \mathrm{Int}\,(1 - \lambda)) \\ = & = \beta - \mathrm{cl}\,(\lambda) + \beta - \mathrm{Int}\,(1 - \lambda) = \beta - \mathrm{cl}\,(\lambda) + (1 - \beta - \mathrm{cl}\,(\lambda)) = 1. \end{aligned}$$

(c) $\Rightarrow$ (d). Assume for any fuzzy open set λ, $\beta - \mathrm{cl}\,(\lambda) + \beta - \mathrm{cl}\,(1 - \beta - \mathrm{cl}\,(\lambda)) = 1$. Suppose λ and μ be any two fuzzy open sets such that

$$\beta - \mathrm{cl}\,(\lambda) + \mu = 1.$$

Then

$$\begin{aligned} \beta - \mathrm{cl}\,(\lambda) + \mu &= 1 = \beta - \mathrm{cl}\,(\lambda) + \beta - \mathrm{cl}\,(1 - \beta - \mathrm{cl}\,(\lambda)) \\ \Rightarrow \mu &= \beta - \mathrm{cl}\,(1 - \beta - \mathrm{cl}\,(\lambda)) = 1 - \beta - \mathrm{cl}\,(\lambda). \end{aligned} \tag{A}$$

Thus we find $\mu = \beta - \mathrm{cl}\,(\mu)$. Then from (A) we have $\beta - \mathrm{cl}\,(\mu) = 1 - \beta - \mathrm{cl}\,(\lambda)$. That is $1 = \beta - \mathrm{cl}\,(\lambda) + \beta - \mathrm{cl}\,(\mu)$.

(d) $\Rightarrow$ (a). Let λ be any fuzzy open set and put $\beta - \mathrm{cl}\,(\lambda) + \mu = 1$. That is $\mu = 1 - \beta - \mathrm{cl}\,(\lambda)$. By (d) $\beta - \mathrm{cl}\,(\mu) + \beta - \mathrm{cl}\,(\lambda) = 1$. Therefore $\beta - \mathrm{cl}\,(\lambda)$ is fuzzy β-open in X. That is X is fuzzy β-extremally disconnected.

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