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A NEW PERIODIC MULTIRATE MODEL REFERENCE ADAPTIVE CONTROLLER FOR POSSIBLY NON STABLY INVERTIBLE PLANTS

KOSTAS G. ARVANITIS AND GRIGORIS KALOGEROPOULOS

An indirect adaptive algorithm is derived for model reference control of linear continuous–time systems with unknown parameters. The control structure proposed relies on a periodic controller, which suitably modulates the sampled output and discrete reference signals by a multirate periodically time–varying function. Such a control strategy, allows us to assign an arbitrary discrete–time transfer function for the sampled closed–loop system and does not make assumptions on the plant other than controllability, observability and known order. On the basis of the proposed adaptive algorithm, the model reference adaptive control problem is reduced to the solution of a non–homogeneous algebraic matrix equation. Known indirect model reference adaptive control techniques usually resort to the direct computation of dynamic controllers, through the solution of polynomial Diophantine equations. Moreover persistency of exitation of the continuous–time plant under control, is ensured without making any special richness assumption on the reference signal.

1. INTRODUCTION

In the last 20 years, much research has been reported, which is intended to the use of periodically time–varying and/or multirate digital compensators in controlling continuous–time linear systems. Several digital control schemes were proposed in the literature, among them periodically varying gain controllers [7, 8, 12], multirate–input controllers [1], intersample–data controllers [13], multirate–output controllers [9], generalized sampled–data hold functions (GSHF) [11], multirate GSHF [2], etc. These classes of digital controllers have been applied successfully in solving many important control problems, providing various advantages over ordinary digital feedback control schemes, such as classical state feedback, dynamic compensation or state observers (see [3, 4, 6, 14], for an extensive overview of the applications and the advantages of periodic and/or multirate controllers).

In a recent paper [5] a digital multirate–input controller (MRIC) for continuous–time systems is explored, which suitably modulates the sampled output and discrete reference signals by a multirate periodically time–varying function. This approach is considered as an alternative to standard dynamic compensation. Under certain conditions, the modulating functions can be tailored to a given system in such a way that for the sampled closed–loop system a discrete–time transfer function matrix can arbitrarily be assigned. A main feature of the approach reported in [5], is that the model matching is obtained without the requirement of pole–zero cancellation.

The purpose of the present paper is to explore the possibility cf extending the MRIC based approach presented in [5] to the control of linear time–invariant plants with unknown parameters. In particular, we use the certainty equivalence principle to combine a MRIC based model matching structure, which could be used to meet the control objective in the case of plants with known parameters, with an on-line identification procedure. The motivation for studying an adaptive version of the particular controller structure given in [5], is manifold. Foremostly, since it does not rely on pole–zero cancellation, it may be readily applicable for solving the model reference adaptive control problem for nonstably invertible plants and with reference models having arbitrary poles and zeros and relative degree. Furthermore, the degrees of freedom in the choice of the modulating function provide a solution to the problem of assuring persistency of excitation of the continuous–time plant under control, without imposing any special assumption on the reference signals (except boundedness), as in known model reference adaptive control schemes, and without declining the achievment of discrete–time asymptotic model following. Finally, on the basis of the approach reported in [5], the solution of the model matching problem is reduced to the solution of a simple non–homogeneous algebraic matrix equation, rather than a complicated polynomial Diophantine equation as is needed in standard indirect model reference adaptive technique. In this paper, we address all these problems and, as a result, we present a globally stable indirect periodic multirate model reference adaptive controller for possibly nonstably invertible plants that does not rely on pole–zero cancellations or on richness assumptions on the reference signal. The only a priori knowledge needed to implement the proposed model reference adaptive controller is controllability and observability of the continuous and sampled system and known order.

2. PRELIMINARIES AND PROBLEM STATEMENT

Consider the continuous–time, linear time–invariant single–input, single–output (SISO) system having the following state–space representation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t) \quad , \quad y(t) = \boldsymbol{c}^\top \boldsymbol{x}(t) \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$, are the state vector, control and output signals, respectively, and $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}^\top$ are real matrices having appropriate dimensions. With regard to system (2.1) we make the following two assumptions:

Assumption 2.1. System (2.1) is controllable and observable and of known order n.

Assumption 2.2. There is a sampling period $T_0 \in \mathbb{R}^+$, such as the discretized system $\left(\exp(\boldsymbol{A}T_0),\ \int_0^{T_0} \exp[\boldsymbol{A}(T_0 - \lambda)]\boldsymbol{b}\,\mathrm{d}\lambda,\ \boldsymbol{c}^\top\right)$ is controllable and observable.

Except for this prior information, the matrix triplet $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}^{\top})$ is arbitrary and unknown. In particular, no assumption is made here, on the relative degree of the plant or its stable invertibility.

Now, consider applying to system (2.1) the multirate control strategy depicted in Figure 1. With regard to the sampling mechanism, we assume that all samplers start simultaneously at $t = 0$. The sampling period T^* has rational ratio, i.e. $T^* = T_0/N$, where T_0 is the *frame period* and $N \in \mathbb{Z}^+$ is the *input multiplicity* of the sampling. The hold circuits H_N and H_0 are the zeroth order holds with holding times T^* and T_0, respectively. Finally, the compensator $f(t)$ is a periodically time-varying controller with period T_0. That is

$$f(t + T_0) = f(t). \tag{2.2}$$

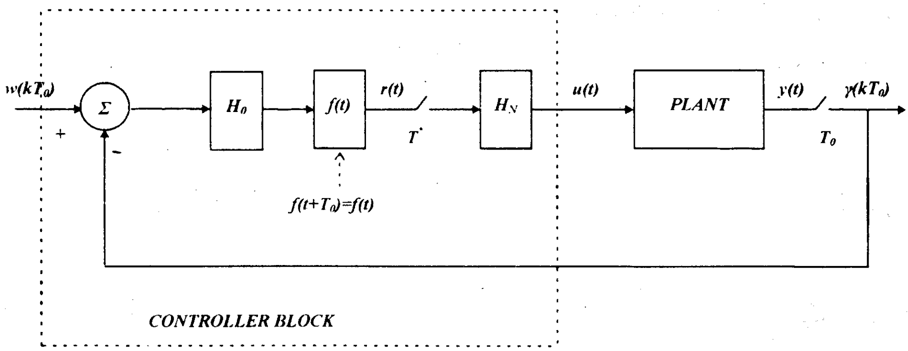

Fig. 1. Control strategy in the nonadaptive case.

The resulting closed-loop system is described by the following state space equations

$$\xi[(k+1)\boldsymbol{T}_0] = [\Phi - \boldsymbol{k}\boldsymbol{c}^{\top}]\xi(kT_0) + \boldsymbol{k}w(kT_0), \quad \gamma(kT_0) = \boldsymbol{c}^{\top}\xi(kT_0), \quad k \geq 0$$

where $\xi(kT_0) \in \mathbb{R}^n$ and $\gamma(kT_0) \in \mathbb{R}$ are discrete measurement quantities obtained by sampling $\boldsymbol{x}(t)$ and $y(t)$, respectively, with sampling period T_0 and the matrices Φ and $\boldsymbol{k}$ are defined as

$$\Phi = \exp(\boldsymbol{A}T_0), \quad \boldsymbol{k} = \int_0^{T_0} \exp[\boldsymbol{A}(T_0 - \lambda)]\, \boldsymbol{b} f(\lambda)\, \mathrm{d}\lambda \tag{2.3}$$

The model reference adaptive control problem treated in the paper is as follows: Given a discrete-time linear reference model M of the form

$$Z\{\gamma^*(kT_0)\} = H_m(z) Z\{w(kT_0)\} \tag{2.4a}$$

with

$$H_m(z) = \frac{b(z)}{a(z)} = \frac{b_{n-1}z^{n-1} + \ldots + b_1 z + b_0}{z^n + a_1 z^{n-1} + \ldots + a_1 z + a_0} \tag{2.4b}$$

where, $a(z)$ is strictly stable, $Z\{\cdot\}$ denotes the usual Z-transform, $\gamma^*(kT_0)$ is the output of the reference model and $w(kT_0) \in \mathbb{R}$ is an arbitrary uniformly bounded reference sequence, find a periodic controller $f(t)$, which when applied to system (2.1), in the sense of Figure 1, achieves discrete–time asymptotic model following, i. e.

1. $\lim_{k\to\infty} [\gamma(kT_0) - \gamma^*(kT_0)] = 0$.
2. All signals in the control loop are bounded.

With regard to the reference model M, it is emphasized that, no assumptions are made here on its poles and zeros or its relative degree.

To solve the above problem, an indirect adaptive control scheme is exhibited in the sequel. In particular, we first solve the model matching problem, namely, the exact matching of system (2.1) to the model (2.4). This is done in Section 3. Next, using these results, the exact model matching problem is solved for the configuration given in Figure 2, wherein the periodic controller $f(t)$ is with prespecified periodic behaviour and a persistent excitation signal is introduced in the control loop for future identification purposes. This is done in Section 4. It is remarked that the motivation for modifying the control strategy as in Figure 2, is that it facilitates the derivation of the indirect adaptive control scheme sought. The derivation of the adaptive scheme is presented in Section 5, where the global stability of the proposed scheme is also studied.

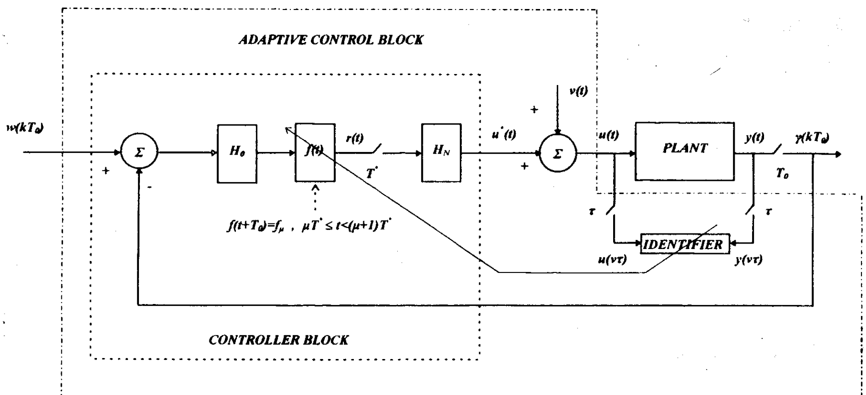

Fig. 2. The structure of the adaptive control system.

3. SOLUTION OF THE EXACT MODEL MATCHING PROBLEM FOR KNOWN SYSTEMS

In this section, our purpose is to present a new technique for the solution of the exact model matching problem via the control strategy of Figure 1, in the case where the system under control has known parameters. This technique is as follows:

System (2.1) can match system (2.2) using a periodic multirate sampled–data controller of the form (2.2), iff

$$H_c(z) = \boldsymbol{c}^\top \left[z\boldsymbol{I} - \Phi + \boldsymbol{k}\boldsymbol{c}^\top\right]^{-1} \boldsymbol{k} \equiv H_m(z). \tag{3.1}$$

Expand both sides of (3.1) in series of negative powers of z to yield

$$\sum_{j=1}^{\infty} z^{-j} \boldsymbol{c}^{\mathsf{T}} \left[\Phi - \boldsymbol{k}\boldsymbol{c}^{\mathsf{T}}\right]^{j-1} \boldsymbol{k} = \sum_{j=1}^{\infty} z^{-j} L_j \tag{3.2}$$

where L_j, $\forall j \geq 1$ are the coefficients of the expansion of $H_m(z)$ in formal Laurent series. Equating coefficients of like powers of z^{-1} in (3.2) we get

$$\boldsymbol{c}^{\mathsf{T}} \left[\Phi - \boldsymbol{k}\boldsymbol{c}^{\mathsf{T}}\right]^{j-1} \boldsymbol{k} = L_j, \quad \forall j \geq 1 \tag{3.3}$$

In (3.3) it is sufficient to keep only the first $2n+1$ equations (see [10] for the details). Then, relation (3.3) reduces to

$$\boldsymbol{c}^{\mathsf{T}} \left[\Phi - \boldsymbol{k}\boldsymbol{c}^{\mathsf{T}}\right]^{j-1} \boldsymbol{k} = L_j, \text{ for } j = 1, 2, \ldots, 2n+1. \tag{3.4}$$

Next, define the scalars M_j, for $j = 1, 2, \ldots, 2n+1$, as follows

$$\begin{aligned} M_1 &= L_1 \\ M_{j+1} &= L_{j+1} + \sum_{i=1}^{j} L_i M_{j-i+1}, \quad \forall j = 1, 2, \ldots, 2n. \end{aligned} \tag{3.5}$$

Manipulating appropriately (3.4) on the basis of (3.5), the system of equations (3.4) can be transformed to the following equivalent system of equations

$$\boldsymbol{c}^{\mathsf{T}} \Phi^{j-1} \boldsymbol{k} = M_j, \quad \text{for } j = 1, 2, \ldots, 2n+1. \tag{3.6}$$

Relation (3.6), can be written in a compact matrix form as follows

$$\Pi \boldsymbol{k} = \boldsymbol{h} \tag{3.7}$$

where, the matrix Π and the vector $\boldsymbol{h}$ have the following forms

$$\Pi = \begin{bmatrix} \boldsymbol{c}^{\mathsf{T}} \\ \boldsymbol{c}^{\mathsf{T}}\Phi \\ \vdots \\ \boldsymbol{c}^{\mathsf{T}}\Phi^{2n} \end{bmatrix}, \quad \boldsymbol{h} = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{2n+1} \end{bmatrix}.$$

Relation (3.7), is a linear nonhomogeneous algebraic system of equations. The matrix $\Pi \in \mathbb{R}^{(2n+1)\times n}$ and the vector $\boldsymbol{h} \in \mathbb{R}^{(2n+1)}$ are known and depend upon the matrices $\boldsymbol{c}^{\mathsf{T}}$ and Φ and upon the Markov parameters M_j, for $j = 1, 2, \ldots, 2n+1$, of the desired model, respectively. Clearly, the solution of the exact model matching problem (3.1) is now reduced to that of solving (3.7). With regard to the solution of (3.7), we remark that since the pair $\left(\exp(\boldsymbol{A}T_0),\ \boldsymbol{c}^{\mathsf{T}}\right)$ is observable, the matrix Π has full column rank equal to n. The above remark leads to the conclusion that (3.7) has a solution *iff*

$$\operatorname{rank}\left[\Pi \,\vdots\, \boldsymbol{h}\right] = n. \tag{3.8}$$

Furthermore, using the Cauley–Hamilton theorem and taking into account (3.8), we can easily conclude that a solution of (3.7) has the form

$$\boldsymbol{k} = \boldsymbol{R}^{-1}\boldsymbol{h}^* \tag{3.9}$$

where

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{c}^\top \\ \boldsymbol{c}^\top \Phi \\ \vdots \\ \boldsymbol{c}^\top \Phi^{n-1} \end{bmatrix}, \quad \boldsymbol{h}^* = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix}. \tag{3.10}$$

Using the vector $\boldsymbol{k}$ as specified by (3.9), we can determine the modulating function $f(t)$, by solving (2.3) (for a detailed analysis of this issue, see [11], [5]).

4. SOLUTION OF THE MODEL MATCHING PROBLEM APPROPRIATE FOR THE ADAPTIVE CASE

To obtain a solution of the exact model matching problem which will be more appropriate for application in the case of systems with unknown parameters, we slightly modify in the sequel the control strategy of Figure 1 as it is shown in Figure 2. In particular, we focus our attention on the special class of the time–varying T_0–periodic modulating functions $f(t)$, which are piecewise constant over intervals of length T^*, i.e.

$$f(t) = f_\mu, \quad \forall\, t \in [\mu T^*, (\mu+1)T^*), \; \mu = 0, 1, \ldots, N-1. \tag{4.1}$$

The persistent excitation signal $v(t)$ is defined as

$$v(t) = \boldsymbol{g}^\top(t)\,\boldsymbol{v}, \;\; \boldsymbol{g}^\top(t) = [g_0(t), \ldots, g_{N-1}(t)]$$

Here, $\boldsymbol{g}(t)$ is the T^*–periodic vector function with elements having the form

$$g_q(t) = g_{q,\mu}, \text{ for } t \in [\mu T^*, (\mu+1)T^*), \; q = 0, 1, \ldots, N-1, \; \mu = 0, 1, \ldots, N-1. \tag{4.2}$$

where $g_{q,\mu}$ is constant taking the following values

$$g_{q,\mu} = \begin{cases} 1, & \text{for } \mu = q \\ 0, & \text{for } \mu \neq q \end{cases} \tag{4.3}$$

and where $\boldsymbol{v}$ is as yet unknown. We remark that the additive term $v(t) = \boldsymbol{g}^\top(t)\,\boldsymbol{v}$, in the input of the continuous–time system, is used only for identification purposes and as it will be shown later, it is selected such as it will not influence the exact model matching problem. Furthermore, define

$$\hat{\boldsymbol{A}} \triangleq \exp(AT^*), \;\; \hat{\boldsymbol{b}} \triangleq \int_0^{T^*} \exp(\boldsymbol{A}\lambda)\boldsymbol{b}\,\mathrm{d}\lambda.$$

We are now able to establish the following theorems.

Theorem 4.1. For a modulating function of the form (4.1), the resulting closed loop system takes on the form

$$\xi[(k+1)T_0] = \left[\Phi - \hat{\boldsymbol{B}}\hat{\boldsymbol{f}}\boldsymbol{c}^\top\right]\xi(kT_0) + \hat{\boldsymbol{B}}\hat{\boldsymbol{f}}w(kT_0) + \boldsymbol{B}^*\boldsymbol{v}, \quad \gamma(kT_0) = \boldsymbol{c}^\top\xi(kT_0), \quad k \geq 0 \tag{4.4}$$

where $\hat{\boldsymbol{B}}$ is the $n \times N$ matrix having the form

$$\hat{\boldsymbol{B}} = \left[\hat{\boldsymbol{b}} \,|\, \hat{\boldsymbol{A}}\hat{\boldsymbol{b}} \vdots \ldots \vdots \hat{\boldsymbol{A}}^{N-1}\hat{\boldsymbol{b}}\right]$$

$\hat{\boldsymbol{f}}$ is the N–dimensional vector of the form

$$\hat{\boldsymbol{f}} = \begin{bmatrix} f_{N-1} \\ \vdots \\ f_0 \end{bmatrix} \tag{4.5}$$

and $\boldsymbol{B}^*$ is the $n \times N$ matrix having the form

$$\boldsymbol{B}^* = \left[\hat{\boldsymbol{A}}^{N-1}\hat{\boldsymbol{b}} \vdots \ldots \vdots \hat{\boldsymbol{A}}\hat{\boldsymbol{b}} \vdots \hat{\boldsymbol{b}}\right].$$

Proof. To show that the closed–loop system can be written in the form (4.4), we start by discretizing system (2.1) with sampling period T_0, t_0 yield

$$\xi[(k+1)T_0] = \Phi\xi(kT_0) + \int_{kT_0}^{(k+1)T_0} \exp\{\boldsymbol{A}[(k+1)T_0 - \lambda]\}\boldsymbol{b}u(\lambda)\,\mathrm{d}\lambda. \tag{4.6}$$

By observing that $u(t) = r_i(t) + \boldsymbol{g}^\top(t)\,\boldsymbol{v}$ and taking into account the structure of the control scheme of Figure 2, we obtain

$$u(t) = f(t)\,\varepsilon(kT_0) + \boldsymbol{g}^\top(t)\,\boldsymbol{v}, \quad \text{for } t \in [\mu T^*, (\mu+1)T^*) \tag{4.7}$$

where

$$\varepsilon(kT_0) = w(kT_0) - \gamma(kT_0) = w(kT_0) - \boldsymbol{c}^\top\xi(kT_0) \tag{4.8}$$

Combining relations (4.6)–(4.8), we obtain the following relationship

$$\xi[(l+1)T_0] = \left[\Phi - \boldsymbol{k}\boldsymbol{c}^\top\right]\xi(kT_0) + \boldsymbol{k}w(kT_0) + \Gamma\boldsymbol{v} \tag{4.9}$$

where

$$\Gamma = \int_{kT_0}^{(k+1)T_0} \exp\{\boldsymbol{A}[(k+1)T_0 - \lambda]\}\boldsymbol{b}\boldsymbol{g}^\top(\lambda)\,\mathrm{d}\lambda.$$

Now, partition Γ as follows

$$\Gamma = [\Gamma_1 \vdots \Gamma_2 \vdots \ldots \vdots \Gamma_N].$$

Then, the $(q+1)$th column of the matrix Γ, denoted by Γ_{q+1}, for $q = 0, 1, \ldots, N-1$, can be expresses as

$$\Gamma_{q+1} = \int_0^{T_0} \exp\left[\boldsymbol{A}(T_0 - \lambda)\right] \boldsymbol{b} g_q(\lambda)\, \mathrm{d}\lambda, \quad \text{for } q = 0, 1, \ldots, N-1. \tag{4.10}$$

Introducing (4.2) in (4.10), yields

$$\Gamma_{q+1} = \sum_{\mu=0}^{N-1} \int_{\mu T^*}^{(\mu+1)T^*} \exp[\boldsymbol{A}(T_0 - \lambda)]\, \boldsymbol{b} q_{q,\mu}\, \mathrm{d}\lambda, \text{ for } q = 0, 1, \ldots, N-1. \tag{4.11}$$

Relation (4.11) may further be written as

$$\begin{aligned} \Gamma_{q+1} &= \sum_{\mu=0}^{N-1} g_{q,\mu} \exp\left\{\boldsymbol{A}(N-1-\mu)T^*\right\} \int_0^{T^*} \exp\left[\boldsymbol{A}(T^* - \lambda)\right] \boldsymbol{b}\, \mathrm{d}\lambda \\ &= \left\{ \sum_{\xi=1}^{N} g_{q,N-\xi} \hat{\boldsymbol{A}}^{\xi-1} \right\} \hat{\boldsymbol{b}}. \end{aligned}$$

By making use of relation (4.3), we arrive at the following relationship

$$\Gamma_{q+1} = \hat{\boldsymbol{A}}^{N-q-1} \hat{\boldsymbol{b}}$$

Clearly $\Gamma \equiv \boldsymbol{B}^*$. Application of the above algorithm to the first two terms of (4.9) gives $\hat{\boldsymbol{B}}\hat{\boldsymbol{f}} \equiv \boldsymbol{k}$ (see [1] for details). This completes the proof of the theorem. □

Theorem 4.2. If N is chosen such that $N > n$, the matrix $\hat{\boldsymbol{B}}$ has full row rank, for almost every T_0.

Proof. The proof of the theorem is given in the Appendix. □

Thus far, we have established that the exact model matching controller vector $\boldsymbol{k}$ is related to the vector $\hat{\boldsymbol{f}}$ via relation $\boldsymbol{k} = \hat{\boldsymbol{B}}\hat{\boldsymbol{f}}$. It remains to determine $\hat{\boldsymbol{f}}$. To this end, let $\hat{\boldsymbol{S}}$ be the following $n \times n$ matrix

$$\hat{\boldsymbol{S}} = \left[\hat{\boldsymbol{b}} \vdots \hat{\boldsymbol{A}}\hat{\boldsymbol{b}} \vdots \ldots \vdots \hat{\boldsymbol{A}}^{n-1}\hat{\boldsymbol{b}}\right]. \tag{4.12}$$

On the basis of the results presented in the Appendix, the matrix $\hat{\boldsymbol{S}}$ is nonsingular for almost every T_0. Let also $\boldsymbol{E}$ be the $N \times N$ nonsingular permutation matrix with the property $\boldsymbol{E}^{-1} \equiv \boldsymbol{E}^\top$ and having the form

$$\boldsymbol{E} = [\boldsymbol{E}_1 \vdots \boldsymbol{E}_2]^\top$$

where

$$\boldsymbol{E}_1 = [\varepsilon_1 \vdots \varepsilon_2 \vdots \ldots \vdots \varepsilon_n], \quad \boldsymbol{E}_2 = [\varepsilon_{n+1} \vdots \varepsilon_{n+2} \vdots \ldots \vdots \varepsilon_N]$$

where in general $\varepsilon_j \in \mathbb{R}^N$ is the column vector whose elements are zeros except to a unity appearing in the jth position. Also, let

$$\tilde{\boldsymbol{B}} \triangleq \hat{\boldsymbol{B}}\boldsymbol{E}^{-1} \equiv \left[\hat{\boldsymbol{S}} \vdots \hat{\boldsymbol{Q}}\right]$$

where the matrix $\hat{\boldsymbol{Q}}$ has the form

$$\hat{\boldsymbol{Q}} = \left[\hat{\boldsymbol{A}}^n\hat{\boldsymbol{b}} \vdots \hat{\boldsymbol{A}}^{n+1}\hat{\boldsymbol{b}} \vdots \dots \vdots \hat{\boldsymbol{A}}^{N-1}\hat{\boldsymbol{b}}\right].$$

Also let Δ be the $N \times N$ nonsingular permutation matrix with the property $\Delta^{-1} \equiv \Delta^{\mathsf{T}}$ and having the form

$$\Delta = [\Delta_1 \vdots \Delta_2 \vdots \Delta_3]^{\mathsf{T}}$$

where

$$\Delta_1 = [\varepsilon_{N-n+1} \vdots \varepsilon_{N-n+2} \vdots \dots \vdots \varepsilon_n], \quad \Delta_2 = \varepsilon_{N-n}, \quad \Delta_3 = [\varepsilon_1 \vdots \varepsilon_2 \vdots \dots \vdots \varepsilon_{N-n-1}].$$

Furthermore, let

$$\tilde{\boldsymbol{B}}^* \triangleq \boldsymbol{B}^*\Delta^{-1} \equiv \left[\hat{\boldsymbol{S}}^* \vdots \hat{\boldsymbol{A}}^n\hat{\boldsymbol{b}} \vdots \hat{\boldsymbol{Q}}^*\right]$$

whe e

$$\hat{\boldsymbol{S}}^* = \left[\hat{\boldsymbol{A}}^{n-1}\hat{\boldsymbol{b}} \vdots \hat{\boldsymbol{A}}^{n-2}\hat{\boldsymbol{b}} \vdots \dots \vdots \hat{\boldsymbol{b}}\right], \quad \hat{\boldsymbol{Q}}^* = \left[\hat{\boldsymbol{A}}^{N-1}\hat{\boldsymbol{b}} \vdots \hat{\boldsymbol{A}}^{N-2}\hat{\boldsymbol{b}} \vdots \dots \vdots \hat{\boldsymbol{A}}^{n+1}\hat{\boldsymbol{b}}\right]. \tag{4.13}$$

Using the above definitions, one may determine $\hat{\boldsymbol{f}}$ by inspection, to have the form

$$\hat{\boldsymbol{f}} = \boldsymbol{E}^{\mathsf{T}} \begin{bmatrix} \hat{\boldsymbol{S}}^{-1}\boldsymbol{R}^{-1}\boldsymbol{h}^* \\ \dots\dots\dots \\ \boldsymbol{0} \end{bmatrix}. \tag{4.14}$$

It only remains to determine the appropriate vector $\boldsymbol{v}$ which does not influence the exact model matching problem. In other words $\boldsymbol{v} \in \operatorname{Ker} \boldsymbol{B}^*$, or $\boldsymbol{B}^*\boldsymbol{v} = 0$. An obvious selection of such $\boldsymbol{v}$ obtained also by inspection is the following

$$\boldsymbol{v} = \Delta^{\mathsf{T}} \begin{bmatrix} -\hat{\boldsymbol{S}}^{*-1}\hat{\boldsymbol{A}}^n\hat{\boldsymbol{b}} \\ \dots\dots\dots \\ 1 \\ \dots\dots\dots \\ 0_{(N-n-1)\times 1} \end{bmatrix}. \tag{4.15}$$

It is noted that the N-dimensional vector $\boldsymbol{v}$, eventhough does not affect the discrete model matching problem, it provides persistent excitation useful for the identification of the system, as it will be shown in the following section.

Clearly, the modulating function $f(t)$ in the configuration dipicted in Figure 2 has the form

$$f(t) = e_{N-\mu}\hat{\boldsymbol{f}}, \quad \forall\, t \in [\mu T^*, (\mu+1)T^*) \tag{4.16}$$

for $\mu = 0, 1, \ldots, N-1$, where $e_{N-\mu}$ is the N-dimensional row vector defined as $e_{N-\mu} = e_{N-\mu}^{\mathsf{T}}$. Note that, the above periodic function is largely affected upon the multirate mechanism, while the modulating function $f(t)$ of Figure 1, is not. Furthermore, the introduction of the excitation signal $v(t)$ in the control loop, greatly facilitates the estimation of the plant parameters in the case of unknown systems. For these reasons, the control strategy of Figure 2 is more appropriate than the control strategy of Figure 1 for the development of the indirect adaptive control scheme that follows.

5. CONTROL STRATEGY FOR THE ADAPTIVE CASE

The control scheme presented above has a corresponding scheme in the case where the system is unknown. For the case of unknown systems, the control strategy relies on the computation of the vector $\hat{\boldsymbol{f}}$ and the vector $\boldsymbol{v}$ from suitable estimates of the parameters of the plant with updating taking place every kT_0, $k \geq 0$, and results to a globally stable closed loop system whose output assymptotically follows the output of the desired model.

5.1. Identification of the system

System (2.1), when descretized with sampling period $\tau = T^*/(2n+1)$, takes on the form

$$\xi[(v+1)\tau] = \Phi_\tau \xi(v\tau) + \hat{\boldsymbol{b}}_\tau u(v\tau), \quad y(v\tau) = \boldsymbol{c}^{\mathsf{T}} \xi(v\tau), \ v = 0, 1, \ldots \tag{5.1}$$

where

$$\Phi_\tau = \exp(\boldsymbol{A}\tau), \quad \hat{\boldsymbol{b}}_\tau = \int_0^\tau \exp[\boldsymbol{A}(\tau - \lambda)]\, \boldsymbol{b}\, \mathrm{d}\lambda. \tag{5.2}$$

Iterating equation (5.1), $2n+1$ times and observing that $u(v\tau)$ is constant for $v\tau \in [mT^*, (m+1)T^*)$, $m = 0, 1, \ldots$, yields

$$\xi[(m+1)T^*] = \hat{\boldsymbol{A}}\xi(mT^*) + \hat{\boldsymbol{b}}u(mT^*), \quad y(mT^*) = \boldsymbol{c}^{\mathsf{T}}\xi(mT^*), \quad m = 0, 1, \ldots$$

where

$$\hat{\boldsymbol{A}} = \Phi_\tau^{2n+1} \ \text{ and } \ \hat{\boldsymbol{b}} = \sum_{\rho=0}^{2n} \Phi_\tau^\rho \hat{\boldsymbol{b}}_\tau. \tag{5.3}$$

We also note that matrix Φ is given by

$$\Phi \equiv \hat{\boldsymbol{A}}^N \equiv \Phi_\tau^{(2n+1)N}. \tag{5.4}$$

From the above analysis, it is clear that the matrices Φ, $\hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{A}}$ (which are the only matrices involved in computing the vectors $\boldsymbol{k}$ and $\boldsymbol{v}$), can be computed on the basis of the pair $(\Phi_\tau, \hat{\boldsymbol{b}}_\tau)$. Moreover, fixing the coordinate system such that

$$\Phi_\tau = \begin{bmatrix} 0 & 0 & \ldots & 0 & -\alpha_n \\ 1 & 0 & \ldots & 0 & -\alpha_{n-1} \\ 0 & 1 & \ldots & 0 & -\alpha_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -\alpha_1 \end{bmatrix}, \quad \hat{\boldsymbol{b}} = \begin{bmatrix} \beta_n \\ \beta_{n-1} \\ \beta_{n-2} \\ \vdots \\ \beta_1 \end{bmatrix}, \quad \boldsymbol{c}^{\mathsf{T}} = [0\ 0\ \ldots\ 0\ 1] \tag{5.5}$$

only α_i and β_i, $i = 1, 2, \ldots, n$ are considered as unknown parameters. Note that relations (5.1) and (5.5), are equivalent to the following difference equation

$$y(v\tau) + \sum_{\rho=1}^{n} \alpha_i y(v\tau - \rho\tau) = \sum_{\rho=1}^{n} \beta_i u(v\tau - \rho\tau), \quad v = 0, 1, \ldots \tag{5.6}$$

Relation (5.6), can now be used for the identification of the parameters of the unknown system. To this end, relation (5.6) can be written in the following linear regression form

$$y(v\tau) = \phi^{\mathsf{T}}(v\tau)\,\theta$$

where

$$\phi(v\tau) = [-y(v\tau), \ldots, -y(\nu\tau - n\tau), u(v\tau - \tau), \ldots, u(v\tau - n\tau)]^{\mathsf{T}}$$

and

$$\theta = [\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n].$$

Define

$$\begin{aligned} \boldsymbol{Y}(kT_0) &= [y(kT_0), y(kT_0 - \tau), \ldots, y[(k-1)T_0]] \\ \boldsymbol{Z}(kT_0) &= [\phi(kT_0), \phi(kT_0 - \tau), \ldots, \phi[(k-1)T_0]] \end{aligned}$$

and

$$\hat{\theta}(kT_0) = [\alpha_1(kT_0), \ldots, \alpha_n(kT_0), \beta_1(kT_0), \ldots, \beta_n(kT_0)].$$

Clearly, we have the relation

$$Y(kT_0) = \boldsymbol{Z}^{\mathsf{T}}(kT_0)\,\theta(kT_0).$$

We now choose the recursive algorithm for the estimation of $\hat{\theta}_k$ as

$$\begin{aligned} \hat{\theta}(kT_0 + T_0) &= \hat{\theta}(kT_0) - \left[a\boldsymbol{I} + \boldsymbol{Z}(kT_0)\,\boldsymbol{Z}^{\mathsf{T}}(kT_0)\right]^{-1} \cdot \\ &\qquad \boldsymbol{Z}(kT_0)\left[\boldsymbol{Z}^{\mathsf{T}}(kT_0)\,\hat{\theta}(kT_0) - \boldsymbol{Y}(kT_0)\right] \end{aligned} \tag{5.7}$$

where $a > 0$ and $\hat{\theta}(0)$ is given.

5.2. Adaptive controller synthesis algorithm

On the basis of the estimated parameter vector $\hat{\theta}(kT_0)$, obtained from (5.7), as well as on the basis of the relations (5.3)–(5.5), one can take the estimates, which are needed for the computation of the matrices $\boldsymbol{c}^{\mathsf{T}}\left(\hat{\theta}(kT_0)\right)$, $\hat{\boldsymbol{A}} \equiv \hat{\boldsymbol{A}}\left(\hat{\theta}(kT_0)\right)$, $\Phi \equiv \Phi\left(\hat{\theta}(kT_0)\right)$, $\hat{\boldsymbol{b}} \equiv \hat{\boldsymbol{b}}\left(\hat{\theta}(kT_0)\right)$, which are involved in the algorithms presented in the previous sections. Moreover, since the matrix Π can be constructed on the basis of the matrices $\Phi\left(\hat{\theta}(kT_0)\right)$ and $\boldsymbol{c}^{\mathsf{T}}\left(\hat{\theta}(kT_0)\right)$, then provided that the triplet

$\left(\hat{\boldsymbol{A}}\left(\hat{\theta}(kT_0)\right), \hat{\boldsymbol{b}}\left(\hat{\theta}(kT_0)\right), \boldsymbol{c}^\top\left(\hat{\theta}(kT_0)\right)\right)$ is controllable and observable and the relation (3.8) holds for any possible value of $\hat{\theta}(kT_0)$ (whereas no update is taken otherwise), we can obtain the following results sought:

$$\hat{\boldsymbol{f}} \equiv \hat{\boldsymbol{f}}\left(\hat{\theta}(kT_0)\right), \quad \boldsymbol{v} \equiv \boldsymbol{v}\left(\hat{\theta}(kT_0)\right) \tag{5.8}$$

Overall, the procedure for the synthesis of a multirate periodic model reference adaptive controller of the form (4.1), consists on the main steps given bellow:

Step 1. Choose the sampling period τ such that $\tau = T_0/(2n+1)N = T^*/(2n+1)$.
Step 2. Update the estimates using (5.7).
Step 3. Use (5.5) to compute the matrices $\Phi_\tau, \hat{\boldsymbol{b}}_\tau$ and $\boldsymbol{c}^\top$.
Step 4. Use (5.3) and (5.4) to compute the matrices $\hat{\boldsymbol{A}}, \hat{\boldsymbol{b}}$ and Φ.
Step 5. Conpute the vector $\boldsymbol{k}$ using relation (3.9).
Step 6. Compute the matrices $\hat{\boldsymbol{S}}$ and $\hat{\boldsymbol{S}}^*$ using relations (4.12) and (4.13).
Step 7. Compute the vectors $\hat{\boldsymbol{f}}$ and $\boldsymbol{v}$ on the basis of relations (4.14) and (4.15).
Step 8. Implement the periodic multirate modulating function $f(t)$ using (4.16).

5.3. Stability analysis of the adaptive control scheme

With regard to the stability of the proposed adaptive scheme, we establish the following fundamental theorem.

Theorem 5.1. The regressor sequence $\phi(v\tau)$ is persistently exciting. i. e. there is a $\delta > 0$, such that

$$\boldsymbol{Z}(kT_0)\boldsymbol{Z}^\top(kT_0) = \sum_{v=0}^{(2n+1)N} \phi(kT_0 - v\tau)\,\phi^\top(kT_0 - v\tau) \geq \delta \boldsymbol{I}. \tag{5.9}$$

Proof. Let $u(t) = \boldsymbol{g}^\top(t)\,\boldsymbol{v}$ and observe that, introducing the pseudovariable $\zeta(v\tau)$, equation (5.6) can be decomposed as follows

$$\zeta(v\tau) + \sum_{i=1}^{n} \alpha_i \zeta(v\tau - i\tau) = u(v\tau), \ y(v\tau) = \sum_{i=1}^{n} \beta_i \zeta(v\tau - i\tau), \quad v = 1, 2, \ldots \tag{5.10}$$

Defining the following vectors

$$\begin{aligned} \hat{\phi}(v\tau) &= [u(v\tau), \ldots, u(v\tau - n\tau), \, y(v\tau - \tau), \ldots, y(v\tau - n\tau)]^\top \\ \hat{\zeta}(v\tau) &= [\zeta(v\tau), \ldots, \zeta(v\tau - 2n\tau)]^\top \end{aligned}$$

it is easy to see that

$$\hat{\phi}(v\tau) = \boldsymbol{P}\hat{\zeta}(v\tau) \tag{5.11}$$

where $\boldsymbol{P}$ is a Sylvester–matrix which is nonsingular due to Assumption 2.2, and has the following form

$$\boldsymbol{P} = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_n & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \alpha_1 & \alpha_2 & \dots & \alpha_{n-1} & \alpha_n & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & \alpha_1 & \alpha_2 & \dots & \alpha_n \\ 0 & 0 & \beta_1 & \beta_2 & \dots & \beta_{n-1} & \beta_n & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \beta_1 & \dots & \beta_{n-2} & \beta_{n-1} & \beta_n & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \beta_1 & \beta_2 & \beta_3 & \dots & \beta_n \end{bmatrix}.$$

Observe also that the vectors $\phi(v\tau)$ and $\hat{\phi}(v\tau)$ are inter–related upon the following relation

$$\phi(v\tau) = \boldsymbol{T}\hat{\phi}(v\tau) \tag{5.12}$$

where $\boldsymbol{T} \in \mathbb{R}^{2n\times(2n+1)}$ is the full rank matrix of the form

$$\boldsymbol{T} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

It is now obvious that excitation of $\hat{\zeta}(v\tau)$ implies excitation of $\phi(v\tau)$. Therefore, we next investigate excitation of $\hat{\zeta}(v\tau)$. To this end, observe that from relation (5.10), we can write

$$\gamma^{\mathsf{T}}\hat{\zeta}(v\tau) = u(v\tau) \tag{5.13}$$

where $\gamma^{\mathsf{T}} \in \mathbb{R}^{2n+1}$ is the following vector

$$\gamma^{\mathsf{T}} = [1, \alpha_1, \alpha_2, \dots, \alpha_n, 0, \dots, 0].$$

Now, let $\boldsymbol{X}(v\tau) \in \mathbb{R}^{2n\times 2n}$ be the following symmetric matrix

$$\boldsymbol{X}(v\tau) = \left[\hat{\zeta}(v\tau) \vdots \hat{\zeta}(v\tau - \tau) \vdots \dots \vdots \hat{\zeta}(v\tau - 2n\tau)\right] \tag{5.14}$$

and $\hat{\boldsymbol{u}}(v\tau) \in \mathbb{R}^{2n}$ be the following vector

$$\hat{\boldsymbol{u}}(v\tau) = [u(v\tau), u(v\tau - \tau), \dots, u(v\tau - 2n\tau)]^{\mathsf{T}}. \tag{5.15}$$

Combining relations (5.13) – (5.15), we obtain

$$\gamma^{\mathsf{T}}\boldsymbol{X}(v\tau) = \hat{\boldsymbol{u}}^{\mathsf{T}}(v\tau).$$

Therefore, for every column vector η, with norm equal to unity, we have

$$\left|\eta^{\top}\hat{\boldsymbol{u}}(v\tau)\right|^2 = \left|\eta^{\top}\boldsymbol{X}^{\top}(v\tau)\,\gamma\right|^2 = \left|\gamma^{\top}\boldsymbol{X}(v\tau)\,\eta\right|^2 \le \|\gamma\|^2 \left\|\boldsymbol{X}^{\top}(v\tau)\,\eta\right\|^2 .$$

Summing over the interval $[kT_0 + (2n+1)\,\tau, kT_0 + (4n+1)\,\tau]$ and observing that

$$[\hat{\boldsymbol{u}}(kT_0 + (2n+1)\,\tau), \hat{\boldsymbol{u}}(kT_0 + (2n+2)\,\tau), \ldots, \hat{\boldsymbol{u}}(kT_0 + (4n+1)\,\tau)]$$
$$= \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} = \hat{\boldsymbol{U}}(kT_0)$$

we obtain

$$\sum_{v=2n+1}^{4n+1} \left|\eta^{\top}\hat{\boldsymbol{u}}(kT_0 + v\tau)\right|^2 = \left\|\hat{\boldsymbol{U}}(kT_0)\,\eta\right\|^2 \le \|\gamma\|^2 \sum_{v=2n+1}^{4n+1} \left\|\boldsymbol{X}^{\top}(v\tau)\,\eta\right\|^2$$
$$\le \|\gamma\|^2 (2n+1) \sum_{v=1}^{4n+1} \left[\hat{\zeta}(kT_0 + v\tau)\,\eta\right]^2 .$$

Therefore

$$\sum_{v=1}^{4n+1} \left[\hat{\zeta}^{\top}(kT_0 + v\tau)\,\eta\right]^2 \ge \left(\|\gamma\|^2 (2n+1)\right)^{-1} \left\|\hat{\boldsymbol{U}}(kT_0)\,\eta\right\|^2 .$$

Since the smallest singular value of $\hat{\boldsymbol{U}}(kT_0)$ is greater than a constant, there is a constant $\delta > 0$, such that

$$\sum_{v=1}^{4n+1} \hat{\zeta}(kT_0 + v\tau)\,\hat{\zeta}^{\top}(kT_0 + v\tau) \ge \delta$$

Therefore, the vector $\hat{\zeta}(v\tau)$ is persistently exciting. According to (5.11) and (5.12), the regressor sequence $\phi(v\tau)$, is also persistenly exciting. This completes the proof of the theorem. □

Since the regressor sequence is persistently exciting, the difference $\hat{\theta}(kT_0) - \theta$, where θ is the true value of the parameters, converges to zero. This guarantees convergence of the controller parameter estimates to their true values, uniform boundedness of $\xi(kT_0)$, $y(kT_0)$, $\forall\, k = 0, 1, \ldots$ and $y(t)$ and assymptotic discrete model following. Moreover, the adaptive scheme ensures exponential convergence of the estimated parameters, since

$$\hat{\theta}(kT_0 + T_0) - \theta = \left[1 + a^{-1}\boldsymbol{Z}(kT_0)\,\boldsymbol{Z}^{\top}(kT_0)\right]\left(\hat{\theta}(kT_0) - \theta\right) \qquad (5.16)$$

Relation (5.16) together with (5.9), ensures that $\hat{\theta}(kT_0) \to \theta$ exponentially as $k \to \infty$.

6. CONCLUSIONS

A new periodic multirate sampled–data model reference adaptive controller for continuous–time linear systems has been exhibited in the present paper. The proposed control strategy has several advantages over known indirect model reference adaptive control techniques. The main of them are:

(a) It is readily applicable to nonstably invertible plants and to reference models having arbitrary poles and zeros and relative degree. This is due to the fact that the approach used to solve the problem does not rely on pole–zero cancellations.

(b) Following the proposed technique a gain controller is essentially needed to be designed as compared to dynamic compensators or state observers needed by known indirect adaptive control technique. Consequently, no exogenous dynamics are introduced in the control loop by our technique, whereas in many known techniques the dynamics introduced are of high order. This improves the computational aspect of the problem, since the proposed technique does not require many on–line computations and its practical implementation requires computer memory only for storing the modulating function $f(t)$ over one period of time.

(c) It reduces the solution of the problem to that of solving a simple nonhomogeneous algebraic system of equations, rather than polynomial Diophantine equations as is needed in standard indirect adaptive techniques.

(d) Finally, persistency of excitation of the plant under control and hence parameter convergence, is provided, without making any assumption on the richeness of the reference signals (except boundedness), as compared to known indirect model reference adaptive control schemes.

The present paper gives some new insights to the model reference adaptive control problem of linear systems. The proposed technique can be extended to solve other important problems in the area of adaptive control, such as adaptive pole placement problem, adaptive LQG regulation, etc., and for other type of systems, such as time–varying periodic and non–periodic linear multivariable systems. Adaptive control schemes based on alternative parameter–estimation algorithms or on alternate multirate controllers are currently under investigation.

APPENDIX: PROOF OF THEOREM 4.2

In order to prove Theorem 4.2, it suffices to prove that the matrix $\hat{\boldsymbol{S}}$, given by (4.12), is nonsingular for almost every T_0. To this end, define

$$\pi(T^*) = \det \hat{\boldsymbol{S}}.$$

Since $\pi(T^*)$ is an analytic function of T^*, in order to prove that $\pi(T^*) \neq 0$, it suffices to prove that $\pi^{(k)}(0) \neq 0$ holds true for some positive integer k. To this end we work as follows:

First, observe that, by the formula about the differential of a determinant, we obtain

$$\pi^{(k)}(T^*) = \sum \frac{k!}{\prod_{i=1}^{n} \lambda_i!} \det \left[\hat{\boldsymbol{b}}^{(\lambda_1)} \vdots \left(\hat{\boldsymbol{A}}\hat{\boldsymbol{b}}\right)^{(\lambda_2)} \vdots \ldots \vdots \left(\hat{\boldsymbol{A}}^{n-1}\hat{\boldsymbol{b}}\right)^{(\lambda_n)} \right] \tag{A.1}$$

where the summation is made for

$$\lambda_i \geq 0, \quad \sum_{i=1}^{n} \lambda_i = k.$$

Since

$$\left(\hat{\boldsymbol{A}}^{i-1}\hat{\boldsymbol{b}}\right)^{(k)}\bigg|_{T^*=0} = \begin{cases} 0 & \text{for } k = 0 \\ \left\{ i^k - (i-1)^k \right\} \boldsymbol{A}^{k-1}\boldsymbol{b} & \text{for } k \geq 1 \end{cases} \tag{A.2}$$

we obtain

$$\pi^{(k)}(0) = \sum \frac{k!}{\prod_{i=1}^{n} \lambda_i!} \prod_{i=1}^{n} \left\{ i^{\lambda_i} - (i-1)^{\lambda_i} \right\} \det \left[\boldsymbol{A}^{\lambda_1 - 1}\boldsymbol{b} \vdots \boldsymbol{A}^{\lambda_2 - 1}\boldsymbol{b} \vdots \ldots \vdots \boldsymbol{A}^{\lambda_n - 1}\boldsymbol{b} \right] \tag{A.3}$$

where the summation is made for

$$\lambda_i \geq 1, \quad \sum_{i=1}^{n} \lambda_i = k. \tag{A.4}$$

We next focus our attention to the k_0th derivative where

$$k_0 = \sum_{j=1}^{n} j = n(n+1)/2. \tag{A.5}$$

The determinant in (A.3) does not reduce to zero if the λ_i's satisfy

$$\lambda_i \neq \lambda_j \quad (i \neq j) \tag{A.6}$$

and from (A.4) and (A.5) it must be hold

$$\sum_{i=1}^{n} \lambda_i = n(n+1)/2. \tag{A.7}$$

From (A.6) and (A.7) we obtain the following set relation

$$\{\lambda_1, \lambda_1, \ldots, \lambda_n\} \equiv \{1, 2, \ldots, n\}. \tag{A.8}$$

This means that, for $k = k_0 = n(n+1)/2$, the summation in (A.3) needs to be made for λ_i, $i = 1, 2, \ldots, n$, given by (A.8). Next, denote by σ the permutation

$$\sigma := \begin{pmatrix} 1 & 2 & & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}$$

and let $\lambda_i = \sigma(i)$. Let also $\mathbb{S}$ be the symmetric group of permutations σ. Then, the summation in (A.3), for $k = k_0 = n(n+1)/2$, can be written as

$$\pi^{(k_0)}(0) = \sum \frac{k_0!}{\prod_{i=1}^{n} \sigma(i)!} \prod_{i=1}^{n} \left\{ i^{\sigma(i)} - (i-1)^{\sigma(i)} \right\} \det \left[\boldsymbol{A}^{\sigma(1)-1}\boldsymbol{b} \vdots \boldsymbol{A}^{\sigma(2)-1}\boldsymbol{b} \vdots \dots \vdots \boldsymbol{A}^{\sigma(n)-1}\boldsymbol{b} \right]. \tag{A.9}$$

Relation (A.9) can be rewritten as

$$\pi^{(k_0)}(0) = \frac{k_0!}{\prod_{i=1}^{n} i!} \left\{ \sum_{\sigma \in \mathbb{S}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \left\{ i^{\sigma(i)} - (i-1)^{\sigma(i)} \right\} \right\} \det \left[\boldsymbol{b} \vdots \boldsymbol{A}\boldsymbol{b} \vdots \dots \vdots \boldsymbol{A}^{n-1}\boldsymbol{b} \right]. \tag{A.10}$$

Observe now that

$$\sum_{\sigma \in \mathbb{S}} sgn(\sigma) \prod_{i=1}^{n} \left\{ i^{\sigma(i)} - (i-1)^{\sigma(i)} \right\} = \det \Theta \tag{A.11}$$

where

$$\Theta = \begin{bmatrix} 1^1 - 0^1 & \dots & 1^n - 0^n \\ 2^1 - 1^1 & \dots & 2^n - 1^n \\ \vdots & & \vdots \\ n^1 - (n-1)^1 & \dots & n^n - (n-1)^n \end{bmatrix}. \tag{A.12}$$

Introducing (A.11) in (A.10), yields

$$\pi^{(k_0)}(0) = \frac{k_0!}{\prod_{i=1}^{n} i!} \det \Theta \det \left[\boldsymbol{b} \vdots \boldsymbol{A}\boldsymbol{b} \vdots \dots \vdots \boldsymbol{A}^{n-1}\boldsymbol{b} \right] \tag{A.13}$$

Observe also that

$$\det \Theta = \det \begin{bmatrix} 1^1 & 1^2 & \dots & 1^n \\ 2^1 & 2^2 & \dots & 2^n \\ \vdots & & & \vdots \\ n^1 & n^2 & \dots & n^n \end{bmatrix} = n! \prod_{1 \le \rho < q \le n} (q - \rho) = \prod_{i=1}^{n} i!. \tag{A.14}$$

Introducing (A.14) in (A.13), we obtain

$$\pi^{(k_0)}(0) = k_0! \det \left[\boldsymbol{b} \vdots \boldsymbol{A}\boldsymbol{b} \vdots \dots \vdots \boldsymbol{A}^{n-1}\boldsymbol{b} \right].$$

Since system (2.1) is assumed to be controllable we finally obtain that

$$\pi^{(k_0)}(0) \neq 0. \tag{A.15}$$

This completes the proof of the theorem. □

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REFERENCES

[1] M. Araki and T. Hagiwara: Pole assignment by multirate sampled data output feedback. Internat. J. Control *44* (1986), 1661–1673.

[2] K. G. Arvanitis: Model reference adaptive control of MIMO linear continuous-time systems using multirate generalized sampled-data hold functions. In: Proc. 2nd IEEE Mediter. Symp. on New Directions in Control and Automation, Chania, Greece 1994, pp. 332–339.

[3] K. G. Arvanitis: Adaptive decoupling of linear systems using multirate generalized sampled-data hold functions. IMA J. Math. Control Inform. *12* (1995), 157–177.

[4] K. G. Arvanitis: An indirect adaptive pole placer for MIMO systems based on multirate sampling of the plant output. IMA J. Math. Control Inform. *12* (1995), 363–394.

[5] K. G. Arvanitis and P. N. Paraskevopoulos: Exact model matching of linear systems using multirate digital controllers. In: Proc. 2nd E.C.C., Groningen, The Netherlands 1992, vol. 3, pp. 1648–1652.

[6] K. G. Arvanitis and P. N. Paraskevopoulos: Sampled-data minimum H^{∞}-norm regulation of continuous-time linear systems using multirate-output controllers. J. Optim. Theory Appl. *87* (1995), 2, 235–267.

[7] A. B. Chammas and C. T. Leondes: Pole assignment by piecewise constant output feedback. Internat. J. Control *29* (1979), 31–38.

[8] J. P. Greshak and G. C. Vergese: Periodically varying compensation of time-invariant systems. Systems Control Lett. *2* (1982), 88–93.

[9] T. Hagiwara and M. Araki: Design of a stable state feedback controller based on the multirate sampling of the plant output. IEEE Trans. Automat. Control *AC-33* (1988), 812–819.

[10] B. L. Ho and R. E. Kalman: Effective construction of linear state variable models from input/output functions. Regelungstechnik *14* (1966), 545–548.

[11] P. T. Kabamba: Control of linear systems using generalised sampled-data hold functions. IEEE Trans. Automat. Control *AC-32* (1987), 772–783.

[12] P. P. Khargonekar, K. Poolla and A. Tannenbaum: Robust control of linear time-invariant plants using periodic compensation. IEEE Trans. Automat. Control *AC-30* (1985), 1088–1096.

[13] T. Mita, B. C. Pang and K. Z. Liu: Design of optimal strongly stable digital control systems and application to output feedback control of mechanical systems. Internat. J. Control *45* (1987), 2071–2082.

[14] P. N. Paraskevopoulos and K. G. Arvanitis: Exact model matching of linear systems using generalized sampled-data hold functions. Automatica *30* (1994), 503–506.

Dr. Kostas G. Arvanitis, National Technical University of Athens, Department of Electrical and Computer Engineering, Division of Computer Science, Zographou 155773, Athens. Greece.

Prof. Grigoris Kalogeropoulos, University of Athens, Department of Mathematics, Section of Mathematical Analysis, Panepistimiopolis 15784, Athens. Greece.