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ON THE ANALYSIS OF PERIODIC LINEAR SYSTEMS[1]

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In this paper, both discrete-time and continuous-time periodic linear systems are analysed and discussed. The concepts of eigenvalue, eigenvector, characteristic multiplier, steady-state response, and of blocking zero are stated in a unique framework for both classes of systems.

1. INTRODUCTION AND NOTATION

The interest of considering periodic linear systems is motivated by the large variety of processes that can be modelled by (difference or differential) linear equations with periodic coefficients (see, e. g., [1,2,10,28,29,31,33] for the continuous-time ones and see [3,7,8,11,12,20,21,22,23,32] for the discrete-time ones). A control theory is developing for periodic linear systems, and contributions on several control problems have been given, including eigenvalue assignment, state and output dead-beat control, disturbance localisation, model matching, robust tracking and regulation, block decoupling, and adaptive control [6,13,14,15,16,17,18,19,25,26,27,30,34].

The aim of this paper is to express in a unique framework the concepts of eigenvalue, eigenvector, characteristic multiplier, steady-state response, and of blocking zero both for continuos time and discrete-time linear systems.

The class of the linear periodic systems of period ω (briefly, ω-periodic) that are considered in this paper, is described by:

$$\Delta x(t) = A(t)\, x(t) + B(t)\, u(t), \tag{1.1}$$

$$y(t) = C(t)\, x(t) + D(t)\, u(t), \tag{1.2}$$

where Δ is either the differentiation operator (i. e., the operator such that $\Delta x(t) = \frac{\mathrm{d}\,x(t)}{\mathrm{d}t}$, $t \in \mathbb{R}$) or the one-step forward-shift operator (i. e., the operator such that $\Delta x(t) = x(t+1)$, $t \in \mathbb{Z}$), $t \in T$, $T = \mathbb{R}$ if Δ is the differentiation operator or $T = \mathbb{Z}$ if Δ is the one-step forward-shift operator, $\omega \in T$, $\omega > 0$, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the control input, $y(t) \in \mathbb{R}^q$ is the output to be controlled (which is assumed to be measured), and $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ are real matrices that are ω-periodic [continuous, if $T = \mathbb{R}$] functions of $t \in T$.

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2. ANALYSIS OF ω-PERIODIC HOMOGENEOUS LINEAR SYSTEMS

Consider an ω-periodic homogeneous linear system described by the following equation:

$$\Delta x(t) = A(t)\, x(t). \tag{2.1}$$

Definition 1. (See [24] for the case $T = \mathbb{R}$ and $t_0 = 0$.) A complex λ_{t_0} is an *eigenvalue at the initial time* $t_0 \in T$ of the ω-periodic matrix $A(t)$ if and only if there exists an ω-periodic [differentiable with respect to $t \in T$, if $T = \mathbb{R}$] vector function $v_{t_0}(\cdot) \in \mathbb{C}^n$ of $t \in T$, $v_{t_0}(t) \neq 0$ for all $t \in T$, which is referred to as a *right eigenvector at the initial time* t_0 of $A(t)$, such that the vector function $\xi_{t_0}(\cdot)$ of $t \in T$, defined as follows

$$\xi_{t_0}(t) := \begin{cases} v_{t_0}(t)\, \mathrm{e}^{\lambda_{t_0}(t-t_0)}, & \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{R}, \\ v_{t_0}(t)\, \lambda_{t_0}^{(t-t_0)}, & \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{Z}, \end{cases} \tag{2.2}$$

is solution of (2.1) from the initial time $t = t_0$; such a vector function $\xi_{t_0}(\cdot)$ is called an *eigensolution at the initial time* t_0 of (2.1) with eigenvalue λ_{t_0}.

Since $[0, \omega]$ is a closed interval [and $v_{t_0}(\cdot)$ is a continuous vector function of $t \in T$, if $T = \mathbb{R}$], then the following relation holds:

$$\hat{v} := \max_{t \in [0,\omega]} \{|v_{t_0}(t)|\} < \infty,$$

whence

$$\begin{aligned} |\xi_{t_0}(t)| &\leq \hat{v}\mathrm{e}^{\mathrm{re}[\lambda_{t_0}](t-t_0)}, & \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{R}, \\ |\xi_{t_0}(t)| &\leq \hat{v}(|\lambda_{t_0}|)^{(t-t_0)}, & \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{Z}. \end{aligned}$$

The following lemma is classical (see, e. g., [24]).

Lemma 1. System (2.1) is exponentially stable if and only if all the eigenvalues at the initial time $t_0 \in T$ of matrix $A(t)$ have negative real part if $T = \mathbb{R}$, or modulus smaller than 1 if $T = \mathbb{Z}$, for all $t_0 \in T$.

The following lemmas can be easily stated and proved.

Lemma 2. (See [24] for the case $T = \mathbb{R}$ and $t_0 = 0$.) Let $v_{t_0}(\cdot) \in \mathbb{C}^n$ be an ω-periodic vector function of $t \in T$ [which is assumed to be differentiable, if $T = \mathbb{R}$], different from the zero vector for all $t \in T$. Then, $v_{t_0}(t)$ is a right eigenvector at the initial time $t_0 \in T$ of $A(t)$ with eigenvalue $\lambda_{t_0} \in \mathbb{C}$ at the initial time $t_0 \in T$ if and only if the following relations hold:

$$\dot{v}_{t_0}(t) = [A(t) - \lambda_{t_0} I]\, v_{t_0}(t), \quad \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{R}, \tag{2.3}$$

$$v_{t_0}(t+1) = \frac{1}{\lambda_{t_0}} A(t)\, v_{t_0}(t), \quad \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{Z},\ \lambda_{t_0} \neq 0, \tag{2.4}$$

$$0 = A(t_0)\, v_{t_0}(t_0 + h\omega), \quad \forall h \in \mathbb{Z},\ \text{if } T = \mathbb{Z},\ \lambda_{t_0} = 0. \tag{2.5}$$

Proof. (Necessity) If the vector function $\xi_{t_0}(\cdot)$ of $t \in T$ defined in (2.2) is solution of (2.1) from the initial time $t = t_0 \in T$, then the following relations hold

$$\dot{v}_{t_0}(t)\,\mathrm{e}^{\lambda_{t_0}(t-t_0)} + \lambda_{t_0} v_{t_0}(t)\,\mathrm{e}^{\lambda_{t_0}(t-t_0)} = A(t)\,v_{t_0}(t)\,\mathrm{e}^{\lambda_{t_0}(t-t_0)}, \quad \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{R}, \tag{2.6}$$

$$v_{t_0}(t+1)\,\lambda_{t_0}^{(t+1-t_0)} = A(t)\,v_{t_0}(t)\,\lambda_{t_0}^{(t-t_0)}, \quad \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{Z}. \tag{2.7}$$

Relation (2.3) is obtained from (2.6), taking into account that $\mathrm{e}^{\lambda_{t_0}(t-t_0)} \neq 0$ for all $t \in \mathbb{R}$. For $\lambda_{t_0} \neq 0$, relation (2.4) is obtained from (2.7), taking into account that $\lambda_{t_0}^{(t-t_0)} \neq 0$ for all $t \in \mathbb{Z}$. For $\lambda_{t_0} = 0$, relation (2.5) is obtained from (2.7), by the ω-periodicity of $v_{t_0}(\cdot)$, taking into account that $\lambda_{t_0}^{(t+1-t_0)} = 0$ and $\lambda_{t_0}^{(t-t_0)} = 1$ for $t = t_0$.

(Sufficiency) If the non-zero ω-periodic [differentiable, if $T = \mathbb{R}$] vector function $v_{t_0}(\cdot)$ of $t \in T$ satisfies (2.3)–(2.5) (with $v_{t_0}(t)$ that is arbitrarily chosen different from the zero vector for all $t \in T$, $t \neq t_0 + h\omega$, if $T = \mathbb{Z}$ and $\lambda_{t_0} = 0$), then relations (2.6), (2.7) hold, whence $\xi_{t_0}(t)$ defined in (2.2) is solution of (2.1) from the initial time $t = t_0 \in T$.

Remark 1. By Lemma 2, in the cases $T = \mathbb{R}$ and $T = \mathbb{Z}$, $\lambda_{t_0} \neq 0$, a solution $z_{t_0}(t)$ of (2.3) (if $T = \mathbb{R}$), or of (2.4) (if $T = \mathbb{Z}$, $\lambda_{t_0} \neq 0$), is a right eigenvector at the initial time $t_0 \in T$ of $A(t)$ with eigenvalue λ_{t_0} if and only if it is an ω-periodic function of $t \in T$ different from the zero vector for all $t \in T$.

Lemma 3. Let $v_{t_0}(t) \in \mathbb{C}^n$ be a right eigenvector at the initial time $t_0 \in T$ of $A(t)$. Then, the value $\lambda_{t_0} \in \mathbb{C}$ such that (2.3) (if $T = \mathbb{R}$), or (2.4) (if $T = \mathbb{Z}$ and $\lambda_{t_0} \neq 0$), or (2.5) (if $T = \mathbb{Z}$ and $\lambda_{t_0} = 0$), hold, is uniquely determined.

Proof. Consider the case $T = \mathbb{R}$. Suppose there exists two values $\hat{\lambda}_{t_0} \in \mathbb{C}$ and $\tilde{\lambda}_{t_0} \in \mathbb{C}$ such that (2.3) holds with $\lambda_{t_0} = \hat{\lambda}_{t_0}$ and $\lambda_{t_0} = \tilde{\lambda}_{t_0}$. Then, by subtraction one obtains:

$$[\tilde{\lambda}_{t_0} - \hat{\lambda}_{t_0}]\,v_{t_0}(t) = 0, \quad \forall t \in \mathbb{R}. \tag{2.8}$$

Since $v_{t_0}(t) \neq 0$ for all $t \in \mathbb{R}$, equation (2.8) implies $\tilde{\lambda}_{t_0} = \hat{\lambda}_{t_0}$, as was to be proved.

Consider the case $T = \mathbb{Z}$. Suppose there exist two values $\hat{\lambda}_{t_0} \in \mathbb{C}$, $\hat{\lambda}_{t_0} \neq 0$, and $\tilde{\lambda}_{t_0} \in \mathbb{C}$, $\tilde{\lambda}_{t_0} \neq 0$, such that (2.4) holds with $\lambda_{t_0} = \hat{\lambda}_{t_0}$ and $\lambda_{t_0} = \tilde{\lambda}_{t_0}$, i.e., such that

$$\hat{\lambda}_{t_0} v_{t_0}(t+1) = A(t)\,v_{t_0}(t), \quad \forall t \in \mathbb{Z}, \tag{2.9}$$

$$\tilde{\lambda}_{t_0} v_{t_0}(t+1) = A(t)\,v_{t_0}(t), \quad \forall t \in \mathbb{Z}. \tag{2.10}$$

By subtracting (2.9) from (2.10), one obtains

$$[\tilde{\lambda}_{t_0} - \hat{\lambda}_{t_0}]\,v_{t_0}(t+1) = 0, \quad \forall t \in \mathbb{Z}. \tag{2.11}$$

Since $v_{t_0}(t) \neq 0$ for all $t \in \mathbb{Z}$, the equation (2.11) implies $\tilde{\lambda}_{t_0} = \hat{\lambda}_{t_0}$, as was to be proved.

Finally, by absurd, suppose that (2.4) and (2.5) hold, with $\lambda_{t_0} \neq 0$. Replacing in (2.4) t by t_0, and in (2.5) h by 0, one obtains the following relations

$$\begin{aligned} v_{t_0}(t_0+1) &= \frac{1}{\lambda_{t_0}} A(t_0)\, v_{t_0}(t_0), \\ 0 &= A(t_0)\, v_{t_0}(t_0), \end{aligned}$$

which imply $v_{t_0}(t_0+1) = 0$, in contradiction with the hypothesis that $v_{t_0}(t) \neq 0$, $\forall t \in T$. □

Lemma 4. Let $v_{t_0}(\cdot) \in \mathbb{C}^n$ be an ω-periodic vector function of $t \in T$ [which is assumed to be differentiable, if $T = \mathbb{R}$], solution of (2.3) (if $T = \mathbb{R}$) or of (2.4) (if $T = \mathbb{Z}$ and $\lambda_{t_0} \neq 0$), from the initial time $t = t_0 \in T$. If $v_{t_0}(t)$ is different from the zero vector for some $t = \tilde{t} \in T$, then it is different from the zero vector for all $t \in T$.

Proof. The lemma is proved by showing that if $v_{t_0}(t)$ is equal to the zero vector for some $t = \hat{t} \in T$, then $v_{t_0}(t)$ is equal to the zero vector for $t = \tilde{t}$, that is a contradiction of the hypothesis of the lemma. By (2.3) (if $T = \mathbb{R}$), or by (2.4) (if $T = \mathbb{Z}$ and $\lambda_{t_0} \neq 0$), if $v_{t_0}(t)$ is equal to the zero vector for $t = \hat{t}$, then it is equal to the zero vector for all $t \geq \hat{t}$, whence for $t = \tilde{t}+k\omega$, for some $k \in \mathbb{Z}$ such that $\tilde{t}+k\omega \geq \hat{t}$. The proof follows by the ω-periodicity of $v_{t_0}(\cdot)$ that implies $v_{t_0}(\tilde{t}) = v_{t_0}(\tilde{t}+k\omega) = 0$. □

Let $\hat{\Phi}(t,\tau,\lambda_{t_0})$, $t,\tau \in T$, be the state transition matrix of (2.3) (if $T = \mathbb{R}$) or of (2.4) (if $T = \mathbb{Z}$ and $\lambda_{t_0} \neq 0$), i. e., a matrix such that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Phi}(t,\tau,\lambda_{t_0}) = [A(t) - \lambda_{t_0} I]\,\hat{\Phi}(t,\tau,\lambda_{t_0}), \quad \forall t,\tau \in T,\ t \geq \tau,\ \text{if } T = \mathbb{R}, \tag{2.12}$$

$$\hat{\Phi}(t+1,\tau,\lambda_{t_0}) = \frac{1}{\lambda_{t_0}} A(t)\,\hat{\Phi}(t,\tau,\lambda_{t_0}), \quad \forall t,\tau \in T,\ t \geq \tau,\ \text{if } T = \mathbb{Z},\ \lambda_{t_0} \neq 0, \tag{2.13}$$

$$\hat{\Phi}(\tau,\tau,\lambda_{t_0}) = I, \quad \forall \tau \in T. \tag{2.14}$$

By the ω-periodicity of $A(\cdot)$, it is stressed that

$$\hat{\Phi}(t+\omega,\tau+\omega,\lambda_{t_0}) = \hat{\Phi}(t,\tau,\lambda_{t_0}), \quad \forall t,\tau \in T,\ t \geq \tau. \tag{2.15}$$

The following lemma gives conditions for a complex λ_{t_0} to be an eigenvalue at the initial time $t_0 \in T$ of $A(t)$, in the cases $T = \mathbb{R}$, and $T = \mathbb{Z}$, $\lambda_{t_0} \neq 0$.

Lemma 5. (See [24] for the case $T = \mathbb{R}$ and $t_0 = 0$.) In the case $T = \mathbb{R}$, or $T = \mathbb{Z}$ and $\lambda_{t_0} \neq 0$, the complex λ_{t_0} is an eigenvalue at the initial time $t_0 \in T$ of $A(t)$ if and only if the following relation holds:

$$\det[\hat{\Phi}(t_0+\omega,t_0,\lambda_{t_0}) - I] = 0. \tag{2.16}$$

Proof. (Sufficiency) By Lemma 2, number $\lambda_{t_0} \in \mathbb{C}$ is an eigenvalue at the initial time $t_0 \in T$ of $A(t)$ if and only if there exists an ω-periodic [differentiable, if $T = \mathbb{R}$] vector function $v_{t_0}(\cdot) \in \mathbb{C}^n$ of $t \in T$, $v_{t_0}(t) \neq 0$ for all $t \in T$, such that (2.3) holds (if $T = \mathbb{R}$), or (2.4) holds (if $T = \mathbb{Z}$ and $\lambda_{t_0} \neq 0$). Any solution of (2.3) (if $T = \mathbb{R}$), or of (2.4) (if $T = \mathbb{Z}$ and $\lambda_{t_0} \neq 0$), can be written as follows:

$$v_{t_0}(t) = \hat{\Phi}(t, t_0, \lambda_{t_0})\, v_{t_0}(t_0), \quad \forall\, t \in T,\ t \geq t_0. \tag{2.17}$$

One can replace in (2.17) t by $t + \omega$, to obtain (by virtue of (2.15))

$$\begin{aligned} v_{t_0}(t+\omega) &= \hat{\Phi}(t+\omega, t_0, \lambda_{t_0})\, v_{t_0}(t_0) \\ &= \hat{\Phi}(t+\omega, t_0+\omega, \lambda_{t_0})\, \hat{\Phi}(t_0+\omega, t_0, \lambda_{t_0})\, v_{t_0}(t_0) \\ &= \hat{\Phi}(t, t_0, \lambda_{t_0})\, \hat{\Phi}(t_0+\omega, t_0, \lambda_{t_0})\, v_{t_0}(t_0), \quad \forall\, t \in T,\ t \geq t_0. \end{aligned} \tag{2.18}$$

If (2.16) holds, then there exists a vector $v_{t_0}(t_0) \neq 0$ such that

$$\hat{\Phi}(t_0+\omega, t_0, \lambda_{t_0})\, v_{t_0}(t_0) = v_{t_0}(t_0). \tag{2.19}$$

Taking into account (2.19), equation (2.18) becomes

$$\begin{aligned} v_{t_0}(t+\omega) &= \hat{\Phi}(t, t_0, \lambda_{t_0})\, v_{t_0}(t_0) \\ &= v_{t_0}(t), \quad \forall\, t \in T,\ t \geq t_0. \end{aligned} \tag{2.20}$$

The ω-periodicity of $v_{t_0}(\cdot)$ implied by (2.20), and the property that $v_{t_0}(t) \neq 0$, for all $t \in T$, implied by $v_{t_0}(t_0) \neq 0$ and by Lemma 4, prove the sufficiency of condition (2.16).

(Necessity) If the solution $v_{t_0}(t) \in \mathbb{C}^n$, $v_{t_0}(t) \neq 0$ for all $t \in T$, of (2.3) (if $T = \mathbb{R}$), or of (2.4) (if $T = \mathbb{Z}$ and $\lambda_{t_0} \neq 0$), is ω-periodic, then by replacing in (2.17) t by $t_0 + \omega$, one obtains

$$v_{t_0}(t_0) = \hat{\Phi}(t_0+\omega, t_0, \lambda_{t_0})\, v_{t_0}(t_0),$$

with $v_{t_0}(t_0) \neq 0$, which implies the necessity of condition (2.16). □

Example 1. Consider the following ω-periodic system of the form (2.1):

$$\Delta x(t) = a\left(1 - \cos\left(\frac{2\pi t}{\omega}\right)\right) x(t), \tag{2.21}$$

where a is a suitable real. For system (2.21), equations (2.3), (2.4) (with $\lambda_{t_0} \neq 0$ if $T = \mathbb{Z}$) take the following form, respectively,

$$\dot{v}_{t_0}(t) = \left(a - \lambda_{t_0} - a\cos\left(\frac{2\pi t}{\omega}\right)\right) v_{t_0}(t), \quad \text{if } T = \mathbb{R}, \tag{2.22}$$

$$v_{t_0}(t+1) = \frac{a}{\lambda_{t_0}}\left(1 - \cos\left(\frac{2\pi t}{\omega}\right)\right) v_{t_0}(t), \quad \text{if } T = \mathbb{Z}. \tag{2.23}$$

The state transition functions $\hat{\Phi}(t,\tau,\lambda_{t_0})$ of (2.22) and (2.23) are, respectively,

$$\hat{\Phi}(t,\tau,\lambda_{t_0}) = \exp\left((a-\lambda_{t_0})(t-\tau) - \frac{\omega a}{2\pi}\left(\sin\left(\frac{2\pi t}{\omega}\right) - \sin\left(\frac{2\pi\tau}{\omega}\right)\right)\right),$$
$$t,\tau \in T,\ t \geq \tau,\ \text{if } T = \mathbb{R}, \tag{2.24}$$
$$\hat{\Phi}(t,\tau,\lambda_{t_0}) = \left(\frac{a}{\lambda_{t_0}}\right)^{t-\tau} \prod_{h=\tau}^{t-1}\left(1-\cos\left(\frac{2\pi h}{\omega}\right)\right),$$
$$t,\tau \in T,\ t > \tau,\ \text{if } T = \mathbb{Z}. \tag{2.25}$$

From (2.24), (2.25), one can compute the state transition functions $\hat{\Phi}(t_0+\omega, t_0, \lambda_{t_0})$ of (2.22) and (2.23) over the period $[t_0, t_0+\omega)$ (in the case $T=\mathbb{Z}$, for the integer $h \in [t_0, t_0+\omega)$ that is multiple of ω, one has $1-\cos(\frac{2\pi h}{\omega}) = 0$):

$$\hat{\Phi}(t_0+\omega, t_0, \lambda_{t_0}) = \exp\left((a-\lambda_{t_0})\omega\right), \quad \text{if } T = \mathbb{R}, \tag{2.26}$$
$$\hat{\Phi}(t_0+\omega, t_0, \lambda_{t_0}) = 0, \quad \text{if } T = \mathbb{Z}. \tag{2.27}$$

In the case $T=\mathbb{R}$, through (2.26) one can see that the only real λ_{t_0} such that (2.16) holds, is $\lambda_{t_0} = a$; relation (2.16) is also satisfied by the complex number $\lambda_{t_0} = a + \jmath\frac{2\pi h}{\omega}$, for all $h \in \mathbb{Z}$.

In the case $T=\mathbb{Z}$, through (2.27) one can see that (2.16) is not satisfied for all complex numbers $\lambda_{t_0} \neq 0$; system (2.21) may have only the eigenvalue $\lambda_{t_0} = 0$.

In the case $T=\mathbb{R}$, a real ω-periodic solution of (2.22) with $\lambda_{t_0} = a$ is

$$v_{t_0}(t) = \exp\left(-\frac{\omega a}{2\pi}\left(\sin\left(\frac{2\pi t}{\omega}\right) - \sin\left(\frac{2\pi t_0}{\omega}\right)\right)\right) v_{t_0}(t_0)$$

for all $v_{t_0}(t_0) \in \mathbb{R}$.

Let $\Phi(t,\tau)$, $t,\tau \in T$, be the state transition matrix of (2.1), i.e., a matrix such that:

$$\Delta\Phi(t,\tau) = A(t)\,\Phi(t,\tau), \quad \forall t,\tau \in T,\ t \geq \tau, \tag{2.28}$$
$$\Phi(\tau,\tau) = I, \quad \forall \tau \in T. \tag{2.29}$$

By the ω-periodicity of $A(\cdot)$, it is stressed that

$$\Phi(t+\omega, \tau+\omega) = \Phi(t,\tau), \quad \forall t,\tau \in T,\ t \geq \tau. \tag{2.30}$$

Lemma 6. In the case $T=\mathbb{R}$, or $T=\mathbb{Z}$, $\lambda_{t_0} \neq 0$, the state transition matrices $\hat{\Phi}(t,\tau,\lambda_{t_0})$ and $\Phi(t,\tau)$ satisfy the following relations:

$$\hat{\Phi}(t,\tau,\lambda_{t_0}) = \Phi(t,\tau)\,\mathrm{e}^{-\lambda_{t_0}(t-\tau)}, \quad \forall t,\tau \in T,\ t \geq \tau,\ \text{if } T = \mathbb{R}, \tag{2.31}$$
$$\hat{\Phi}(t,\tau,\lambda_{t_0}) = \Phi(t,\tau)\,\lambda_{t_0}^{-(t-\tau)}, \quad \forall t,\tau \in T,\ t \geq \tau,\ \text{if } T = \mathbb{Z}, \lambda_{t_0} \neq 0. \tag{2.32}$$

Proof. By virtue of (2.14) and (2.29), a simple substitution shows that (2.31) and (2.32) hold for $t = \tau$.

Consider the case $T = \mathbb{R}$. One can take the time derivative of both sides of (2.31), to obtain (by virtue of (2.28))

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Phi}(t,\tau,\lambda_{t_0}) &= \mathrm{e}^{-\lambda_{t_0}(t-\tau)}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,\tau) - \lambda_{t_0}\Phi(t,\tau)\,\mathrm{e}^{-\lambda_{t_0}(t-\tau)} \\ &= A(t)\,\mathrm{e}^{-\lambda_{t_0}(t-\tau)}\Phi(t,\tau) - \lambda_{t_0}\Phi(t,\tau)\,\mathrm{e}^{-\lambda_{t_0}(t-\tau)} \\ &= [A(t) - \lambda_{t_0}I]\,\hat{\Phi}(t,\tau,\lambda_{t_0}), \quad \forall t,\tau \in \mathbb{R},\ t \geq \tau,\end{aligned}$$

as was to be shown on the basis of (2.12).

Consider now the case $T = \mathbb{Z}$, $\lambda_{t_0} \neq 0$. One can replace t by $t+1$ in both sides of (2.32) to obtain (by virtue of (2.28))

$$\begin{aligned}\hat{\Phi}(t+1,\tau,\lambda_{t_0}) &= \Phi(t+1,\tau)\,\lambda_{t_0}^{-(t+1-\tau)} \\ &= \frac{1}{\lambda_{t_0}}A(t)\,\Phi(t,\tau)\,\lambda_{t_0}^{-(t-\tau)} \\ &= \frac{1}{\lambda_{t_0}}A(t)\,\hat{\Phi}(t+1,\tau,\lambda_{t_0}), \quad \forall t,\tau \in \mathbb{Z},\ t \geq \tau,\end{aligned}$$

as was to be shown on the basis of (2.13). □

Remark 2. The analytic computation of $\Phi(t,\tau)$ is not an easy task, in general. For the case $T = \mathbb{Z}$, the state transition matrix $\Phi(t,\tau)$ is simply given by:

$$\Phi(t,\tau) = \prod_{\theta=\tau}^{t-1} A(\theta), \quad \forall t,\tau \in \mathbb{Z},\ t \geq \tau+1.$$

For the case $T = \mathbb{Z}$, the state transition matrix $\Phi(t,\tau)$ can be simply computed when matrices $A(t)$ and $\int_\tau^t A(\theta)\,\mathrm{d}\theta$ commute for all $t,\tau \in \mathbb{R}$, $t \geq \tau$, in the matrix product (see, e.g., [5]), i.e. when

$$A(t)\int_\tau^t A(\theta)\,\mathrm{d}\theta = \int_\tau^t A(\theta)\,\mathrm{d}\theta A(t), \quad \forall t,\tau \in \mathbb{R},\ t \geq \tau;$$

in such a case, one has

$$\Phi(t,\tau) = \exp\left(\int_\tau^t A(\theta)\,\mathrm{d}\theta\right).$$

The following lemma gives conditions for a complex λ_{t_0} to be an eigenvalue at the initial time $t_0 \in T$ of $A(t)$.

Lemma 7. (See [9] for the case $T = \mathbb{Z}$.) The complex λ_{t_0} is an eigenvalue at the initial time $t_0 \in T$ of $A(t)$ if and only if (only if, in the case $T = \mathbb{Z}$ and $\lambda_{t_0} = 0$) the following relation holds:

$$\det[\Phi(t_0+\omega,t_0) - \eta_{t_0}I] = 0, \tag{2.33}$$

where $\eta_{t_0} := e^{\lambda_{t_0}\omega}$ (if $T = \mathbb{R}$) or $\eta_{t_0} := \lambda_{t_0}^{\omega}$ (if $T = \mathbb{Z}$).

Proof. For $T = \mathbb{Z}$ and $\lambda_{t_0} = 0$, the proof is trivial since (2.33) reduces to $\det[\Phi(t_0 + \omega, t_0)] = 0$, which is implied by (2.5) for $h = 0$, i.e., by $\det[A(t_0)] = 0$.

For $T = \mathbb{R}$, and for $T = \mathbb{Z}$, $\lambda_{t_0} \neq 0$, the lemma is proved by the following relations, which are yielded by (2.16) and (2.31), (2.32), taking into account that $\eta_{t_0} \neq 0$:

$$\begin{aligned} 0 &= \det[\hat{\Phi}(t_0 + \omega, t_0, \lambda_{t_0}) - I] \\ &= \det[\Phi(t_0 + \omega, t_0)\eta_{t_0}^{-1} - I] \\ &= \eta_{t_0}^{-n} \det[\Phi(t_0 + \omega, t_0) - \eta_{t_0} I]. \end{aligned} \tag{2.34}$$

By virtue of (2.34), if $T = \mathbb{R}$, or $T = \mathbb{Z}$, $\lambda_{t_0} \neq 0$, then $\eta_{t_0} \neq 0$ and therefore relation (2.16) implies, and is implied, by relation (2.33). □

Definition 2. (See [24] for the case $T = \mathbb{R}$, $t_0 = 0$, and [9] for the case $T = \mathbb{Z}$.) The polynomial

$$p_{t_0}(\eta_{t_0}) := \det[\Phi(t_0 + \omega, t_0) - \eta_{t_0} I] \tag{2.35}$$

is referred to as the *characteristic polynomial at the initial time* t_0 of the ω-periodic matrix $A(t)$, and the n roots $\eta_{t_0,i}$, $i = 1, 2, \ldots, n$, of $p_{t_0}(\eta_{t_0}) = 0$ are referred to as the *characteristic multipliers at the initial time* t_0 of $A(t)$.

Lemma 8. (See [9] for the case $T = \mathbb{Z}$.) The characteristic multipliers at the initial time t_0 of the ω-periodic matrix $A(t)$ are independent of the initial time t_0.

Proof. Consider the case $T = \mathbb{R}$, and $T = \mathbb{Z}$, $\lambda_{t_0} \neq 0$. If η_{t_0} is a characteristic multiplier at the initial time t_0 of matrix $A(t)$, then equation (2.33) implies the existence of a nonzero vector $v_{t_0}(t_0)$ such that

$$[\Phi(t_0 + \omega, t_0) - \eta_{t_0} I]\, v_{t_0}(t_0) = 0. \tag{2.36}$$

The solution $v_{t_0}(t)$ of (2.3) (if $T = \mathbb{R}$), or of (2.4) (if $T = \mathbb{Z}$, $\lambda_{t_0} \neq 0$), from the initial time $t = t_0$ and such an initial condition $v_{t_0}(t_0)$ is a right eigenvector at the initial time t_0 of $A(t)$ with characteristic multiplier η_{t_0}.

For any $t_1 \in T$, $t_1 \geq t_0$, one can left multiply (2.36) by matrix $\Phi(t_1 + \omega, t_0 + \omega)$, to obtain (by virtue of (2.30))

$$\begin{aligned} 0 &= [\Phi(t_1 + \omega, t_0) - \eta_{t_0}\Phi(t_1 + \omega, t_0 + \omega)]\, v_{t_0}(t_0) \\ &= [\Phi(t_1 + \omega, t_1)\Phi(t_1, t_0) - \eta_{t_0}\Phi(t_1, t_0)]\, v_{t_0}(t_0) \\ &= [\Phi(t_1 + \omega, t_1) - \eta_{t_0}]\hat{v}(t_1), \end{aligned} \tag{2.37}$$

where $\hat{v}(t_1) := \Phi(t_1, t_0)\, v_{t_0}(t_0)$. If $\hat{v}(t_1) \neq 0$, then (2.37) implies that

$$\det[\Phi(t_1 + \omega, t_1) - \eta_{t_0} I] = 0, \tag{2.38}$$

namely that η_{t_0} is a characteristic multiplier at the initial time t_1 of $A(t)$, as to be shown. If $T = \mathbb{R}$, then $\Phi(t_1, t_0)$ is non-singular, whence $v_{t_0}(t_0) \neq 0$ implies $\hat{v}(t_1) \neq 0$. If $T = \mathbb{Z}$, $\eta_{t_0} \neq 0$, suppose, by absurd, that $\hat{v}(t_1) = 0$. Then,

$$\Phi(t_1, t_0)\, v_{t_0}(t_0) = 0. \tag{2.39}$$

From (2.39), one obtains

$$v_{t_0}(t_1) \;=\; \eta_{t_0}^{-\frac{t_1 - t_0}{\omega}} \Phi(t_1, t_0)\, v_{t_0}(t_0) \;=\; 0$$

which is a contradiction of the property that if η_{t_0} is a characteristic multiplier at the initial time t_0 of matrix $A(t)$, then the following vector function

$$v_{t_0}(t) = \eta_{t_0}^{-\frac{t - t_0}{\omega}} \Phi(t, t_0)\, v_{t_0}(t_0), \quad t \in \mathbb{Z},\ t \geq t_0,$$

is an eigenvector at the initial time $t_0 \in T$ of $A(t)$, with characteristic multiplier η_{t_0}.

Consider the case $T = \mathbb{Z}$, $\lambda_{t_0} = 0$. Since

$$\det[\Phi(t_0 + \omega, t_0)] = \prod_{k=0}^{\omega - 1} \det[A(k)], \quad \forall\, t_0 \in \mathbb{Z},$$

the property $\det[\Phi(t_0 + \omega, t_0)] = 0$ for some $t_0 = \hat{t}_0 \in T$ implies that $\det[\Phi(t_0 + \omega, t_0)] = 0$ for all $t_0 \in T$, i.e., $\eta_{t_0} = 0$ is a characteristic multiplier of $A(t)$ at each $t_0 \in \mathbb{Z}$, as was to be shown. □

Remark 3. By the proof of Lemma 8, one has obtained the following properties. In the cases $T = \mathbb{R}$, and $T = \mathbb{Z}$, $\lambda_{t_0} \neq 0$, if λ_{t_0} is an eigenvalue at the initial time t_0 of $A(t)$ for some $t_0 = \hat{t}_0 \in T$, then it is an eigenvalue at the initial time t_0 of $A(t)$ for all $t_0 \in T$. In the case $T = \mathbb{Z}$, $\lambda_{t_0} = 0$, if $\lambda_{t_0} = 0$ is an eigenvalue at the initial time $t_0 \in T$ of $A(t)$ for some $t_0 = \hat{t}_0 \in T$, the property that $\lambda_{t_0} = 0$ is an eigenvalue at the initial time $t_0 \in T$ of $A(t)$ for all $t_0 \in T$, is not necessarily true (the property $\det[A(t_0)] = 0$ for some $t_0 = \hat{t}_0 \in T$ does not imply the property $\det[A(t_0)] = 0$ for all $t_0 \in T$).

Lemma 8 allows the following definition to be introduced.

Definition 3. (See [24] for the case $T = \mathbb{R}$, and [9] for the case $T = \mathbb{Z}$.) The following polynomial

$$p(\eta) := \det[\Phi(\omega, 0) - \eta I]$$

is referred to as the *characteristic polynomial* of the ω-periodic matrix $A(t)$, and the n roots of $p(\eta) = 0$ are referred to as the *characteristic multipliers* of $A(t)$.

Example 2. The characteristic polynomial of (2.21) is

$$\begin{aligned} p(\eta) &= \mathrm{e}^{a\omega} - \eta, \quad \text{if } T = \mathbb{R}, \\ p(\eta) &= -\eta, \quad \text{if } T = \mathbb{Z}. \end{aligned}$$

The characteristic multipliers of (2.21) are

$$\begin{aligned} \eta &= \mathrm{e}^{a\omega}, \quad \text{if } T = \mathbb{R}, \\ \eta &= 0, \quad \text{if } T = \mathbb{Z}. \end{aligned}$$

3. ANALYSIS OF ω–PERIODIC INHOMOGENEOUS LINEAR SYSTEMS

Consider an ω-periodic inhomogeneous linear system described by the following equations

$$\Delta x(t) = A(t)\, x(t) + B(t)\, u(t), \tag{3.1}$$
$$y(t) = C(t)\, x(t) + D(t)\, u(t). \tag{3.2}$$

Assumption 1. The complex number $\alpha \in \mathbb{C}$ ($\alpha \neq 0$, if $T = \mathbb{Z}$) is such that $\sigma := \mathrm{e}^{\alpha\omega}$ (if $T = \mathbb{R}$), or $\sigma := \alpha^{\omega}$ (if $T = \mathbb{Z}$), is not a characteristic multiplier of $A(t)$.

Definition 4. (See [4] for the case $T = \mathbb{R}$ and $\alpha = 0$.) Under Assumption 1, the vector function $\zeta_{t_0}(\cdot)$ of $t \in T$, which is defined as follows

$$\zeta_{t_0}(t) := \begin{cases} z_{t_0}(t)\, \mathrm{e}^{\alpha(t-t_0)}, & \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{R}, \\ z_{t_0}(t)\, \alpha^{(t-t_0)}, & \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{Z}, \end{cases} \tag{3.3}$$

where $t_0 \in \mathbb{Z}$, $z_{t_0}(\cdot) \in \mathbb{C}^n$ is an ω-periodic [differentiable, if $T = \mathbb{R}$] vector function of $t \in T$, is an *exosolution at the initial time* t_0 of (3.1) if and only if it is the solution of (3.1) from the initial time $t = t_0$ corresponding to the following input vector function

$$u(t) := \begin{cases} w(t)\, \mathrm{e}^{\alpha(t-t_0)}, & \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{R}, \\ w(t)\, \alpha^{(t-t_0)}, & \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{Z}, \end{cases} \tag{3.4}$$

where $w(\cdot) \in \mathcal{S}^p$, and $\mathcal{S}^p$ is a set of complex-valued ω-periodic [continuous, if $T = \mathbb{Z}$] p-dimensional vector functions of $t \in T$; vector $z_{t_0}(t)$ is referred to as a *right exovector at the initial time* t_0 of (3.1) corresponding to the input vector function (3.4).

Lemma 9. Let $z_{t_0}(\cdot) \in \mathbb{C}^n$ be an ω-periodic vector function of $t \in T$ [which is assumed to be differentiable, if $T = \mathbb{R}$]. Then, under Assumption 1, the vector function $\zeta_{t_0}(\cdot)$ of $t \in T$ defined in (3.3), is an exosolution at the initial time $t_0 \in T$

of (3.1) corresponding to the input vector function (3.4) if and only if the following relations hold:

$$\dot{z}_{t_0}(t) = [A(t) - \alpha I]\, z_{t_0}(t) + B(t)\, w(t), \quad \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{R}, \tag{3.5}$$
$$z_{t_0}(t+1) = \frac{1}{\alpha} A(t)\, z_{t_0}(t) + \frac{1}{\alpha} B(t)\, w(t), \quad \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{Z}. \tag{3.6}$$

Proof. (Necessity) If the vector function $\zeta_{t_0}(\cdot)$ of $t \in T$ defined in (3.3) is the solution of (3.1) from the initial time $t = t_0 \in T$ corresponding to the input vector function (3.4), then the following relations hold

$$\dot{z}_{t_0}(t)\, \mathrm{e}^{\alpha(t-t_0)} + \alpha z_{t_0}(t)\, \mathrm{e}^{\alpha(t-t_0)} = A(t)\, z_{t_0}(t)\, \mathrm{e}^{\alpha(t-t_0)} + B(t)\, w(t)\, \mathrm{e}^{\alpha(t-t_0)}, \quad \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{R}, \tag{3.7}$$
$$z_{t_0}(t+1)\, \alpha^{(t+1-t_0)} = A(t)\, z_{t_0}(t)\, \alpha^{(t-t_0)} + B(t)\, w(t)\, \alpha^{(t-t_0)}, \quad \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{Z}. \tag{3.8}$$

Relation (3.5) is obtained from (3.7), taking into account that $\mathrm{e}^{\alpha(t-t_0)} \neq 0$ for all $t \in \mathbb{R}$, while relation (3.6) is obtained from (3.8), taking into account that (by Assumption 1) $\alpha^{(t-t_0)} \neq 0$ for all $t \in \mathbb{Z}$.

(Sufficiency) If the ω-periodic [differentiable, if $T = \mathbb{R}$] vector function $z_{t_0}(\cdot)$ of $t \in T$ satisfies (3.5)–(3.6), then relations (3.7), (3.8) hold, whence $\zeta_{t_0}(t)$ defined in (3.3) is a solution of (3.1) from the initial time $t = t_0 \in T$ corresponding to the input vector function (3.4). □

It is now possible to state the following corollary to Lemma 9.

Corollary 1. Under Assumption 1, a [differentiable, if $T = \mathbb{R}$] vector function $z_{t_0}(\cdot) \in \mathbb{C}^n$ of $t \in T$ is a right exovector at the initial time $t_0 \in T$ of (3.1) corresponding to the input vector function (3.4), if and only if it is an ω-periodic solution of (3.5), (3.6) from the initial time $t = t_0$.

The following three lemmas specify the conditions under which a right exovector is unique, is independent of the initial time $t_0 \in T$, and exists.

Lemma 10. Under Assumption 1, if there exists a right exovector $z_{t_0}(t)$ at the initial time $t_0 \in T$ of (3.1) corresponding to the input vector function (3.4), then it is uniquely determined.

Proof. By absurd, suppose there exist two ω-periodic vector functions $\hat{z}_{t_0}(\cdot)$ and $\tilde{z}_{t_0}(\cdot)$ of $t \in T$ [which are assumed to be differentiable, if $T = \mathbb{R}$], such that (3.5), (3.6) hold with $z_{t_0}(\cdot) = \hat{z}_{t_0}(\cdot)$ and $z_{t_0}(\cdot) = \tilde{z}_{t_0}(\cdot)$. By subtraction of the equations thus obtained, one obtains

$$\dot{\overline{z}}_{t_0}(t) = [A(t) - \alpha I]\, \overline{z}_{t_0}(t), \quad \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{R}, \tag{3.9}$$
$$\overline{z}_{t_0}(t+1) = \frac{1}{\alpha} A(t)\, \overline{z}_{t_0}(t), \quad \forall t \in T,\ t \geq t_0, \quad \text{if } T = \mathbb{Z}, \tag{3.10}$$

where $\overline{z}_{t_0}(t) := \hat{z}_{t_0}(t) - \tilde{z}_{t_0}(t)$. Since, by Assumption 1, the complex number α is such that $\sigma = e^{\alpha\omega}$ (if $T = \mathbb{R}$), or $\sigma = \alpha^{\omega}$ (if $T = \mathbb{Z}$), is not a characteristic multiplier of $A(t)$, Lemmas 2, 3 and 4 imply that $\overline{z}_{t_0}(t) = 0$ for all $t \in T$. □

Lemma 11. Under Assumption 1, if $z_{t_0}(t)$ is a right exovector at the initial time t_0 of (3.1) corresponding to the input vector function (3.4) for some $t_0 = \hat{t}_0 \in T$, then it is a right exovector at the initial time t_0 of (3.1) corresponding to the input vector function (3.4), for all $t_0 \in T$.

Proof. By Lemma 9, equations (3.5), (3.6) hold for $t_0 = \hat{t}_0$. Since, by Assumption 1, $e^{\alpha(t-t_0)} \neq 0$ for all $t \in T$, $t \geq t_0$ (if $T = \mathbb{R}$), or $\alpha^{(t-t_0)} \neq 0$ for all $t \in T$, $t \geq t_0$ (if $T = \mathbb{Z}$), with t_0 being an arbitrary element of T, one can multiply both sides of (3.5) rewritten with $t_0 = \hat{t}_0$ for $e^{\alpha(t-t_0)}$, and both sides of (3.5) rewritten with $t_0 = \tilde{t}_0$ for $\alpha^{(t-t_0)}$. The resulting equations imply that the following vector function

$$\zeta_{t_0}(t) := \begin{cases} z_{\tilde{t}_0}(t)\, e^{\alpha(t-t_0)}, & \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{R}, \\ z_{\tilde{t}_0}(t)\, \alpha^{(t-t_0)}, & \forall t \in T,\ t \geq t_0,\ \text{if } T = \mathbb{Z}, \end{cases}$$

is a solution of (3.1) from the initial time $t = t_0$ corresponding to the input vector function (3.4). The arbitrariness of t_0 proves the lemma. □

Lemmas 10 and 11 allow the term *exovector* to be used, instead of the term *exovector at the initial time* t_0, to represent an ω-periodic solution of (3.5), (3.6), which (if any) is uniquely determined. From now on, subscript t_0 will be omitted in the representation $z_{t_0}(t)$ of a right exovector.

Under Assumption 1, the following relation:

$$\begin{aligned} \det[I - \Phi(t+\omega, t)\sigma^{-1}] &= \sigma^{-n}\det[\sigma I - \Phi(t+\omega, t)] \\ &\neq 0, \quad \forall t \in T, \end{aligned}$$

where $\Phi(t, \tau)$ is the state transition matrix of (2.1), implies that matrix $[I - \Phi(t+\omega, t)\sigma^{-1}]$ is non-singular for all $t \in T$.

Lemma 12. (See [4] for the case $T = \mathbb{R}$ and $\alpha = 0$.) Under Assumption 1, the exovector $z(t)$ of (3.1) corresponding to the input vector function (3.4) exists and is given by:

$$\begin{aligned} z(t) &= [I - \Phi(t+\omega, t)e^{-\alpha\omega}]^{-1} \\ &\quad \int_t^{t+\omega} \Phi(t+\omega, \tau)\, e^{-\alpha(t+\omega-\tau)} B(\tau) w(\tau)\, d\tau, \quad \text{if } T = \mathbb{R}, \qquad (3.11) \\ z(t) &= \frac{1}{\alpha}[I - \Phi(t+\omega, t)\, \alpha^{-\omega}]^{-1} \\ &\quad \sum_{\tau=t}^{t+\omega-1} \Phi(t+\omega, \tau)\, e^{-\alpha(t+\omega-\tau)} B(\tau)\, w(\tau), \quad \text{if } T = \mathbb{Z}, \qquad (3.12) \end{aligned}$$

for all $t \in T$.

Proof. Consider the case $T = \mathbb{R}$. Any solution of (3.5) can be written as follows for an arbitrary $t_0 \in \mathbb{R}$:

$$\begin{aligned} z(t) &= \Phi(t,t_0)\,\mathrm{e}^{-\alpha(t-t_0)}z(t_0) \\ &\quad + \int_{t_0}^{t} \Phi(t,\tau)\,\mathrm{e}^{-\alpha(t-\tau)}B(\tau)\,w(\tau)\mathrm{d}\tau, \quad t\in\mathbb{R},\ t\geq t_0. \end{aligned} \tag{3.13}$$

(Necessity) Assume that the vector function $z(\cdot)$ in (3.13) is ω-periodic. Replacing in (3.13) t by $t_0+\omega$, and taking into account (by virtue of the hypotheses of the lemma) that matrix $[I-\Phi(\omega,t_0)\,\mathrm{e}^{-\alpha\omega}]$ is non singular and that $z(\cdot)$ is an ω-periodic vector function of $t\in\mathbb{R}$, one obtains that the following relation

$$\begin{aligned} z(t_0+\omega) &= \Phi(t_0+\omega,t_0)\,\mathrm{e}^{-\alpha\omega}z(t_0) \\ &\quad + \int_{t_0}^{t_0+\omega} \Phi(t_0+\omega,\tau)\mathrm{e}^{-\alpha(t_0+\omega-\tau)}B(\tau)\,w(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{3.14}$$

implies condition (3.11) for $t=t_0$. The arbitrariness of $t_0\in\mathbb{R}$ completes the proof of the necessity.

(Sufficiency) The following relations (which are obtained through (2.30), (3.11), and the ω-periodicity of $w(\cdot)$) will be useful for the sufficiency proof:

$$\begin{aligned} &\Phi(t+\omega,t_0)\,\mathrm{e}^{-\alpha(t+\omega-t_0)}\,z(t_0) \\ =\ &\Phi(t+\omega,t_0+\omega)\,\mathrm{e}^{-\alpha(t-t_0)}\,\Phi(t_0+\omega,t_0)\mathrm{e}^{-\alpha\omega}\,z(t_0) \\ =\ &\Phi(t+\omega,t_0+\omega)\,\mathrm{e}^{-\alpha(t-t_0)}z(t_0) \\ &-\int_{t_0}^{t_0+\omega} \Phi(t+\omega,t_0+\omega)\,\Phi(t_0+\omega,\tau)\,\mathrm{e}^{-\alpha(t+\omega-\tau)}B(\tau)w(\tau)\,\mathrm{d}\tau \\ =\ &\Phi(t,t_0)\,\mathrm{e}^{-\alpha(t-t_0)}z(t_0) \\ &-\int_{t_0}^{t_0+\omega} \Phi(t+\omega,\tau)\mathrm{e}^{-\alpha(t+\omega-\tau)}B(\tau)\,w(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\int_{t_0+\omega}^{t+\omega} \Phi(t+\omega,\tau)\,\mathrm{e}^{-\alpha(t+\omega-\tau)}B(\tau)w(\tau)\,\mathrm{d}\tau \\ =\ &\int_{t_0}^{t} \Phi(t+\omega,\tau+\omega)\,\mathrm{e}^{-\alpha(t-\tau)}B(\tau+\omega)\,w(\tau+\omega)\,\mathrm{d}\tau \\ =\ &\int_{t_0}^{t} \Phi(t,\tau)\,\mathrm{e}^{-\alpha(t-\tau)}B(\tau)w(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{3.16}$$

$$\int_{t_0}^{t+\omega} \Phi(t+\omega,\tau)\,\mathrm{e}^{-\alpha(t+\omega-\tau)}B(\tau)w(\tau)\,\mathrm{d}\tau$$

$$
\begin{aligned}
&= \int_{t_0}^{t_0+\omega} \Phi(t+\omega,\tau)\,\mathrm{e}^{-\alpha(t+\omega-\tau)}B(\tau)w(\tau)\,\mathrm{d}\tau \\
&\quad + \int_{t_0+\omega}^{t+\omega} \Phi(t+\omega,\tau)\,\mathrm{e}^{-\alpha(t+\omega-\tau)}B(\tau)w(\tau)\,\mathrm{d}\tau \\
&= \int_{t_0}^{t_0+\omega} \Phi(t+\omega,\tau)\,\mathrm{e}^{-\alpha(t+\omega-\tau)}B(\tau)w(\tau)\,\mathrm{d}\tau \\
&\quad + \int_{t_0}^{t} \Phi(t,\tau)\,\mathrm{e}^{-\alpha(t-\tau)}B(\tau)\;w(\tau)\,\mathrm{d}\tau. \qquad (3.17)
\end{aligned}
$$

One can replace in (3.13) t by $t+\omega$, to obtain

$$
\begin{aligned}
z(t+\omega) &= \Phi(t+\omega,t_0)\,\mathrm{e}^{-\alpha(t+\omega-t_0)}z(t_0) \\
&\quad + \int_{t_0}^{t+\omega} \Phi(t+\omega,\tau)\,\mathrm{e}^{-\alpha(t+\omega-\tau)}B(\tau)w(\tau)\,\mathrm{d}\tau. \qquad (3.18)
\end{aligned}
$$

From (3.18), taking into account (3.15)−(3.17), one obtains that $z(t+\omega)=z(t)$ for all $t\in\mathbb{R}$. Relation (3.11) is obtained from (3.18) by replacing t_0 by t.

Consider the case $T=\mathbb{Z}$. Any solution of (3.6) can be written as follows for an arbitrary $t_0\in\mathbb{Z}$:

$$
\begin{aligned}
z(t) &= \Phi(t,t_0)\,\alpha^{-(t-t_0)}z(t_0) \\
&\quad + \sum_{\tau=t_0}^{t-1} \Phi(t,\tau)\,\mathrm{e}^{-\alpha(t-\tau)}B(\tau)\,w(\tau), \quad t\in\mathbb{Z},\ t\geq t_0+1. \qquad (3.19)
\end{aligned}
$$

(Necessity) Assume that the vector function $z(\cdot)$ in (3.19) is ω-periodic. Replacing in (3.19) t by $t_0+\omega$, and taking into account (by virtue of the hypotheses of the lemma) that matrix $[I-\Phi(\omega,t_0)\,\alpha^{-\omega}]$ is non singular and that $z(\cdot)$ is an ω-periodic vector function of $t\in\mathbb{Z}$, one obtains that the following relation

$$
\begin{aligned}
z(t_0+\omega) &= \Phi(t_0+\omega,t_0)\,\alpha^{-\omega}z(t_0) \\
&\quad + \frac{1}{\alpha}\sum_{\tau=t_0}^{t_0+\omega-1} \Phi(t_0+\omega,\tau)\,\alpha^{-(t_0+\omega-\tau)}B(\tau)\,w(\tau), \qquad (3.20)
\end{aligned}
$$

implies condition (3.12) for $t=t_0$. The arbitrariness of $t_0\in\mathbb{Z}$ completes the proof of the necessity.

(Sufficiency) The following relations (which are obtained through (2.30), (3.12), and the ω-periodicity of $w(\cdot)$) will be useful for the sufficiency proof:

$$
\begin{aligned}
&\Phi(t+\omega,t_0)\,\alpha^{-(t+\omega-t_0)}z(t_0) \\
&= \Phi(t+\omega,t_0+\omega)\,\alpha^{-(t-t_0)}\Phi(t_0+\omega,t_0)\,\alpha^{-\omega}z(t_0) \\
&= \Phi(t+\omega,t_0+\omega)\,\alpha^{-(t-t_0)}z(t_0) \\
&\quad - \frac{1}{\alpha}\sum_{\tau=t_0}^{t_0+\omega-1} \Phi(t+\omega,t_0+\omega)\Phi(t_0+\omega,\tau)\,\alpha^{-(t+\omega-\tau)}B(\tau)\,w(\tau)
\end{aligned}
$$

$$= \Phi(t,t_0)\,\alpha^{-(t-t_0)}z(t_0) - \frac{1}{\alpha}\sum_{\tau=t_0}^{t_0+\omega-1} \Phi(t+\omega,\tau)\alpha^{-(t+\omega-\tau)}B(\tau)\,w(\tau), \tag{3.21}$$

$$\begin{aligned}
&\sum_{\tau=t_0+\omega}^{t+\omega-1} \Phi(t+\omega,\tau)\,\alpha^{-(t+\omega-\tau)}B(\tau)\,w(\tau)\\
&= \sum_{\tau=t_0}^{t-1} \Phi(t+\omega,\tau+\omega)\,\alpha^{-(t-\tau)}B(\tau+\omega)w(\tau+\omega)\\
&= \sum_{\tau=t_0}^{t-1} \Phi(t,\tau)\,\alpha^{-(t-\tau)}B(\tau)\,w(\tau),
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
&\sum_{\tau=t_0}^{t+\omega-1} \Phi(t+\omega,\tau)\,\alpha^{-(t+\omega-\tau)}B(\tau)\,w(\tau)\\
&= \sum_{\tau=t_0}^{t_0+\omega-1} \Phi(t+\omega,\tau)\,\alpha^{-(t+\omega-\tau)}B(\tau)\,w(\tau) + \sum_{\tau=t_0+\omega}^{t+\omega-1} \Phi(t+\omega,\tau)\,\alpha^{-(t+\omega-\tau)}B(\tau)\,w(\tau)\\
&= \sum_{\tau=t_0}^{t_0+\omega-1} \Phi(t+\omega,\tau)\,\alpha^{-(t+\omega-\tau)}B(\tau)\,w(\tau) + \sum_{\tau=t_0}^{t-1} \Phi(t,\tau)\,\alpha^{-(t-\tau)}B(\tau)\,w(\tau).
\end{aligned} \tag{3.23}$$

One can replace in (3.19) t by $t+\omega$, to obtain

$$z(t+\omega) = \Phi(t+\omega,t_0)\,\alpha^{-(t+\omega-t_0)}z(t_0) + \sum_{\tau=t_0}^{t+\omega-1} \Phi(t+\omega,\tau)\alpha^{-(t+\omega-\tau)}B(\tau)\,w(\tau). \tag{3.24}$$

From (3.24), taking into account (3.21)–(3.23), one obtains that $z(t+\omega)=z(t)$ for all $t\in\mathbb{Z}$. Relation (3.12) is obtained from (3.24) by replacing t_0 by t. □

The following lemma gives conditions for the exosolution of (3.1) corresponding to the input vector function (3.4) to be attractive.

Lemma 13. Under Assumption 1, let $z(t)\in\mathbb{C}^n$ be the exovector of (3.1) corresponding to the input vector function (3.4). Let $\zeta(t)\in\mathbb{C}^n$ be the solution of (3.1) from the initial time $t = t_0 \in T$, from an arbitrary initial state $\zeta(t_0) = x_0$, and corresponding to the input vector function (3.4). The function $\tilde{\zeta}(t) := \zeta(t) - \zeta_{t_0}(t)$, with $\zeta_{t_0}(t)$ being defined in (3.3), exponentially goes to zero for all $x_0 \in \mathbb{R}^n$, if and only if all the characteristic multipliers of $A(t)$ have modulus smaller than 1.

Proof. It is easy to see that $\tilde{\zeta}(t)$ satisfies the following equation

$$\Delta\tilde{\zeta}(t) = A(t)\tilde{\zeta}(t). \tag{3.25}$$

Function $\tilde{\zeta}(t)$ exponentially goes to zero for all $\tilde{\zeta}(t_0) \in \mathbb{R}^n$, if and only if all the characteristic multipliers of $A(t)$ have modulus smaller than 1, as was to be proved. □

The definition of steady-state solution of an ω-periodic system of the form (3.1) given in [4] for ω-period input vector functions, is extended for input vector functions of the form (3.4), by the following definition.

Definition 5. Under Assumption 1, let $\zeta_{t_0}(t)$ be the exosolution at the initial time $t_0 \in T$ of system (3.1) corresponding to the input vector function (3.4). Vector function $\zeta_{t_0}(t)$ is referred to as the *state steady-state solution* of (3.1), (3.2) corresponding to the input vector function (3.4) if and only if $\mathrm{re}[\alpha] \geq 0$ (if $T = \mathbb{R}$), or $|\alpha| \geq 1$ (if $T = \mathbb{Z}$), and all the characteristic multipliers of $A(t)$ have modulus smaller than 1. The output solution of (3.1), (3.2) corresponding to such a $\zeta_{t_0}(t)$ is referred to as the *output steady-state solution* of (3.1), (3.2).

The notion of *blocking zero* that has been given (see, e.g., [5]) for time-invariant linear systems, is extended for ω-periodic systems under the following assumption.

Assumption 2. The set $\mathcal{S}^p$ of the ω-periodic [continuous, if $T = \mathbb{R}$] functions $w(\cdot)$ has a ρ-dimensional base

$$\mathcal{B}_\rho := \{w_0(\cdot), w_1(\cdot), \ldots, w_{\rho-1}(\cdot)\},$$

for some $\rho \in \mathbb{Z}, \rho \geq 0$, i.e., for each $w(\cdot) \in \mathcal{S}^p$ there exist ρ complex numbers $c_0, c_1, \ldots, c_{\rho-1}$ such that the following relation is satisfied

$$w(t) = c_0 w_0(t) + c_1 w_1(t) + \ldots + c_{\rho-1} w_{\rho-1}(t), \quad \forall t \in T. \tag{3.26}$$

Remark 4. Assumption 2 does not seem to be restrictive.

Consider the case $T = \mathbb{R}$. For $i = 0, 1, \ldots, \omega - 1$, consider the ω-periodic function $w_i(\cdot) : \mathbb{Z} \to \{0, 1\}$ that is defined as follows

$$w_i(h + k\omega) := \delta(h - i), \quad h \in \{0, 1, \ldots, \omega - 1\}, \quad k \in \mathbb{Z},$$

where $\delta(\cdot) : \mathbb{Z} \to \{0, 1\}$ is such that

$$\delta(t) = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0. \end{cases}$$

For each ω-periodic function $w(\cdot)$, define ω complex numbers as follows:

$$c_i := w(i), \quad i = 0, 1, \ldots, \omega - 1.$$

Then, it is easy to see that (3.26) holds with $\rho := \omega$.

Consider the case $T = \mathbb{R}$ and $\alpha \in \mathbb{R}$. Assume that the input vector function (3.4) is the output free solution from the initial time $t = t_0 \in \mathbb{R}$ of the following ω-periodic system (which is usually referred to as the *exosystem*):

$$\begin{aligned} \dot{x}_u(t) &= A_u(t)\, x_u(t), & (3.27)\\ u(t) &= C_u(t)\, x_u(t), & (3.28) \end{aligned}$$

where $x_u(t) \in \mathbb{R}^\rho$ is the state, $A_u(\cdot)$ and $C_u(\cdot)$ are real matrices that are ω-periodic functions of $t \in \mathbb{R}$. Assume that $A_u(t)$ is *reducible* (see, e. g., [4]) in the sense of Floquet-Lyapunov to the diagonal constant matrix $\Lambda = \alpha I$, through the ω-periodic Lyapunov transformation matrix $W(t)$. Then, the input vector function (3.4) can be expressed by

$$u(t) = C_u(t)\, W(t)\, c\, \mathrm{e}^{\alpha(t-t_0)}, \tag{3.29}$$

where $c \in \mathbb{R}^\rho$ is a vector dependent on the initial conditions of (3.27). By comparing (3.4) with (3.29), one obtains

$$w(t) = C_u(t)\, W(t)\, c;$$

whence condition (3.26) is satisfied once the ρ column vectors of $C_u(t)\, W(t)$ are taken as the base functions $w_i(\cdot), i \in \{0, 1, \ldots, \omega - 1\}$, of $\mathcal{B}_\rho$, and the ρ entries of vector c are taken as the coefficients c_i, $i \in \{0, 1, \ldots, \omega - 1\}$.

Definition 6. Under Assumptions 1 and 2, the complex α is a *blocking zero* of system (3.1), (3.2) from the input $u(t)$ to the output $y(t)$ if and only if the output steady-state solution of (3.1), (3.2) corresponding to the input vector function (3.4) is constant and equal to zero for all the ω-periodic vector functions $w(\cdot) \in \mathcal{S}^p$.

The following lemma gives necessary and sufficient conditions for a complex α to be a blocking zero of system (3.1), (3.2).

Lemma 14. Under Assumptions 1 and 2, the complex α is a blocking zero of system (3.1), (3.2) from the input $u(t)$ to the output $y(t)$ if and only if the following conditions hold for $i = 0, 1, \ldots, \rho - 1$:

$$\mathrm{Im}\begin{bmatrix} \int_t^{t+\omega} \Phi(t+\omega,\tau)\, \mathrm{e}^{-\alpha(t-\tau)} B(\tau) w_i(\tau)\, \mathrm{d}\tau \\ D(t)\, w_i(t) \end{bmatrix} \subseteq \mathrm{Im}\begin{bmatrix} \Phi(t+\omega,t) - I\mathrm{e}^{\alpha\omega} \\ C(t) \end{bmatrix}$$
$$\forall\, t \in T, \text{ if } T = \mathbb{R}, \tag{3.30}$$

$$\mathrm{Im}\begin{bmatrix} \sum_{\tau=t}^{t+\omega-1} \Phi(t+\omega,\tau)\, \mathrm{e}^{-\alpha(t-\tau)} B(\tau)\, w_i(\tau) \\ D(t)\, w_i(t) \end{bmatrix} \subseteq \mathrm{Im}\begin{bmatrix} \Phi(t+\omega,t) - I\alpha^{\omega} \\ C(t) \end{bmatrix},$$
$$\forall\, t \in T, \text{ if } T = \mathbb{Z}. \tag{3.31}$$

Proof of Lemma 14. Since, by Assumption 1, $e^{\alpha t} \neq 0$ and $\alpha^t \neq 0$ for all $t \in T$, the complex α is a blocking zero of system (3.1), (3.2) if and only if the output response of the following system (if $T = \mathbb{R}$)

$$\dot{z}(t) = (A(t) - \alpha I)\, z(t) + B(t)\, w(t), \tag{3.32}$$

$$\overline{y}(t) = C(t)\, z(t) + D(t)\, w(t), \tag{3.33}$$

or of the following system (if $T = \mathbb{Z}$)

$$z(t+1) = \frac{1}{\alpha} A(t)\, z(t) + \frac{1}{\alpha} B(t)\, w(t), \tag{3.34}$$

$$\overline{y}(t) = C(t)\, z(t) + D(t)\, w(t), \tag{3.35}$$

to each input vector function $w(\cdot) \in \mathcal{S}^p$, is constant and equal to zero in correspondence to the unique solution of (3.32) (if $T = \mathbb{R}$), or of (3.34) (if $T = \mathbb{Z}$), that is ω-periodic. In view of Assumption 2, the linearity of systems (3.32), (3.33) and (3.34), (3.35) implies that the complex α is a blocking zero of system (3.1), (3.2) if and only if the output response of (3.32), (3.33) (if $T = \mathbb{R}$), or of (3.34), (3.35) (if $T = \mathbb{Z}$), with $w(t) = w_i(t)$, $i = 0, 1, \ldots, \rho - 1$, is constant and equal to zero in correspondence to its unique state solution that is ω-periodic. Through a reasoning similar to the one used in the proof of Lemma 12, the unique ω-periodic state and output solution of (3.32), (3.33), with $w(t) = w_i(t)$, $i = 0, 1, \ldots, \rho - 1$, can be rewritten as follows for all $t \in \mathbb{R}$:

$$z(t) = \Phi(t+\omega, t) e^{-\alpha\omega} z(t) + \int_t^{t+\omega} \Phi(t+\omega, \tau) e^{-\alpha(t+\omega-\tau)} B(\tau) w_i(\tau)\, d\tau, \tag{3.36}$$

$$\overline{y}(t) = C(t)\, z(t) + D(t)\, w_i(t), \tag{3.37}$$

and the unique ω-periodic state and output solution of (3.34), (3.35), with $w(t) = w_i(t)$, $i = 0, 1, \ldots, \rho - 1$, can be rewritten as follows for all $t \in \mathbb{Z}$:

$$z(t+\omega) = \Phi(t+\omega, t)\, \alpha^{-\omega} z(t) + \frac{1}{\alpha} \sum_{\tau=t}^{t+\omega-1} \Phi(t+\omega, \tau)\, \alpha^{-(t+\omega-\tau)} B(\tau)\, w_i(\tau), \tag{3.38}$$

$$\overline{y}(t) = C(t)\, z(t) + D(t)\, w_i(t). \tag{3.39}$$

(Necessity) If $\overline{y}(t) = 0$ for all $t \in T$, then equations (3.36), (3.37) imply the necessity of condition (3.30), while equations (3.38), (3.39) imply the necessity of condition (3.31).

(Sufficiency) Since both sides of relations (3.30), (3.31) are ω-periodic functions of $t \in T$, then conditions (3.30), (3.31) imply the existence of an ω-periodic vector function $z(t)$ such that (3.36), (3.37) (if $T = \mathbb{R}$), or (3.38), (3.39) (if $T = \mathbb{Z}$), hold with $\overline{y}(t) = 0$ for all $t \in T$. Since the exovector $z(t)$ is uniquely determined, the sufficiency proof is completed. □

4. CONCLUSIONS

In this paper we have analysed a class of linear periodic systems, both discrete-time and continuous-time. For these systems, we have introduced well known notions for linear time-invariant systems, such as the concepts of eigenvalue, eigenvector, characteristic multiplier, steady-state response, and of blocking zero. Future work will regard the possibility of using such notions for the statement of an algebraic version of the internal model principle for periodic linear systems.

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REFERENCES

[1] M. Araki and K. Yamamoto: Multivariable multirate sampled-data systems: state space description, transfer characteristics, and Nyquist criterion. IEEE Trans. Automat. Control *AC-31* (1986), 145–154.

[2] S. Bittanti: Deterministic and stochastic linear periodic systems. In: Time Series and Linear Systems (S. Bittanti, ed.), Springer–Verlag, Berlin 1986, pp. 141–182.

[3] S. Bittanti and P. Bolzern: Discrete-time linear periodic systems: gramian and modal criteria for reachability and controllability. Internat. J. Control *41* (1985), 909–928.

[4] R. W. Brockett: Finite Dimensional Linear Systems. Wiley, New York 1970.

[5] C. T. Chen: Linear System Theory and Design. Holt, Rinehart and Winston, New York 1984.

[6] P. Colaneri: Zero-error regulation of discrete-time linear periodic systems. Systems Control Lett. *15* (1990), 2, 161–167.

[7] G. Conte, A. M. Perdon and S. Longhi: Invertibility and inversion of linear periodic systems. In: Preprints of the 11th IFAC World Congress, Vol. 2, Tallin 1990, pp. 241–244.

[8] G. Conte, A. M. Perdon and S. Longhi: Zero structure at infinity of linear periodic systems. Circuits Systems Signal Process. *10* (1991), 1, 91–100.

[9] D. S. Evans: Finite-dimensional realizations of discrete-time weighting patterns. SIAM J. Appl. Math. *22* (1972), 45–67.

[10] M. Fjeld: Optimal control of multivariable periodic processes. Automatica *5* (1969), 497–506.

[11] O. M. Grasselli: A canonical decomposition of linear periodic discrete-time systems. Internat. J. Control *40* (1984), 1, 201–214.

[12] O. M. Grasselli: Dead-beat observers of reduced order for linear periodic discrete-time systems with inaccessible inputs. Internat. J. Control *40* (1984), 731–745.

[13] O. M. Grasselli and F. Lampariello: Dead-beat control of linear periodic discrete-time systems. Internat. J. Control *33* (1981), 1091–1106.

[14] O. M. Grasselli and S. Longhi: Disturbance localization with dead-beat control for linear periodic discrete-time systems. Internat. J. Control *44* (1986), 5, 1319–1347.

[15] O. M. Grasselli and S. Longhi: Output dead-beat controllers and function dead-beat observers for linear periodic discrete-time systems. Internat. J. Control *43* (1986), 2, 517–537.

[16] O. M. Grasselli and S. Longhi: Disturbance localization by measurement feedback for linear periodic discrete-time systems. Automatica *24* (1988), 3, 375–385.

[17] O. M. Grasselli and S. Longhi: Pole placement for non-reachable periodic discrete-time systems. Math. Control Signals Systems *4* (1991), 439–455.

[18] O. M. Grasselli and S. Longhi: Robust tracking and regulation of linear periodic discrete-time systems. Internat. J. Control *54* (1991), 3, 613–633.

[19] O. M. Grasselli and S. Longhi: Block decoupling with stability of linear periodic systems. J. Math. Systems Estimation Control *3* (1993), 4, 427–458.

[20] O. M. Grasselli and S. Longhi: Linear function dead-beat observers with disturbance localization for linear periodic discrete-time systems. Internat. J. Control *45* (1987), 5, 1603–1627.

[21] O. M. Grasselli and S. Longhi: Zeros and poles of linear periodic multivariable discrete-time systems. Circuits Systems Signal Process. *7* (1988), 3, 361–380.

[22] O. M. Grasselli and S. Longhi: Finite zero structure of linear periodic discrete–time systems. Internat. J. Systems Sci. *22* (1991), 10, 1785–1806.

[23] O. M. Grasselli and S. Longhi: The geometric approach for linear periodic discrete-time systems. Linear Algebra Appl. *158* (1991), 27–60.

[24] J. Hale: Theory of Functional Differential Equations. Springer–Verlag, New York 1977.

[25] M. Kono: Eigenvalue assignment in linear periodic discrete–time systems. Internat. J. Control *32* (1980), 149–158.

[26] M. Kono, T. Suzuki and T. Morishita: Block decoupling problem of linear ω–periodic discrete–time systems. IEEE Trans. Automat. Control *35* (1990), 1262–1265.

[27] S. Longhi, A. M. Perdon and G. Conte: Geometric and algebraic structure at infinity of discrete–time linear periodic systems. Linear Algebra Appl. *122/123/124* (1989), 245–271.

[28] R. A. Meyer and C. S. Burrus: A unified analysis of multirate and periodically time-varying digital filters. IEEE Trans. Circuits and Systems *22* (1975), 162–168.

[29] C. Nikias: A general realization scheme of periodically time–varying digital filters. IEEE Trans. Circuits and Systems *32* (1985), 204–207.

[30] B. Park and E. I. Verriest: Canonical forms on discrete linear periodically time–varying systems and a control application. In: Proc. of the 28th IEEE CDC, Tampa 1989, pp. 1220–1225.

[31] J. A. Richards: Analysis of Periodically Time–varying Systems. Springer–Verlag, Berlin 1983.

[32] E. I. Verriest: The operational transfer function and parametrization of N–periodic systems. In: Proc. of the 27th IEEE CDC, Austin 1988, 1994–1999.

[33] J. Vlach, K. Singhal and M. Vlach: Computer oriented formulation of equations and analysis of switched–capacitor networks. IEEE Trans. Circuits and Systems *31* (1984), 753–765.

[34] L. Zheng: Discrete–time adaptive control for periodically time–varying systems. In: Proc. of the IFAC Symp. on Adaptive Systems in Control and Signal Processing, Lund 1986, pp. 131–135.

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